Group theory

Group theory study notes

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目录

第一章 Basic Group Knowledge

1.1 Group

定义 1.1 (Equivalence Relation) If X is a set, $\sim$ is **equivalence relation**. For $\forall a, b, c \in X$.

- $a \sim a$
- $a \sim b \quad \rightarrow b \sim a$
- $a \sim b \& b \sim c \quad \rightarrow a \sim c$

定义 1.2 (equivalence class) $[a] = \{x \in X, x \sim a\}$

定义 1.3 (Group) A group is a quartet, (G,m,I,e)

- G is a set
- m: $G \times G \rightarrow G$ is a map, called group multiplication map.
- I: $G \rightarrow G$ is a map, called inverse map.
- $e \in G$ is a distinguished element of G called identity element.

They have three significant property:

- associative: $\forall g_1 \ g_2 \ g_3 \in G \ m(m(g_1, g_2), g_3) = m(g_1, m(g_2, g_3))$
- $\forall g \in G, \quad m(g, e) = m(e, g) = g$
- $\forall g \in G \quad m(I(g), g) = m(g, I(g)) = e$

定义 1.4 (abelian group) A group G is called abelian group if

$$\forall g_1 \quad g_2 \in G, \qquad g_1 g_2 = g_2 g_1 \tag{1.1}$$

定义 1.5 (Subgroup) (G,m,I,e) is a group, if $H \subset G$. m,I perserve H, which means $m : H \times H \rightarrow H$, $I : H \rightarrow H$ $(e \in H)$. Then we say that (H,m,I,e) is a subgroup of (G,m,I,e).

$$H < G \tag{1.2}$$

定义 1.6 (Group representation) A representation of Group G is a mapping D which maps elements of G onto a set of linear operators satisfying

- D(e)= $\mathcal{I}$ where $\mathcal{I}$ is the identity operator on the Linear space where the linear operators act.
- $D(g_1)D(g_2) = D(g_1 g_2)$

The dimension of linear operator acts is called the dimension of representation. (The dimension of linear operator equals to the dimension of the space of it acts.)

定义 1.7 (General Linear Group) $M_n(k)$ means a set of $n \times n$ matrixs with entries belongs to $K = \mathbb{R}$ or $k = \mathbb{C}$. However it is unital monoid, But it might not be a group cause some matrix don't have inverse matrix. So we can define a group called General Linear Group

$$GL(n, k) \equiv \{A | A = n \times n \text{ invertible matrix over k}\} \subset M_n(k) \tag{1.3}$$

There are many important groups:

$$SL(n,k) := \{A \in GL(n,k) \ : \ detA = 1\} \tag{1.4}$$

$$O(n,k) :== \{A \in GL(n,k) \ : \ AA^{tr} = 1\} \tag{1.5}$$

$$SO(n,k) :== \{A \in O(n,k) \ : \ detA = 1\} \tag{1.6}$$

$$U(n) :== \{A \in GL(n,C) \ : \ AA^{\dagger} = 1\} \tag{1.7}$$

$$SU(n) :== \{A \in U(n) \ : \ detA = 1\} \tag{1.8}$$

denote: Modular Group is called $SL(2,Z)$

定义 1.8 (Center of a group) Z(G) is a center of a group, which all elements in this group commute with it.

$$Z(G) \equiv \{z \in G | zg = gz \ \forall g \in G\} \tag{1.9}$$

1.2 Group actions on Sets

定义 1.9 (group action by group) X is a set, G is a group,if X is a Group action by a Group G. We say that X is a G-set.

now, explain what does action actually means.

Left G-action on set X is a map $\varphi : G \times X \longrightarrow X$, which is compatible with group multiplication laws.

- $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 g_2, x)$ (compatible with group multiplication laws)
- $\forall x \in X \ \ \varphi(1_G, x) = x$ (we want $1_G \circ x \mapsto x$ to be identity map)

I just wanna to say these two constrains are not the same, we cannot derive the second one from the first one.

From the first one, we know.

$$\varphi(1_G, \varphi(1_G, x)) = \varphi(1_G, x) \tag{1.10}$$

Does Not mean:

$$\forall x \in X \ \ \varphi(1_G, x) = x \tag{1.11}$$

定义 1.10 (Orbits) If group G acts on set X. We can define a **equivalence relation** on X. We say that.

$$x_1, x_2 \in X \ \ if \ \ \exists g \in G \ \varphi(g, x_1) = x_2 \ \ then \ \ x_1 \sim x_2 \tag{1.12}$$

The equivalence class [x] is called orbit of G through a point x.

$$O_G(x) = \{y : \exists g \in G \ \ s.t. \ \ y = \varphi(g, x)\} \tag{1.13}$$

The set of orbits is denoted by X/G.

proof why this definition of equivalence relation is valid.

First condition of equivalence relation would be $x \sim x$.

It is easy to know that:

$$\varphi(1_G, x) = x \tag{1.14}$$

Second condition of equivalence relation would be $x_1 \sim x_2 \rightarrow x_2 \sim x_1$ we suppose that:

$$\varphi(g_1, x_1) = x_2 \tag{1.15}$$

Then, we consider this term:

$$\varphi(g_2, x_2) = \varphi(g_2, \varphi(g_1, x_1)) = \varphi(g_1 g_2, x_1) \tag{1.16}$$

We only need $g_2 = g_1^{-1}$, then $x_2 \sim x_1$

Third condition is $if x_1 \sim x_2 \ \& \ x_2 \sim x_3 \ \ then \ \ x_1 \sim x_3$
We suppose that:

$$\varphi(g_1, x_1) = x_2 \quad \varphi(g_2, x_2) = x_3 \tag{1.17}$$

It is obvious that:

$$\varphi(g_1 g_2, x_1) = x_3 \tag{1.18}$$

The above relation shows that $x_1 \sim x_3$

Now we consider **Group Actions On Sets Induce Group Actions On Associated Function** Spaces.
Consider there are two sets X and Y. $\mathcal{F}[X \rightarrow Y]$ is a set of functions from $X \rightarrow Y$
Now there is a Left G-action defined by φ

$$\varphi : G \times X \rightarrow X \tag{1.19}$$

G action on set $\mathcal{F}[X \rightarrow Y]$ can be induced by:

$$\tilde{\varphi} : \ G \times \mathcal{F} \rightarrow \mathcal{F} \tag{1.20}$$

which satisfies:

$$\tilde{\varphi}(g, F)(x) = F(\varphi(g^{-1}, x)) \tag{1.21}$$

Now I need to explain why this is true. (Why this kind of map is Group acting on set $\mathcal{F}$)
Consider

$$\tilde{\varphi}(g, F) \circ (x) = F(\varphi(g^{-1}, x)) \tag{1.22}$$

Then consider:

$$\tilde{\varphi}(g_1, \tilde{\varphi}(g_2, F)) \circ (x) \tag{1.23}$$

This would equal to:

$$\begin{aligned}\tilde{\varphi}(g_2, F) \circ \left(\varphi(g_1^{-1}, x)\right) \\ &= F\left(\varphi(g_2^{-1}, \varphi(g_1^{-1}, x))\right) \\ &= F\left(\varphi(g_2^{-1} g_1^{-1}, x)\right) \\ &= F\left(\varphi((g_1 g_2)^{-1}, x)\right) \\ &= \tilde{\varphi}(g_1 g_2, F) \circ (x)\end{aligned} \tag{1.24}$$

This means:

$$\tilde{\varphi}(g_1 g_2, F) = \tilde{\varphi}(g_1, \tilde{\varphi}(g_2, F)) \tag{1.25}$$

1.3 Symmetric Group

定理 1.1 (Cayley's theorem) Any finite group is isomorphic to a subgroup of a permutation group S_N for some N.

Proof

suppose there is a finite group G. $\{g_1, g_2, \cdots\}$. define a map called $L(h)$ $h \in G$.

$$L(h) \ : g \mapsto hg \ g \in G \ L(h) \in S_G \tag{1.26}$$

Why L(h) $\in$ S_G.

After impose $L(h)$ to the group G. we obtain:

$$\{hg_1, hg_2, \cdots\} \tag{1.27}$$

We say that $L(h)$ is a 1-1, onto map.(a permutation)

1-1: if

$$hg_i = hg_j \tag{1.28}$$

apply h^{-1} to the left, then: $g_i = g_j$

onto: We wanna to say that for any g_i, we can alwways find a g_x to let $hg_x = g_i$. This is true cause $g_x = h^{-1} g_i$.

So a 1-1, onto map from G to G is called **permutation**.

Also it is easy to say that element in the set $\{L(h) | h \in G\}$ conservers group multiplication of the group S_G.

$$L(h_1) \circ L(h_2) = L(h_1 h_2) \tag{1.29}$$

So $\{L(h) | h \in G\}$ a subgroup of group S_G, Also， the map $h \mapsto L(h)$ is a **1-1, onto** map.

$$L(h_1) = L(h_2) \ \text{ only when } \ h_1 = h_2 \tag{1.30}$$

So we say that G is isomorphic to a subgroup of S_G

Cyclic permutation and cylcle decomposition

cyclic permutation: let G is a set with **order** has n elements. Suppose that $a_1 \cdots a_l$ are l distinct number between 1 and n. here is the operation (called **cyclic permutation**):

$$g_{a_1} \to g_{a_2} \to g_{a_3} \cdots g_{a_l} \to g_{a_1} \tag{1.31}$$

We call this: (obviously, there are l different ways to write this)

$$\phi : \quad (a_1, a_2 \cdots a_l) \tag{1.32}$$

Cycle decomposition means:Any permutation $\sigma \in S_n$ can be uniquely written as a product of disjoint cycles. For example, there is a cycle decomposition in S_{11}

$$\sigma = (12)(34)(10, 11)(56789) \tag{1.33}$$

Any cycle can be written as a product of transportation

Consider a permutation φ

$$\begin{pmatrix} 1 & 2 & \cdots & k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix} \tag{1.34}$$

Now consider a relation:

$$\varphi \circ (1, 2, \cdots k) \circ \varphi^{-1} = (a_1, a_2 \cdots a_k) \tag{1.35}$$

The relation above can be proved like this, Consider the left hand side of equation:

$$\begin{aligned} &\begin{pmatrix} 1 & 2 & \cdots & k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & k \\ 2 & 3 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ 1 & 2 & \cdots & k \end{pmatrix} \\ &= \begin{pmatrix} 2 & \cdots & k & 1 \\ a_2 & \cdots & a_k & a_1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & k \\ 2 & 3 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ 1 & 2 & \cdots & k \end{pmatrix} = (a_1, a_2 \cdots a_k) \end{aligned} \tag{1.36}$$

However, we can alwawys write $(1, 2, \cdots k)$ as:

$$(1, k)(1, k-1) \cdots (1, 2) = (1, 2, \cdots k) \tag{1.37}$$

This equation can be proved by Mathematical induction...

In this case

$$\varphi \circ (1, 2, \cdots k) \circ \varphi^{-1} = \varphi \circ (1, k) \circ \varphi^{-1} \circ \varphi \circ (1, k-1) \circ \varphi^{-1} \cdots \varphi \circ (1, 2) \circ \varphi^{-1} \tag{1.38}$$

However:

$$\varphi \circ (1, k) \circ \varphi^{-1} = (a_1, a_k) \tag{1.39}$$

In this case:

$$(a_1, a_k)(a_1, a_{k-1}) \cdots (a_1, a_2) = (a_1, a_2 \cdots a_k) \tag{1.40}$$

This is the reason why we say that Any cycle can be written as a product of transportation. (I think transportation means exchange over two elements)

We should notice that every element in permutation group can be represented by a product of transportation. We say that transportation is the generator of the permutation group.

1.4 Generators and Relations

定义 1.11 (Generating set of a group) A subset $S \subset G$ is a generating set of a Group, if every element $g \in G$ can be written as a product of elements of S.

$$g = s_{i_1} s_{i_2} \cdots s_{i_r} \tag{1.41}$$

For **finitely generated group** (elements in generating group is finite). We write: (I think the left side represents the generators)

$$G = \langle g_2 \cdots g_n | R_1 \cdots R_r \rangle \tag{1.42}$$

In this relation, R_i means term represented by elements in S which will be set to 1.

However it is convenient to exclude 1 from the Generating Set S, There are two reasons. we can write $s^0 = 1$. And we can write s^n, for $n < 0$, this means: $s^{-|n|}$

A generating set that contains s^{-1} for every generator s is said to be **symmetric**. For this kind of set, we can construct 1 by: $ss^{-1} = 1_G$

Most General group with one generator and one relation:

$$\langle a | a^N = 1 \rangle \tag{1.43}$$

定义 1.12 (Free group) If there is no relation on generating set S. We can define free group on set S, called $F(S)$. Which is generated by generating group S.

定义 1.13 (Coxter Group) Coxter group can be represented as: (let m be an $n \times n$ symmetric matrix)

$$\langle s_1 \cdots s_n | \forall i, j (s_i s_j)^{m_{ij}} = 1 \rangle \tag{1.44}$$

when $m_{ij} = +\infty$ it means there is no relation!

We have a restriction:

$$m_{ii} = 1 \tag{1.45}$$

Which means:

$$s_i s_i = 1 \tag{1.46}$$

This kind of element (group element that squares to 1) is called **involution**.

Then we consider another situation ($m_{ij} = 2$), In this situation: (we used $m_{ii} = 1$)

$$\begin{gathered} s_i s_j \; s_i s_j = 1 \\ s_i s_j = s_j s_i \end{gathered} \tag{1.47}$$

I think I should study detail into classification of coxter group.

Reflection Group

Actually, we will talk about The Reflection group Generated by Reflections in the plane orthogonal to vector v_i. ($v_i \in \mathbb{R}^N$).

We consider n vectors in the space $\mathbb{R}^N$.(I think it needs to satisfies $N \geq n$) There relation would be:

$$v_i \cdot v_j = -2\cos(\frac{\pi}{m_{i,j}}) \tag{1.48}$$

The reflection is a Map:

$$P_{v_i}: \ v \mapsto v - 2\frac{v \cdot v_i}{v_i \cdot v_i} v_i \tag{1.49}$$

For this , We can let these reflections be generators, and we construct a Reflection group:

$$\langle P_{v_i} | (P_{v_i} P_{v_j})^{m_{i,j}} = 1 \rangle \tag{1.50}$$

Okay, I wanna to say that what does $P_{v_i} P_{v_j}$ means. For simplicity, Consider 3-dimension situation:

$$v_i = \sqrt{2}\hat{i} \tag{1.51}$$

$$v_j = -\sqrt{2}\cos(\frac{\pi}{m_{i,j}})\hat{i} + \sqrt{2}\sin(\frac{\pi}{m_{i,j}})\hat{j} \tag{1.52}$$

$$v = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{1.53}$$

It can be calculated that, After the double reflection, the vector would be: (Without the change of z direction)

$$\begin{pmatrix} \cos(2\frac{\pi}{m_{i,j}}) & \sin(2\frac{\pi}{m_{i,j}}) \\ -\sin(2\frac{\pi}{m_{i,j}}) & \cos(2\frac{\pi}{m_{i,j}}) \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{1.54}$$

This states that $P_{v_i} P_{v_j}$ will make a clock-wise rotation in the plane determined by v_i and v_j. With the angle 2 times of the angle between v_i and v_j.
In this consideration, it is not hard to understand the relation:

$$(P_{v_i} P_{v_j})^{m_{i,j}} = 1 \tag{1.55}$$

With $m_{i,i} = 1$ and need m to be a symmetric matrix.

1.5 Cosets

定义 1.14 (Left Coset (陪集) of H) H is a sub group of G. Then the left coset of H is defined as: (it is not a Group! but a set)

$$gH := \{gh | h \in H\} \subset G \tag{1.56}$$

There are three important properties of Left Coset of H.

1. Left Cosets are identical or disjoint.
2. Every elements $g \in G$ lies in some Left Coset.
3. We can define an equivalence principle $g_1 \sim g_2$ if $\exists h \in H$ s.t. $g_1 = g_2 h$

Rearrangemet lemma (重排定理) To begin with, I need to review rearrangement lemma. Rearrangement lemma states that:

$$\text{For a group H;} \quad hH = H \tag{1.57}$$

证明 It is obvious that 1.hH is close in group multiplication 2. it has inverse element 3. It has Identity element. So it is a Group. But we wanna to state that this group is exactly H. This is because (A) we can find all the element of H in group hH. This is obvious because:

$$\begin{aligned} hh_x &= h_1, \\ h_x &= h^{-1}h_1 \Rightarrow \forall h_1 \in H \;\; \exists h_x \; s.t. hh_x = h_1 \in hH. \end{aligned} \tag{1.58}$$

And also (B) we can find all the element of hH in H.

□

Then we back to the three properties of the Left Coset of H.

1. Left Cosets are identical or disjoint(陪集定理) We prove: Not disjoint means identical.

$$\text{Not disjoint} \Rightarrow \quad \text{Identical} \tag{1.59}$$

证明 if g_1H and g_2H are not disjoint. Then, this means:

$$\exists g \in g_1H \cap g_2H \tag{1.60}$$

Then we would say:

$$\begin{aligned} g &= g_1h_1 \\ g &= g_2h_2 \end{aligned} \tag{1.61}$$

Which leads to:

$$g_1 = g_2h_2h_1^{-1} = g_2h_3 \tag{1.62}$$

What we do means **if two Left Coset deduced by g_1 and g_2 are not disjoint, then** $\exists\ h\ s.t. g_1 = g_2h$ Then we rearrangement lemma, the Left Coset deduced by g_1 and g_2 are the same!

□

2. Every elements $g \in G$ lies in some Left Coset. This property is easy to prove. because H contains identity element. g always lies in Left Coset gH

3. Can define an equivalence principle $g_1 \sim g_2$ if $\exists h \in H$ s.t. $g_1 = g_2h$ Obviously, 1. $g_1 \sim g_1$ 2.$g_1 \sim g_2 \to g_2 \sim g_1$ 3. $if\ g_1 \sim g_2\ \ g_2 \sim g_3 \to g_1 \sim g_3$

Also, this equivalence means somethingeles. From 1st property:

$$\text{Not disjoint} \Rightarrow g_1 \sim g_2 \Rightarrow \text{Identical} \tag{1.63}$$

We would say that if $g_1 \sim g_2$, Then the left coset defined by them are the same, otherwise, they are disjoint.

Lagrange Theorem (拉格朗日定理) From 2st: every g lies in some Coset, From 1st: Coset are identical or disjoint, We can actually define a set of Cosets of Subgroup denoted by G/H. This decompose G into cosets.

$$G/H = \{g_1H,\ g_2H, \cdots\} \quad g_iH \cap g_jH = \emptyset \ \ s.t.\ \ G = \amalg_{i=1}^{m} g_iH \tag{1.64}$$

Lagrange Theorem: if group H is a subgroup of a group G, Then the order of H divides the order of G.

1.6 Conjugate

定义 1.15 (Conjugate) Suppose G is a group, we say that group element h is conjugate to h' if:

$$\exists g \in G \ s.t.\ h' = ghg^{-1} \tag{1.65}$$

□

Conjugate Class (共轭类) Using this definition, we can define **Conjugate Class of h** or should I say equivalence class:

$$C(h) := \{ghg^{-1} \mid g \in G\} \tag{1.66}$$

Element in conjugate class divides order of G . For element g, define $F = \{f | fg = gf \ \ f \in G\}$. F has group structure. $f_1 \in F$; $f_2 \in F \to f_1 f_2 \in F$; $e \in F$; $f_1 \in F \to f_1^{-1} \in F$. F is a subgroup of G. G can be expanded with cosets of subgroup F.

$$G = F \cup g_1F \cup g_2F \cdots \tag{1.67}$$

Each coset leads to same conjugate element: $g_if \ \ g \ \ (g_if)^{-1} = g_if \ \ g \ \ f^{-1}(g_i)^{-1} = g_igg_i^{-1}$. Means element in conjugate class divides order of G (equal to element of cosets).

Conjugate subgroup if there is a subgroup of G: $H \subset G \ \ K \subset G$. Then we say that H is conjugate to K (共轭子群) if:

$$\exists g \in G s.t. K = gHg^{-1} := \{ghg^{-1} : h \in H\} \tag{1.68}$$

In abelian Group, every element form a conjugacy class.

Normal(invariant) subgroup and quotient group (不变子群与商群) .

定义 1.16 (normal subgroup / Invariant subgroup) Subgroup H, $H \subseteq G$ is called normal subgroup or invariant subgroup if

$$\forall g \in G \quad gNg^{-1} = N \tag{1.69}$$

Sometimes denoted as: $H \lhd G$

□

If $N \lhd G$ is a normal subgroup, then the set of left cosets $G/N = \{gN|g \in G\}$ has a natural group structure with group multiplication defined by:

$$(g_1 n_1)(g_2 n_2) = (g_3 n_3) \tag{1.70}$$

Group G/N is known as **quotient group(商群)**. Below, prove quotient group has group structure.

1 Identity element: $n_1\ g_i n_2 = g_i g_i^{-1} n_1 g_i n_2 = g_i n_3 \Rightarrow N \circ gN \to gN$

2 Inverse element: $g^{-1} n_i\ g n_j = n_k \Rightarrow g^{-1} N \circ gN = N$

3 group multiplication law, prove by contradict. If $g_i n_\alpha g_j n_\beta = g_k n_\gamma$ & $g_i n'_\alpha g_j n'_\beta = g'_k n'_\gamma$

$$\begin{aligned} \Rightarrow g_i &= g_k n_\gamma n_\beta^{-1} g_j^{-1} n_\alpha^{-1} = g'_k n'_\gamma n_\beta'^{-1} g_j^{-1} n_\alpha'^{-1} \\ g_k n_a g_j^{-1} n_\alpha^{-1} g_j g_j^{-1} &= g'_k n_b g_j^{-1} n_\alpha'^{-1} g_j g_j^{-1} \\ g_k n_a n_c g_j^{-1} &= g'_k n_b n_d g_j^{-1} \\ g_k &= g'_k n_e \end{aligned} \tag{1.71}$$

Coeset generated by g_k and g'_k are the same, contradicts with supposition.

1.7 Homomorphism and Isomorphism

定义 1.17 (Homomorphism and Isomorphism) Consider two groups, They are: (G,m,I,e) and (G',m',I',e')

- A homomorphism（同态）is a map from G to G', which preserve group law:

$$\begin{aligned} \varphi : G \to G' \quad \forall g_1, g_2 \in G \\ \varphi(m(g_1, g_2)) = m'(\varphi(g_1), \varphi(g_2)) \end{aligned} \tag{1.72}$$

- If φ is a 1-1 and onto map. Then it is called **Isomorphism**（同构）
- when G=G' and φ is an Isomorphism, φ is called the automorphism of G.

A common slogan is Isomorphic Group are the same.

□

定义 1.18 (Vector Space) Consider a vector space over field F, There is a non-empty set V. two binary operations.

element in F are called scalars, element in V are called vectors.

There are 8 axioms.

- Associative of vector addition: $u + (v + w) = (u + v) + w$.
- Commutativity of vectors addition: $u + v = v + u$.
- Identity in vector addition: $\exists 0 \in V \quad \forall u \in V\ u + 0 = u$
- Inverse in vector addition: $\forall u \in V,\ \exists -u \in V\ s.t.\ u + (-u) = 0$
- Compatibility of scalar multiplication with field multiplication: $a(bv) = (ab)v$
- Identity element of scalar multiplication: $\exists 1 \in F\ 1v = v$
- Distributivity of scalar multiplication with respect to vector addition $\quad a(u + v) = au + av$

- Distributivity of scalar multiplication with respect to field addition $(a+b)u = au + bu$

□

定义 1.19 (Matrix Representation of group G) A matrix representation of group G is a **homomorphism**（同态）:

$$T:\ G \to GL(n,k) \tag{1.73}$$

for some positive integral n and field k.
V is a vector space over a field k. $GL(V)$ represents all invertible linear transformations of V. Then

$$T : G \to GL(V) \tag{1.74}$$

is a Representation of group G. where V is a carrier space.

□

Automorphism Automorphism group of G:

$$Aut(G) = \{v|v = Iso(G,G)\} \tag{1.75}$$

It satisfies group structure.

Hom-Kernel theorem 同态核定理 For a homomorphism $G \to H$.

1 Its kernel is invariant subgroup. $F \triangleleft G$
2 Quotient group of kernel (G/F) is isomorphic to H.(each coset of F corresponds to one element in H)

1.8 Direct Product and Semi Direct Product

Direct product group Exist group G_1 and G_2, direct product group is :

$$G = G_1 \otimes G_2 = \{(g_{1\alpha}, g_{2\beta})|g_{1\alpha} \in G_1\ g_{2\beta} \in G_2\} \tag{1.76}$$

It has structure of group.

Direct product decomposition G has subgroups G_1, G_2. $\forall g \in G$,exist one $g_{1\alpha} \in G_1$ and one $g_{2\beta} \in G_2$ s.t. $g = g_{1\alpha}g_{2\beta}$, $g_{1\alpha}g_{2\beta} = g_{2\beta}g_{1\alpha}$ Then $G = G_1 \otimes G_2$

Properties of direct product decomposition .

1. $G_1 \cap G_2 = \{e\}$ Prove by contradiction. If $G_1 \cap G_2 = \{e, a\} \Rightarrow (e,a) = (a,e)$ Contradicts to requirement each element in G can only represent in 1 way.

2. G_1 and G_2 are invariant subgroup of G Using commute property between $g_{1\alpha}$ and $g_{2\alpha}$

$$\begin{aligned}(g_{1\alpha}g_{2\beta})g_{1i}(g_{1\alpha}g_{2\beta})^{-1} &= g_{1\alpha}g_{2\beta}g_{1i}g_{2\beta}^{-1}g_{1\alpha}^{-1} \\ &= g_{1\alpha}g_{1i}g_{1\alpha}^{-1} \in G_1\end{aligned} \tag{1.77}$$

Semi direct product Exist group G_1, G_2, $A(G_1) < Aut(G_1)$, $f = Hom\,(G_2, A(G_1))$.

$$G = G_1 \otimes_s G_2 = \{(g_{1\alpha}, g_{2\beta})|g_{1\alpha} \in G_1 \ g_{2\beta} \in G_2\} \tag{1.78}$$

$$(g_{1\alpha}, g_{2\beta})(g_{1\alpha'}, g_{2\beta'}) = (g_{1\alpha}f_{g_{2\beta}}(g_{1\alpha'}), g_{2\beta}g_{2\beta'}) \tag{1.79}$$

It has group structure.

properties of simi direct product (G_1, e_2) is its invariant subgroup, but (e_1, G_2) is not invariant subgroup.

第二章 Representation Theory

2.1 Unitary Theory

A. finite groups have equivalent unitary representations

for finite groups and for compact Lie groups, all representations are equivalent to a unitary one.

proof ...

定理 2.1 every representation of a **finite group** is **equivalent** to a unitary representation.

证明 For representation in **complex space**, define

$$S = \Sigma_{g\in G} D(g)^{\dagger} D(g) \tag{2.1}$$

S is **hermitian** and is **positive semidefinite**. So we can decompose S matrix. (U is unitary matrix)

$$S = U^{\dagger} d U \tag{2.2}$$

$$d = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \end{bmatrix} \tag{2.3}$$

We want to say that$\forall j, d_j > 0$. if $d_j = 0$, then, $\exists \alpha\ s.t.\ S\alpha = 0$, then $\alpha^{\dagger} S\alpha = 0$, which means:

$$\Sigma_{g\in G} \alpha^{\dagger} D(g)^{\dagger} D(g)\alpha = \Sigma_g (D(g)\alpha)^{\dagger}(D(g)\alpha) = \Sigma ||D(g)\alpha||^2 = 0 \tag{2.4}$$

This is not possible because $D(e) = I$, Then we define Hermitian matrix X:

$$X = S^{1/2} = U^{\dagger} \begin{bmatrix} \sqrt{d_1} & & \\ & \sqrt{d_2} & \\ & & \ddots \end{bmatrix} U \tag{2.5}$$

Then we have a similarity transformation ($X^{\dagger} = X$, $(X^{-1})^{\dagger} = X^{-1}$)

$$D'(g) = X D(g) X^{-1} \tag{2.6}$$

Then (S=XX)

$$D'(g)^{\dagger} D'(g) = X^{-1} D(g)^{\dagger} X X D(g) X^{-1} \tag{2.7}$$

however: (gG=G)

$$\begin{aligned} D(g)^{\dagger} X X D(g) &= D(g)^{\dagger} S D(g) \\ &= D(g)^{\dagger}\left(\Sigma_{h\in G} D(h)^{\dagger} D(h)\right) D(g) \\ &= \Sigma_{h\in G} D(hg)^{\dagger} D(hg) \\ &= \Sigma_{h\in G} D(h)^{\dagger} D(h) = S = X^2 \end{aligned} \tag{2.8}$$

Then

$$D'(g)^{\dagger} D'(g) = I \tag{2.9}$$

Which means the representation D is equivalent to a unitary representation D'.

□

B. Unitary representations are always completely reducible.

定理 2.2 Every representation of a **finite group** is **completely reducible**. (Finite group ↔ Unitary group)

Here have two proof,

证明 We need to prove that if V_1 is an invariant subspace, then its complement $V_1^{\perp}$ is an invariant subspace. if $v \in V_1$ and $w \in V_1^{\perp}$ then: $\langle v, w \rangle = 0$ Then:

$$0 = \langle D(g)v, w \rangle = \langle v, D(g)^{\dagger} w \rangle = \langle v, D(g^{-1})w \rangle \tag{2.10}$$

Which means:

$$D(g)w \in V_1^{\perp} \tag{2.11}$$

Then, its complement is an invariant subspace.

□

The below is a proof from the book (Lie groups and particle physics–I don't remember)

证明 If P is a projection operator, and the group is a reducible group, then:

$$PD(g)P = D(g)P \tag{2.12}$$

apply conjugate to the operator above:

$$PD^{\dagger}(g)P = PD(g)^{\dagger} \tag{2.13}$$

As the theory above, We can only consider unitary representation. Which means:

$$D(g)^{\dagger} = D(g)^{-1} = D(g^{-1}) \tag{2.14}$$

(I don't quite understand why $D(g)^{-1} = D(g^{-1})$–Because the definition of the group), We say that (don't know why–Rearrangement Lemma) when g travel through all the element in G, g^{-1} will travel through all the elements too. So,

$$PD(g)P = PD(g) \tag{2.15}$$

however, this equation is equal to:

$$(I - P)D(g)(I - P) = D(g)(I - P) \tag{2.16}$$

So we say that it is completely reducible.

□

C. Every finite group is equivalent to direct sum of ir-Unitary-Representation

2.2 Shur's Lemma

To begin with, we need to define a intertwiner.

定义 2.1 (Intertwiner) And intertwiner between Representation D_1 and D_2 is a linear operator from space V_1 to space V_2. (Suppos two representations have same Field $\mathbb{K} = \mathbb{R}\ or\ \mathbb{C}$)

$$F : V_1 \to V_2 \tag{2.17}$$

Which commutes with G:

$$FD_1(g) = D_2(g)F \tag{2.18}$$

□

1. The kernel and the image of F are invariant subspaces of D_1 and D_2 Firstly, We talk about kernel:

$$if\ v \in ker\ F \Rightarrow FD_1(g)v = D_2(g)Fv = D_2(g)0 = 0 \Rightarrow D_1(g)v \in ker\ F \tag{2.19}$$

This proves that The kernel is an invariant subspace of D_1.

Then we talk about the Image space:

$$if\ w_2 = Fw_1 \Rightarrow D_2(g)w_2 = D_2(g)Fw_1 = FD_1(g)w_1 \in Img\ F \tag{2.20}$$

Which states that the Image space is an invariant subspace of D_2.

2. If D_1 is irreducible, F is injective or zero

$$D_1 \text{ is irreducible \& ker is invariant subspace} \Rightarrow Ker = \{0\}\ or\ V_1 \tag{2.21}$$

$Ker = V_1$ means F is Zero.

$Ker = \{0\}$ means F is injective. P.S Injective means **o**ne to one. We need to know why injective can be derived from $kerF = \{0\}$. Here prove by contradiction let's suppose F is not injective:

$$Fv_1 = v_2\ \&\ Fv_1' = v_2 \quad v_1' \neq v_1 \tag{2.22}$$

This means that: (This is a special case in linear space)

$$F(v_1 - v_1') = 0 \tag{2.23}$$

Then, we know that:

$$v_1 - v_1' \in\ kerF \tag{2.24}$$

As we know that:

$$kerF = \{0\} \tag{2.25}$$

Then:

$$v_1 - v_1' = 0 \to v_1 = v_1' \tag{2.26}$$

Which contradicts to supposition, then F is injective.

3. If D_2 is irreducible, F is either surjective or zero **Surjective** means onto.

$$D_2 \text{is irreducible \& Img is invariant subspace} \Rightarrow \; Img = \{0\} \; or \; V_2 \tag{2.27}$$

F=0 or D_1 equal to D_2 If F is injective(1-1), it can't be zero operator,→ F is 1-1 onto. Otherwise, F is zero.

If F is 1-1 onto, $dim(V_1) = dim(V_2)$, D_1 and D_2 are equivalent

引理 2.1 (Shur's Lemma 1) For representation $D_1(g)$ $D_2(g)$, Exists a intertwiner s.t. $FD_1(g) = D_2(g)F$, then $D_1(g)$ is equivalent with $D_2(g)$ or $F = 0$.

□

Shur's Lemma 2 .

引理 2.2 (Shur's Lemma 2) Consider the situation of $D_1 = D_2$, Then F is an linear operator from V_1 to V_1 .

if D is an **irreducible** finite-dimensional $\mathbb{C}$ Field representation, $\exists F : V \to V$ s.t. $\forall g \in G \; FD(g) = D(g)F$. Then, $F = \lambda I$.

For a less formal representation: (we use the language of matrix to say this) A matrix which commutes with all matrices of an irreducible representation is proportional to unit matrix. Need to know that we need V to be a complex vector space, a real vector space might not have real eigen vector.

□

证明 If F is an matrix, then consider it has eigen value λ and eigen vector v. As we know the constrain for the intertwiner is that it commutes with Representation operator:

$$FD(g) = D(g)F \tag{2.28}$$

However, we notice that

$$F - \lambda I \tag{2.29}$$

is also an intertwiner. But this linear operator has a kernel space:

$$(F - \lambda I)v = 0 \tag{2.30}$$

But as we discussed before, if D is an irreducible representation, it means that the kernel space should be empty or be the vector space V itself. So we need to let this new intertwiner vanish all the vector in the space, which means that:

$$F = \lambda I \tag{2.31}$$

□

If D_1 are not equal to D_2, the intertwiner is unique upto a constant. This is because $F_2^{-1}F_1$ is a self-intertwiner in D_1 which is porpotional to identity operator, then:

$$F_1 = \lambda F_2 \tag{2.32}$$

2.3 Regular Representation

Group Algebra Vector Space with an extra bilinear product operation is called Algebra. We have a vector space of linear combination of group elements: (Field is $\mathbb{C}$, We always let v_g to be complex number)

$$v = \sum_g v_g g \tag{2.33}$$

with an addition operation:

$$v + w = \sum_g (v_g + w_g) g \tag{2.34}$$

vector space with an extra bilinear product operation is called Algebra. The bilinear product operation would be:

$$v \cdot w = \sum_{gg'} v_g w_{g'} gg' = \sum_h (\sum_g v_g w_{g^{-1}h}) h \tag{2.35}$$

I will make slightly explain to this formula. Firstly, it would be easy to realize:

$$g' = g^{-1} h \tag{2.36}$$

while:

$$gg' = h \tag{2.37}$$

okay, then why

$$\sum_g \sum_{g'} = \sum_g \sum_h \tag{2.38}$$

Actually, we only need to know:

$$\sum_{g'} = \sum_h \tag{2.39}$$

this is because:

$$g' = g^{-1} h \tag{2.40}$$

Using Rearrangement lemma, we know $\{g^{-1}h | \ h \in G\}$ contains all the element in G.

Now define inner product:

$$\langle v, w \rangle = \sum_g v_g^* w_g \tag{2.41}$$

Regular representation .

$$D_{reg}(g) : \ v \mapsto g \cdot v \tag{2.42}$$

Actually,

$$g \cdot v = \sum_h v_h gh = \sum_{h'} v_{g^{-1}h'} h' \tag{2.43}$$

Which means:

$$(g \cdot v)_h = v_{g^{-1}h} \tag{2.44}$$

in regular representation, there is only one element in each row and colume.

$$(D_{reg}(g))^i{}_j = 1 \text{ only when },\ g_j = g^{-1}h\ ,\ g_i = h \rightarrow g_i = gg_j \tag{2.45}$$

Regular representation is **unitary** because:

$$\begin{aligned} (D^{\dagger}_{reg}(g))^i{}_j &= 1 \text{ only when } gg_i = g_j \\ (D_{reg}(g^{-1}))^i{}_j &= 1 \text{ only when } g^{-1}g_j = g_i \end{aligned} \tag{2.46}$$

They are the same!

Regular representation is unitary, so it is completely reducible to irreducible components.

2.4 Great Orthogonality Theorem

A. Orthogonality theorem for representation Defines(μ, ν are Irreducible, Inequvilent, Unitary, Complex reps):

$$(A^{(ja)}_{(\mu\nu)})^i{}_b = \sum_g (D_{(\mu)}(g))^i{}_j (D_{(\nu)}(g^{-1}))^a{}_b \tag{2.47}$$

It is a linear map:

$$A_{(\mu\nu)}:\ u^b \in V_\nu \mapsto (A^{(ja)}_{(\mu\nu)})^i{}_b u^b = v^i \in V_{(\mu)} \tag{2.48}$$

Then:

$$\begin{aligned} D_{(\mu)}(g)A_{\mu\nu} &= \sum_{g'} D_{(\mu)}(g)D_{(\mu)}(g')D_{(\nu)}(g'^{-1}) \\ &= \sum_{g'} D_{(\mu)}(gg')D_{(\nu)}(g'^{-1}) \\ &= \sum_h D_{(\mu)}(h)D_{(\nu)}(h^{-1}g) \\ &= \sum_h D_{(\mu)}(h)D_{(\nu)}(h^{-1})D_{(\nu)}(g) \\ &= A_{(\mu\nu)}D_{(\nu)}(g) \end{aligned} \tag{2.49}$$

According to 2.2 (Shur's Lemma 1) , this is the intertwiner. If $\mu = \nu$, $A = \lambda\mathbb{I}$. If $\mu \neq \nu$, $A = 0$. ($=$ means equivalent)

In all we say that:

$$(A^{(kl)}_{(\mu\nu)})^i{}_j = \delta_{\mu\nu}\delta^i_j\ \lambda^{(kl)}_\mu \tag{2.50}$$

$$\sum_g (D_{(\mu)}(g))^i{}_k (D_{(\nu)}(g^{-1}))^l{}_j = \delta_{\mu\nu}\delta^i_j\ \lambda^{(kl)}_\mu \tag{2.51}$$

Taking a trace can find the coefficient:

$$\begin{aligned} tr(A^{(kl)}_{(\mu\nu)}) &= \delta_{\mu\nu}(A^{(kl)}_{\mu\nu})^i{}_i = \delta_{\mu\nu}\sum_g (D_{(\mu)}(g))^i{}_k(D_{(\nu)}(g^{-1}))^l{}_i = \delta_{\mu\nu}N(G)\delta^l_k \\ &= \delta_{\mu\nu}\delta^i_i\,\lambda^{(kl)}_\mu = \delta_{\mu\nu}d_\mu\,\lambda^{(kl)}_\mu \end{aligned} \tag{2.52}$$

Used the property that: $D(g)D(g^{-1}) = \mathbb{I}$.

Then:

$$\lambda^{(kl)}_\mu = \frac{N(G)}{d_\mu}\delta^l_k \tag{2.53}$$

Orthogonality theorem for representation:

$$\sum_g (D_{(\mu)}(g))^i{}_k(D_{(\nu)}(g^{-1}))^l{}_j = \frac{N(G)}{d_\mu}\delta_{\mu\nu}\delta^i_j\delta^l_k \tag{2.54}$$

If the representation is equivalent to **unitary** one (It always does):

$$\sum_g (D_{(\mu)}(g))^i{}_k(D^*_{(\nu)}(g))^j{}_l = \frac{N(G)}{d_\mu}\delta_{\mu\nu}\delta^i_j\delta^l_k \tag{2.55}$$

Write as (Better for memory):

$$\frac{1}{N(G)}\sum_g (D_{(\mu)}(g))^i{}_k(D^*_{(\nu)}(g))^j{}_l = \frac{1}{d_\mu}\delta_{\mu\nu}\delta^i_j\delta^l_k \tag{2.56}$$

Then define a Group Algebra Element

$$v^i_{\mu\ j} = \sqrt{\frac{d_\mu}{N(G)}}\sum_g (D_{(\mu)}(g))^i{}_j\ g \tag{2.57}$$

Since they are orthogonal with eachother:

$$\sum_\mu d^2_\mu \leq N(G) \tag{2.58}$$

B. Regular representation's Subspace Carry ir-Unitary-rep Regular representation space is Group Algebra space (Field is $\mathbb{C}$), Define d_p dimensional Subspace. Sorry that used D before and A here for rep.

$$\left\{ e_\mu = \sum_{i=1}^n A^{p*}_{\mu\nu}(g_i)g_i \mid \mu = 1\cdots d_p \right\}. \tag{2.59}$$

This subspace is invariant under **Regular representation**, And Carries ir-Unitary-rep p. Direct calculation shows

$$\begin{aligned} L(g_j)\left[\sum_{i=1}^n A^{p*}_{\mu\nu}(g_i)g_i\right] &= \sum_{i=1}^n A^{p*}_{\mu\nu}(g_i)g_jg_i = \sum_{k=1}^n A^{p*}_{\mu\nu}(g_j^{-1}g_k)g_k, \\ &= \sum_{k=1}^n A^{p*}_{\mu\alpha}(g_j^{-1})A^{p*}_{\alpha\nu}(g_k)g_k = \sum_{k=1}^n A^p_{\alpha\mu}(g_j)A^{p*}_{\alpha\nu}(g_k)g_k, \\ &= A^p_{\alpha\mu}(g_j)\left(\sum_{k=1}^n A^{p*}_{\alpha\nu}(g_k)g_k\right). \end{aligned} \tag{2.60}$$

也就是对基底 e_μ 作用的结果是

$$e_\mu \Rightarrow A^p_{\alpha\mu}(g_j)e_\alpha. \tag{2.61}$$

它等价于对 Vector in subspace 的作用

$$L(g_j)e_\mu x^\mu = e_\alpha A^p_{\alpha\mu}(g_j)x^\mu. \tag{2.62}$$

Inall, We do find a subspace and its basis, which is invariant under Regular representation and Carryies ir-Unitary-rep d.

C. Completeness In this paragraph, we prove basis mentioned above expand the group algebra space. We prove by contradiction.

Regular representation is Unitary $\Rightarrow$ it's completely reducible. Group algebra can not be expand by basis mentioned above $\Rightarrow$ Exists a different subspace carry irreduciable-Unitary-representation r.

Suppose this subspace basis to be

$$\left\{ e_\alpha = \sum_{j=1}^{N(G)} X_\alpha(g_j)g_j \middle| \alpha = 1\cdots d_r \right\}. \tag{2.63}$$

Carries ir-Unitary-rep r means

$$\begin{aligned} L(g_i)\left(\sum_j X_\alpha(g_j)g_j\right) &= g_i\sum_j X_\alpha(g_j)g_j = \sum_j X_\alpha(g_j)g_ig_j = \sum_k X_\alpha(g_i^{-1}g_k)g_k \\ &= \sum_\beta A^r_{\beta\alpha}(g_i)\left(\sum_{j=1}^{N(G)} X_\beta(g_j)g_j\right). \end{aligned} \tag{2.64}$$

Explicitly:

$$\begin{aligned} \sum_k X_\alpha(g_i^{-1}g_k)g_k &= \sum_\beta A^r_{\beta\alpha}(g_i)\left(\sum_{j=1}^{N(G)} X_\beta(g_j)g_j\right), \\ \sum_k X_\alpha(g_ig_k)g_k &= \sum_\beta A^r_{\beta\alpha}(g_i^{-1})\left(\sum_{j=1}^{N(G)} X_\beta(g_j)g_j\right), \\ &= \sum_\beta A^{r*}_{\alpha\beta}(g_i)\left(\sum_{j=1}^{N(G)} X_\beta(g_j)g_j\right). \end{aligned} \tag{2.65}$$

For the group algebra basis g_0 term,

$$X_\alpha(g_i) = \sum_\beta A^{r*}_{\alpha\beta}(g_i)X_\beta(g_0). \tag{2.66}$$

Construct group algebra vector:

$$\begin{aligned} \sum_i X_\alpha(g_i)g_i &= \sum_i\sum_\beta A^{r*}_{\alpha\beta}(g_i)X_\beta(g_0)g_i, \\ &= \sum_\beta X_\beta(g_0)\left(\sum_i A^{r*}_{\alpha\beta}(g_i)g_i\right) \end{aligned} \tag{2.67}$$

Which means this subspace basis can be expand by previous basis, Contradicts to Assumption.

D. Burnside Theorem The group algebra space can be expand by basis mentioned above, which means its space dimension $N(G)$ is equal to sum of these sub space.

$$N(G) = \sum_{\mu} d_{\mu}^2. \tag{2.68}$$

E. Number of Ir-Unitary-Rep in Regular For each Inequivalent irreducible rep μ, exists d_μ sets of subspace basis. If Regular representation can be expanded by inequivalent Irreducible-Unitary-Representatioin by:

$$D_{reg}(g) = X^{-1}(\underbrace{D_{(\mu1)} \oplus \cdots \oplus}_{d_{\mu1}\text{ term}} \underbrace{D_{(\mu2)} \cdots \oplus}_{d_{\mu2}\text{ term}} \cdots)X. \tag{2.69}$$

F. Great Orthogonality Theorem for characters (column, 列) Great Orthogonality Theorem is:

$$\sum_g (D_{(\mu)}(g))^i{}_k (D_{(\nu)}(g^{-1}))^l{}_j = \frac{N(G)}{d_\mu} \delta_{\mu\nu} \delta^i_j \delta^l_k \tag{2.70}$$

Consider the character: (we take $k = i$ and $l = j$ — Character is the trace of rep matrix)

$$\sum_g \chi_{(\mu)}(g) \chi_{(\nu)}(g^{-1}) = N(G)\delta_{\mu\nu} \tag{2.71}$$

For Unitary Representation, (For Finite Group And Compact Lie Groups, All Representations are equivalent to a unitary one. Equivalent rep has same character.)

$$\sum_g \chi_{(\mu)}(g) \chi^*_{(\nu)}(g) = N\delta_{\mu\nu} \tag{2.72}$$

let use n_r label the number of elements on class k_r, then attain **Great Orthogonality Theorem for characters**

$$\sum_a n_a \chi^a_{(\mu)} \chi^{a*}_{(\nu)} = N(G)\delta_{\mu\nu} \tag{2.73}$$

We can construct a r different orthogonal k-vector (k means the number of classes, r means number of representations)

$$\frac{1}{\sqrt{N(G)}} (\sqrt{n_1}\chi^1_{(\mu)}, \sqrt{n_2}\chi^2_{(\mu)}, \cdots, \sqrt{n_k}\chi^k_{(\mu)}) \tag{2.74}$$

So we say that: (r means number of representation)

$$r \leq k \tag{2.75}$$

Always write as

$$\frac{1}{N(G)} \sum_a n_a \chi^a_{(\mu)} \chi^{a*}_{(\nu)} = \delta_{\mu\nu} \tag{2.76}$$

F. Judge if a representation is irreducible Suppose a representation (or it is equivalent to this representation, $D_{(\mu)}$ means irreducible-Unitary-Representation):

$$D \sim \oplus a^{\mu} D_{(\mu)} \tag{2.77}$$

Then

$$\chi = \sum a^{\mu} \chi_{(\mu)} \tag{2.78}$$

Consider:

$$\begin{aligned} \sum_a n_a \chi^a \chi^{a*} &= \sum_{\mu,\nu} a^{\mu} a^{\nu} \sum_a n_a \chi^a_{(\mu)} \chi^{a*}_{(\nu)} \\ &= \sum_{\mu,\nu} a^{\mu} a^{\nu} N(G) \delta_{\mu\nu} = N(G) \sum_{\mu} (a^{\mu})^2 \end{aligned} \tag{2.79}$$

By calculating

$$\frac{1}{N(G)} \sum_a n_a \chi^a \chi^{a*} = \sum_{\mu} (a^{\mu})^2 \tag{2.80}$$

We can understand whether it is a irreducible representation

G. Find coefficient a^{μ} of each irreducible representations in a reducible representation Consider:

$$\begin{aligned} \sum_a n_a \chi^{a*}_{(\nu)} \chi^a &= \sum_{\mu} a^{\mu} \sum_a n_a \chi^{a*}_{(\nu)} \chi^a_{(\mu)} \\ &= \sum_{\mu} a^{\mu} N(G) \delta_{\mu\nu} = N(G) a^{\nu} \end{aligned} \tag{2.81}$$

Which means:

$$a^{\nu} = \frac{1}{N(G)} \sum_a n_a \chi^{a*}_{(\nu)} \chi^a \tag{2.82}$$

H. r=k Firstly, we define subspace of group algebra space with dimension k. Then we prove this subspace have r basis. This k-dimensional group algebra subspace basis is

$$\left\{ \mathcal{K}_a = \sum_{g \in k_a} g \middle| a = 1 \cdots k \right\}. \tag{2.83}$$

We can construct vector in this subspace from a general vector (coefficient of same class basis are the same)

$$\sum_i f(g_i) g_i \Rightarrow \sum_i f'(g_i) g_i \quad \text{In which} \quad f'(g_i) = \frac{1}{N(G)} \sum_j f(g_j^{-1} g_i g_j). \tag{2.84}$$

What's more, each vector $\sum_i f(g_i) g_i$ in this subspace, must satisfy

$$f(g_i) = \frac{1}{N(G)} \sum_j f(g_j^{-1} g_i g_j). \tag{2.85}$$

For vector in group algebra space, as discussed in part B, Can be expand by basis (2.59)

$$\sum_i f(g_i)g_i = \sum_{\mu\nu p} a^p_{\mu\nu}\left(\sum_i A^{p*}_{\mu\nu}(g_i)g_i\right). \tag{2.86}$$

Consider the Constraint of coefficient 2.85

$$\begin{aligned} f(g_i) =& \frac{1}{N(G)}\sum_j\sum_{\mu\nu p} a^p_{\mu\nu}\left(A^{p*}_{\mu\nu}(g_j^{-1}g_ig_j)\right), \\ =& \sum_{\mu\nu p\alpha\beta} a^p_{\mu\nu}\left(\sum_j \frac{1}{N(G)}A^{p*}_{\mu\alpha}(g_j^{-1})A^{p*}_{\alpha\beta}(g_i)A^{p*}_{\beta\nu}(g_j)\right), \\ =& \sum_{\mu\nu p\alpha\beta} a^p_{\mu\nu}A^{p*}_{\alpha\beta}(g_i)\left(\sum_j \frac{1}{N(G)}A^{p}_{\alpha\mu}(g_j)A^{p*}_{\beta\nu}(g_j)\right), \end{aligned} \tag{2.87}$$

Using the Orthogonality of representation

$$\left\{\sum_j \frac{1}{N(G)}A^p_{\alpha\mu}(g_j)A^{p*}_{\beta\nu}(g_j) = \frac{1}{d_p}\delta_{\alpha\beta}\delta_{\mu\nu}.\right. \tag{2.88}$$

Thus

$$\begin{aligned} f(g_i) =& \sum_{\mu\nu p\alpha\beta} a^p_{\mu\nu}A^{p*}_{\alpha\beta}(g_i)\left(\frac{1}{d_p}\delta_{\alpha\beta}\delta_{\mu\nu}\right), \\ =& \sum_{\mu p\alpha}\frac{1}{d_p}a^p_{\mu\mu}A^{p*}_{\alpha\alpha}(g_i), \\ =& \sum_{\mu p}\frac{1}{d_p}a^p_{\mu\mu}\chi^*_{(p)}(g_i), \\ =& \sum_p a_p\chi^*_{(p)}(g_i). \end{aligned} \tag{2.89}$$

Thus, Every vector in this subspace can be write as

$$\sum_i\sum_p a_p\chi^*_{(p)}(g_i)g_i = \sum_p a_p\left(\sum_i \chi^*_{(p)}(g_i)g_i\right) \tag{2.90}$$

Which means this subspace must can be expanded by basis:

$$\left\{\sum_i \chi^*_{(p)}(g_i)g_i | p = 1\cdots r\right\}. \tag{2.91}$$

Which means

$$k \le r. \tag{2.92}$$

combined with constraint $r \le k$ mentioned in E, Inall

$$r = k. \tag{2.93}$$

I Orthogonal of character in another way Define matrix

$$F_{\mu a} = \sqrt{\frac{n_a}{N(G)}}\chi^a_{(\mu)}. \tag{2.94}$$

Using the property proved before that $r = k$, this is a square matrix. satisfies:

$$(FF^{\dagger})_{\mu\nu} = \sum_a F_{\mu a} F^*_{\nu a} = \sum_a \sqrt{\frac{n_a}{N(G)}} \chi^a_{(\nu)} \sqrt{\frac{n_a}{N(G)}} \chi^{*a}_{(\mu)} = \sum_a \frac{n_a}{N(G)} \chi^{*a}_{(\mu)} \chi^a_{(\nu)} = \delta_{\mu\nu}. \tag{2.95}$$

Which means

$$(F^{\dagger}F)_{ba} \sum_\mu F^*_{\mu b} F_{\mu a} = \sum_\mu \sqrt{\frac{n_b}{N(G)}} \chi^{*b}_{(\mu)} \sqrt{\frac{n_a}{N(G)}} \chi^a_{(\mu)} = \delta_{ab}. \tag{2.96}$$

From which, we get another orthogonal relation

$$\frac{1}{N(G)} \sum_\mu n_a \chi^{*b}_{(\mu)} \chi^a_{(\mu)} = \delta_{ab} \tag{2.97}$$

J (Older Way). r=k for regular representation .

Regular representation decomposition Consider Regular Representation:

$$\chi_{reg}(g) = \begin{cases} N \text{ for } g = e \\ 0 \text{ for } g \neq e \end{cases} \tag{2.98}$$

With this property, by using the conclusion in **D.**, Find a^μ of regular representation

$$a^\mu = \frac{1}{N(G)} \sum_a n_a \chi^{a*}_{(\mu)} \chi^a_{reg} = \frac{1}{N(G)} \chi^{1*}_{(\mu)} \chi^1_{reg} = d_\mu \tag{2.99}$$

Orthogonal property After finding the coefficient of all irreducible representation, We have a useful property: (a=1 means class which contains identical element)

$$\left.\begin{matrix} N \text{ for } a = 1 \\ 0 \text{ for } a \neq 1 \end{matrix}\right\} = (\chi_{reg})_a = \sum_\mu a^\mu \chi^a_{(\mu)} = \sum_\mu d_\mu \chi^a_{(\mu)} = \sum_\mu \chi^1_{(\mu)} \chi^a_{(\mu)} \tag{2.100}$$

We focus on this property:

$$\sum_\mu \chi^1_{(\mu)} \chi^a_{(\mu)} = \begin{cases} N \text{ for } a = 1 \\ 0 \text{ for } a \neq 1 \end{cases} \tag{2.101}$$

Class vector and class Algebra Define class vector in Group Algebra.

$$\mathcal{K}_a = \sum_{g \in k_a} g \tag{2.102}$$

Class vector is invariant under conjugation:

$$g\mathcal{K}_a g^{-1} = \mathcal{K}_a \tag{2.103}$$

Product of Class vector is invariant under conjugation:

$$g\mathcal{K}_a \mathcal{K}_b g^{-1} = g\mathcal{K}_a g^{-1} g\mathcal{K}_b g^{-1} = \mathcal{K}_a \mathcal{K}_b \tag{2.104}$$

We state that: **If a vector is invariant under conjugation,(for all g) it must be a linear combination of class vectors.**

$$\mathcal{K}_a \mathcal{K}_b = \sum_c C_{abc} \mathcal{K}_c \tag{2.105}$$

Hence the $\mathcal{K}_a$ form an algebra themselves, Fixed by coefficients C_{abc}

For a given conjugacy class k_a, there is a class $k_{a'}$ whose elements are the inverse of those in k_a and $n_a = n_{a'}$ (n_a means number of elements in a clas). k_a might be equal to $k_{a'}$. Then $\mathcal{K}_a\mathcal{K}_{a'}$ contains n_a copies of identity.

$$C_{ab1} = \begin{cases} n_a \text{ for } b = a' \\ 0 \text{ for } b \neq a' \end{cases} \tag{2.106}$$

Class algebra representation Consider a matrix:

$$D^a_{(\mu)} = \sum_{g \in K_a} D_{(\mu)}(g) \tag{2.107}$$

(Sorry that I used k_a to denote conjugacy class, and K_a here.) This matrix commutes with all matrix in representation, Using Shur's Lemma.

$$D^a_{(\mu)} = \lambda^a_{(\mu)} \mathbb{I} \tag{2.108}$$

Taking trace we attain:

$$\begin{aligned} tr \sum_{g \in K_a} D_{(\mu)}(g) &= tr\ \lambda^a_{(\mu)} \mathbb{I} \\ n_a \chi^a_{(\mu)} &= \lambda^a_{(\mu)} d_\mu \\ n_a \chi^a_{(\mu)} &= \lambda^a_{(\mu)} \chi^1_{(\mu)} \\ \lambda^a_{(\mu)} &= \frac{n_a \chi^a_{(\mu)}}{\chi^1_{(\mu)}} \end{aligned} \tag{2.109}$$

The matrix we defined satisfies Class algebra 2.105.

$$\lambda^a_{(\mu)} \lambda^b_{(\mu)} = \sum_c C_{abc} \lambda^c_{(\mu)} \tag{2.110}$$

insert 2.109 we attain:

$$\begin{aligned} \frac{n_a \chi^a_{(\mu)}}{\chi^1_{(\mu)}} \frac{n_b \chi^b_{(\mu)}}{\chi^1_{(\mu)}} &= \sum_c C_{abc} \frac{n_c \chi^c_{(\mu)}}{\chi^1_{(\mu)}} \\ \sum_\mu \chi^a_{(\mu)} \chi^b_{(\mu)} &= \sum_c C_{abc} \frac{n_c}{n_a n_b} \sum_\mu \chi^1_{(\mu)} \chi^c_{(\mu)} \end{aligned} \tag{2.111}$$

By using the property of regular representation (2.101)

$$\sum_\mu \chi^a_{(\mu)} \chi^b_{(\mu)} = \sum_c C_{ab1} \frac{n_1}{n_a n_b} N(G) \tag{2.112}$$

Then using the property of the coefficient C (2.106)

$$\sum_\mu \chi^a_{(\mu)} \chi^b_{(\mu)} = \begin{cases} \dfrac{N(G)}{n_a} \text{ for } b = a' \\ 0 \text{ for } b \neq a' \end{cases} \tag{2.113}$$

Summarized as:

$$\sum_\mu \chi^a_{(\mu)} \chi^b_{(\mu)} = \frac{N(G)}{n_a} \delta_{a'b} \tag{2.114}$$

For regular representation (unitary):

$$\chi^{a'}_{(\mu)} = \chi^{a*}_{(\mu)} \tag{2.115}$$

Finally: (row orthogonality)

$$\sum_{\mu} \chi^{a*}_{(\mu)} \chi^{b}_{(\mu)} = \frac{N(G)}{n_a} \delta_{ab} \tag{2.116}$$

Then we can define k different r-dimensional orthogonal vectors:

$$\sqrt{\frac{n_a}{N}} (\chi^{a}_{(1)}, \chi^{a}_{(2)} \cdots \chi^{a}_{(r)}) \tag{2.117}$$

then:

$$k \leq r \tag{2.118}$$

So, for regular representation

$$r = k \tag{2.119}$$

2.5 Induced Representation

For a representation B(h) of G's subgroup H, can induce a representation $U(g)$ of G.

Representation space For representation of subgroup H (W is its representation space)

$$B: \ h \in H \mapsto B(h) \in GL(W). \tag{2.120}$$

Then, define representation space constructed by functions image elements in group G to vectors in Linear space W.

$$V = \{f | g \in G \mapsto f(g) \in W \ , \ f(hg) = B(h) f(g)\} \ . \tag{2.121}$$

Sometimes written as

$$f_\alpha(g), \tag{2.122}$$

in which α is the index in space W.

Representation Induce representation

$$U: \ g \in G \mapsto U(g) \in GL(V) \ s.t. \ U(g) f \circ (g'') = f \circ (g'' g). \tag{2.123}$$

证明 [Proof of image still in representation space V] Requirement of representation space is

$$\left\{ f(hg'') = B(h) f(g''), \right. \tag{2.124}$$

Label transformed function to be φ

$$U(g) f = \varphi. \tag{2.125}$$

transformed function Satisfy requirement of function

$$\varphi \circ (hg'') = f \circ (hg''g) = B(h) f \circ (g''g) = B(h) \varphi \circ (g''.) \tag{2.126}$$

Which means transformed function is still in representation space.

□

证明 [Transformation preserves group multiplication]

$$U(g_1)U(g_2)f = U(g_1)\left(U(g_2)f\right). \tag{2.127}$$

$$\begin{aligned} U(g_1)U(g_2)f \circ (g) &= U(g_1)\left(U(g_2)f\right) \circ (g), \\ &= U(g_2)f \circ (gg_1) = f \circ (gg_1g_2) = U(g_1g_2)f \circ (g). \end{aligned} \tag{2.128}$$

□

Basis of rep space A group can be decomposed into its Right-Cosets

$$G = \{Hg_1, Hg_2. \cdots Hg_l\}. \tag{2.129}$$

Noticed functions in representation space satisfies relation

$$\left\{ f(hg) = B(h)f(g). \right. \tag{2.130}$$

Function in representation space is fixed by its effect on right-coset decompose elemet $f(g_1)\cdots f(g_l)$. Define basis as (e_r span space W, g_k is element of Right-coset decompose element)

$$V = Span_{\mathbb{C}}\left\{e_{rj} | e_{rj}(g_k) = \delta_{jk}\vec{e}_r | r = 1\cdots d, j = 1\cdots l\right\}. \tag{2.131}$$

Vector in representation space can be written as

$$f = \sum_{r,j} f^{rj} e_{rj}. \tag{2.132}$$

f maps elements in group (denote by $h_i g_j$) into vector in space W by

$$f(h_i g_j) = \sum_{r,j} B(h_i) f^{rj} e_{rj}(g_j). \tag{2.133}$$

Explicitly

$$\begin{aligned} f(h_i g_j) &= \sum_{r,k} B(h_i) f^{rk} e_{rk}(g_j), \\ &= \sum_{r,k} B(h_i) f^{rk} \delta_{kj} \vec{e}_r, \\ &= \sum_{r} B(h_i) f^{rj} \vec{e}_r. \end{aligned} \tag{2.134}$$

Consider the effect of $B(h_i)$ on $\vec{e}_r$, in which $\vec{e}_r = (0, \cdots \underbrace{1}_{r\,th} \cdots)^T$, α labels number of coordinate element ($\alpha = 1 \to d$ which is dimension of rep B space W).

$$\left\{ (B(h_i)\vec{e}_r)_\alpha = B(h_i)_{\alpha,r}. \right. \tag{2.135}$$

Thus:

$$f_\alpha(h_i g_j) = \sum_r f^{rj} B(h_i)_{\alpha,r}, \tag{2.136}$$

Effect of induced group element For induced group rep element $U(g)$, its effect on function f shows:

$$U(g)f_\alpha \circ (h_i g_j) = f_\alpha \circ (h_i g_j g). \tag{2.137}$$

Our calculation of image of function f is based on Right-hand Coset decomposition, Suppose

$$h_i g_j g = h_{i'} g_{j'}. \tag{2.138}$$

Which means

$$h_{i'} = h_i g_j g g_{j'}^{-1}. \tag{2.139}$$

证明 [There is only one $(h_{i'}, g_{j'})$ satisfies requirement] According to equation above

$$g_j g g_{j'}^{-1} \in H. \tag{2.140}$$

Assumption: there exists two $g_{j'}$ denoted by g_1, g_2 satisfy this relation.

$$g_1 g_2^{-1} \in H \tag{2.141}$$

means they expand same right-coset, contradicts with requirement.

Sometimes, write as

$$j' = j'(j). \tag{2.142}$$

It is interesting that j' is not influenced by h_i.

The way to find $g_{j'}$ is check all $g_k,\ k = 1 \cdots l$, and find one satisfies

$$g_j g g_{j'}^{-1} \in H \tag{2.143}$$

□

Based on expansion of function f :

$$\left\{ f_\alpha(h_i g_j) = \sum_r f^{rj} B(h_i)_{\alpha,r}. \right. \tag{2.144}$$

We obtain

$$\begin{aligned} U(g)f_\alpha \circ (h_i g_j) =& f_\alpha \circ (h_i g_j g), \\ =& f_\alpha(h_{i'} g_{j'}) = \sum_r f^{rj'} B(h_{i'})_{\alpha,r}, \\ =& \sum_{r,\beta} f^{rj'} B(h_i)_{\alpha,\beta} B(g_j g g_{j'}^{-1})_{\beta,r} = \sum_\beta \left(\sum_r B(g_j g g_{j'}^{-1})_{\beta,r} f^{rj'} \right) B(h_i)_{\alpha,\beta}. \end{aligned} \tag{2.145}$$

We suppose it to be:

$$\varphi \circ (h_i g_j) = \sum_r \varphi^{rj} B(h_i)_{\alpha,r} \tag{2.146}$$

Basis transformation

$$\varphi^{\beta j} = \sum_r B(g_j g g_{j'}^{-1})_{\beta,r} f^{rj'}. \tag{2.147}$$

Denote space j as first space, space β as second space, representation space is a tensor product space of these two

$$V = V_1^{(l)} \otimes V_2^{(d)}. \tag{2.148}$$

Vector in this space can be written as

$$\varphi = \left(\varphi^{11}, \cdots \varphi^{d1}, \cdots \varphi^{1l}, \cdots \varphi^{dl}\right)^T \tag{2.149}$$

Noticed that $\varphi^{\beta 1}$ is only contributed by:

$$\varphi^{\beta j} = \sum_r B(g_j g g_{j'(1)}^{-1})_{\beta,r} f^{rj'(1)} \tag{2.150}$$

Transformation matrix can be written as

$$U(g) = \begin{pmatrix} 0 & \cdots & \underbrace{B(g_1 g g_{j'}^{-1})}_{\text{j'(1)'s row element}} & \cdots & 0 \\ 0 & \underbrace{B(g_2 g g_{j'}^{-1})}_{\text{j'(2)'s row element}} & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{2.151}$$

In text book by Prof. XinZheng Li ,written as (ij is first space index, kl is the second space index)

$$U(g)_{ik,jl} = \dot{B}(g_i g g_j^{-1})_{kl}. \tag{2.152}$$

In which dot B is defined as

$$\dot{B}(g_i g g_j^{-1}) = B(g_i g g_j^{-1}) \quad \text{when} \quad g_i g g_j^{-1} \in H \;\; ; \;\; \text{otherwise} \;\; 0. \tag{2.153}$$

write as matrix:

$$U(g) = \begin{pmatrix} \dot{B}(g_1 g g_1^{-1}) & \cdots & \dot{B}(g_1 g g_j^{-1}) & \cdots & \dot{B}(g_1 g g_l^{-1}) \\ \dot{B}(g_2 g g_1^{-1}) & \cdots & \dot{B}(g_1 g g_j^{-1}) & \cdots & \dot{B}(g_2 g g_l^{-1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dot{B}(g_l g g_1^{-1}) & \cdots & \dot{B}(g_1 g g_j^{-1}) & \cdots & \dot{B}(g_l g g_l^{-1}) \end{pmatrix} \tag{2.154}$$

2.6 Other way to construct representation

Direct product of group representation for group representation A and B of group G, direct product of group representation would be

$$(A \otimes B)(g) = A(g) \otimes B(g). \tag{2.155}$$

It is not irreducible even A and B are irrep.

Direct product Group representation A group is a direct product group of two subgroups. $g = g_{1\alpha} g_{2\beta}$. These two subgroups have representations A and B. The representation of the direct product group would be

$$D(g_{1\alpha} g_{2\beta}) = A(g_{1\alpha}) \otimes B(g_{2\beta}). \tag{2.156}$$

证明 [Rep of direct product group is irrep] Prove by character-orthogonality property

$$\chi_{(D)}(g) = tr\left(D(g)\right) = tr\left(A(g_{1\alpha})\right) tr\left(B(g_{2\beta})\right). \tag{2.157}$$

Check if rep is irrep by calculating (A, B are irrep)

$$\begin{aligned}\frac{1}{N(G)}\sum_g \chi^*_{(D)}(g)\chi_{(D)}(g) =& \sum_{\alpha\beta}\frac{1}{N(G)}\chi_{(A)}(g_{1\alpha})\chi_{(B)}(g_{2\beta})\chi^*_{(A)}(g_{1\alpha})\chi^*_{(B)}(g_{2\beta}) \\ =& \sum_{\alpha\beta}\frac{1}{N(G_1)N(G_2)}\chi_{(A)}(g_{1\alpha})\chi_{(B)}(g_{2\beta})\chi^*_{(A)}(g_{1\alpha})\chi^*_{(B)}(g_{2\beta}) \\ =& 1.\end{aligned} \tag{2.158}$$

□

Minimize group representation D to its subgroup B For representation D of group G. This representation can be minimize to its subgroup H. Noted as $D|_H$.

For irrep D, its minimize group rep $D|_H$ is not always irrep!.

Frobenius lemma

引理 2.3 (Frobenius) .

1. group G carries ir-Unitary-rep A,
2. subgroup H carries ir-Unitary-rep B,
3. group G carries induced rep $U|_B$,
4. subgroup H carries minimized group $A|_H$,
5. numbers of ir-Unitary part A showed in rep $U|_B$ = numbers of ir-Unitary part B showed in rep $A|_H$.

□

证明 Based on orthogonality of character, the 5. statement can be proved by showing following equation

$$\frac{1}{N(G)}\sum_g \chi^*_{(A)}(g)\chi_{(U)}(g) = \frac{1}{N(H)}\sum_h \chi^*_{(B)}(h)\chi_{(A|_H)}(h). \tag{2.159}$$

The LHS can be written as

$$\frac{1}{N(G)}\sum_g \chi^*_{(A)}(g)\chi_{(U)}(g) = \frac{1}{N(G)}\sum_g \chi^*_{(A)}(g)\sum_{i=1}^{l} tr\dot{B}(g_i g g_i^{-1}) \tag{2.160}$$

Noticed

$$\left\{\sum_{i=1}^{l} tr\dot{B}(g_i g g_i^{-1}) = \sum_t^G \frac{1}{N(H)} tr\dot{B}(tgt^{-1}).\right. \tag{2.161}$$

leads to

$$\begin{aligned}
\frac{1}{N(G)}\sum_g \chi^*_{(A)}(g)\chi_{(U)}(g) =&\frac{1}{N(G)}\sum_g \chi^*_{(A)}(g)\sum_{i=1}^{l} tr\dot{B}(g_i g g_i^{-1}) \\
=&\frac{1}{N(G)}\sum_g \chi^*_{(A)}(g)\sum_t \frac{1}{N(H)} tr\dot{B}(tgt^{-1}) \\
=&\frac{1}{N(G)N(H)}\sum_t\sum_g \chi^*_{(A)}(g) tr\dot{B}(tgt^{-1}) \\
=&\frac{1}{N(G)N(H)}\sum_t\sum_s \chi^*_{(A)}(t^{-1}st) tr\dot{B}(s) \quad \text{Why?} \\
=&\frac{1}{N(H)}\sum_s \chi^*_{(A)}(s) tr\dot{B}(s) = \frac{1}{N(H)}\sum_h \chi^*_{(A)}(h) trB(h), \\
=&\frac{1}{N(H)}\sum_h \chi^*_{(A)}(h)\chi_{(B)}(h).
\end{aligned} \tag{2.162}$$

□

第三章 Discrete Group

3.1 Young diagrams

For a symmetric group, the conjugacy will not change the circle structure. So we can find n-tuple ν_p to gives the number of p-circle. Which satisfies:

$$\sum_p p\nu_p = n \tag{3.1}$$

A conjugacy class is formed by all elements of given circle structure (ν)

Also, we can define a n-tuple λ to difine conjugacy class, this is done by:

$$\lambda_p = \sum_{q=p} \lambda_q \tag{3.2}$$

For these λ, they satisfies:

$$\sum_p \lambda_p = n \tag{3.3}$$

Also:

$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \tag{3.4}$$

Conjugacy class are in one to ine correspondence with the partitions of n.

3.2 representation of Symmetric Group

We have shown that the number of irreducible representations is equal to the number of conjugacy classes, and we know that the conjugacy classes are in one-to-one correspondence with the Young tableaux.

第四章 Lie Group

4.1 Some Differential Geometry

A Lie group is a group G. The group and manifold structure are required to be compatible in the sense that product and inverses are continuous maps. This can be combined in the requirement that the map

$$\begin{aligned} G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2^{-1} \end{aligned} \tag{4.1}$$

are continous.

Manifold The **manifold** is defined by the **atlas**. which means we cover the group G by **open sets** U_i. And define a set of **charts** from U_i to $\mathbb{R}^d$ i.e. invertible maps.

$$\varphi_i : G \supset U_i \to V_i \subset \mathbb{R}^d \tag{4.2}$$

we have to require that on the overlap of the open sets $U_{ij} = U_i \cap U_j$, the change of coordinates is differentiable. i.e. the maps:

$$\varphi_i \circ \varphi_j^{-1} : V_j \to V_i \tag{4.3}$$

are differentiable bijections

Then Consider about the functions. we want to talk about the differentiability of the functions:

$$\begin{aligned} f : K \to G \\ g : G \to K^m \end{aligned} \tag{4.4}$$

These functions are differentiable if:

$$\begin{aligned} \varphi_i \circ f : K \to V_i \\ g \circ \varphi_i^{-1} : V_i \to K^m \end{aligned} \tag{4.5}$$

are differentiable.

Vector Fields To begin with

定义 4.1 ($C^\infty(\mathcal{M})$)

$$\text{for } f: \ \mathcal{M} \to \mathbb{R}, \text{ if } f \circ \varphi_i^{-1} \ \forall i \ V_i \to \mathbb{R} \text{ are } C^\infty \Rightarrow f \text{ are } C^\infty(\mathcal{M}) \tag{4.6}$$

vector field is a linear map: $\xi: \ C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$. that satisfies product rule:

$$\xi(fg) = f\xi(g) + \xi(f)g \tag{4.7}$$

We can define vector ξ_x:

$$\xi_x(f) = \xi(f)(x) \tag{4.8}$$

Tangent space Tangent space at a point x of the manifold $\mathcal{M}$:

$$T_x = \{\xi_x|\xi \quad \text{is a (local) vector field}\} \tag{4.9}$$

Tangent Map For a differentiable map $F : M_1 \longrightarrow M_2$, The tangent map $T_xF : T_xM_1 \longrightarrow T_{F(x)}M_2$ is defined by:

$$T_xF(v)(f) \equiv v(f \circ F) \tag{4.10}$$

where: $x \in M_1$, $v \in T_xM_1$, $f \in C^\infty(M_2)$. This is also denoted by F_*

$$(F_*\xi)_{F(x)} = T_xF(\xi_x) \tag{4.11}$$

Where ξ is a vector field on M_1, $F_*\xi$ is a vector field on M_2

Some properties of tangent map .

(i) If there are two differential map $G : M_1 \longrightarrow M_2$ and $F : M_2 \longrightarrow M_3$. Then:

$$T_x(F \circ G) = T_{G(x)}(F)T_x(G) \tag{4.12}$$

证明 Consider : $f \in C^\infty(M_3)$ $v \in T_xM_1$ $w = T_xG(v)$ $w \in T_{G(x)}M_2$

Then:

$$\begin{aligned} T_x(F \circ G)(v)(f) &= v(f \circ F \circ G) = v\big((f \circ F) \circ G\big) \\ &= T_x(G)(v)(f \circ F) \\ &= w(f \circ F) \\ &= T_{G(x)}(F)(w)(f) \\ &= \big[T_{G(x)}(F) \circ T_x(G)\ (v)\big](f) \end{aligned} \tag{4.13}$$

End of Proof

(ii) Tangent map of identity map id_M is $T_x id_M = id_{T_xM}$. Called Identity map on the tangent space.

(iii) if F is invertible map: $F : M_1 \longrightarrow M_2$, Then:

$$(T_xF)^{-1} = T_{F(x)}(F^{-1}) \tag{4.14}$$

local identification of tangent vector Consider chart (ϕ, U) of M,

$$T_x\phi :\ T_xM \longrightarrow T_{\phi(x)}V = T_{\phi(x)}\mathbb{R}^n \tag{4.15}$$

From my perspective, this is a some what definition

$$T_x\phi(v) = v^i(x)\frac{\partial}{\partial x^i} \tag{4.16}$$

ϕ is invertible $\Rightarrow T_x\phi$ is invertible

$$dim(T_xM) = dim(T_{\phi(x)}\mathbb{R}^n) = n \tag{4.17}$$

Tangent map in coordinates Consider differentiable map $F : M \to N$

$$\begin{aligned} M:\ (\phi, U) \quad \phi(x) = (x^1, x^2 \cdots x^n) \\ N:\ (\psi, W) \quad \phi(y) = (y^1, y^2 \cdots y^m) \end{aligned} \tag{4.18}$$

$$\begin{aligned} T_x\phi(v) = v^i(x)\frac{\partial}{\partial x^i} \in T_{\phi(x)}\mathbb{R}^n \\ T_y\phi(w) = w^i(x)\frac{\partial}{\partial y^i} \in T_{\psi(y)}\mathbb{R}^m \end{aligned} \tag{4.19}$$

Consider:

$$\begin{aligned} \mathcal{F} = \psi \circ F \circ \phi^{-1} \\ \mathcal{F} : \mathbb{R}^n \to \mathbb{R}^m \\ \mathcal{F} : x^i \mapsto \mathcal{F}^j \end{aligned} \tag{4.20}$$

And f is a function of $(y^1, y^2 \cdots y^m)$

$$\begin{aligned} &T_{\phi(x)}(\mathcal{F})(v^i\partial_{x^i})(f) \\ &= v^i\partial_{x^i}(f \circ \mathcal{F})(x) \\ &= v^i\frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j}(x) \\ &= v^i(\partial_{x^i}\mathcal{F}^j)(x)\ (\partial_{y^j}f)(y) \\ &= \left[D\mathcal{F}(x)\right]_i^j v^i\partial_{y^j}f(y) \end{aligned} \tag{4.21}$$

Integral curve Integral curve of vector field ξ on manifold $\mathcal{M}$ is a differentiable curve

$$\alpha_\xi :\ [a, b] \to M \ \ t \in [a, b] \mapsto \alpha_\xi(t) \in M \tag{4.22}$$

Such that:

$$\partial_t\alpha_\xi(t) \equiv T_t(\alpha_\xi)(\partial t) = \xi_{\alpha(t)} \tag{4.23}$$

Flow Flow $\phi(t, x) = \phi_t(x)$ of the vector field ξ on M is given by unique integral curve with the initial condition $\phi(0, x) = x$

4.2 Lie Algebra

left translation take a fixed element h, the left translation is defined by:

$$L_h : g \mapsto hg \tag{4.24}$$

In coordinates, the left translation would be:

$$L_h : \alpha^a \mapsto \beta^a(\alpha^a) \tag{4.25}$$

which satisfies:

$$\varphi(g) = \alpha^a \tag{4.26}$$

and

$$\varphi(hg) = \beta^a \tag{4.27}$$

The left translation also acts on functions on manifold. For a function f, the new function is the old function move along the manifold.

$$(L_h f)(hg) = f(g) \tag{4.28}$$

(这一段似乎有问题)It also induces a map on tangent vectors, the differential map (or push-forward), which maps tangent vector X at the point g to the vector $dL_h \circ X$ at the point hg, which satisfies:

$$(dL_h \circ X)[f(hg)] = X[f(g)] \tag{4.29}$$

The differential map allows us to single out a particular kind of vector fields, namely those that are invariant under the differential maps of all left translations

定义 4.2 left-invariant A vector field is called left invariant if:

$$X|_{hg} = dL_h \circ X|_g \quad \text{for all } g, h \in G \tag{4.30}$$

In the difinition, for left-invariant vector fields:

$$X|_{hg}[f(hg)] = X|_g[f(g)] \tag{4.31}$$

下面没问题了

Left-invariant vector field

$$T_x L_g(\xi_x) = \xi_{gx} \ \ \forall x, g \in G, \tag{4.32}$$

Also denote as:

$$(g_* \xi)_{gx} = \xi_{gx} \tag{4.33}$$

Vector space isomorphism The vector space of left invariant fields is isomorphic the tangentspace.

$$\mathcal{L}(G) \xrightarrow{\cong} T_e G \tag{4.34}$$

$$dim(\mathcal{L}(G)) = dim(T_e G) = dim(G) \tag{4.35}$$

Lie algebra is closed under commute operation If ξ, η are left invariant vector fields on G, so is $[\xi, \eta]$

Firstly:

$$(g_*^{-1}\xi)_x(f) = \xi_{gx}(f \circ g^{-1}) = \xi(f \circ g^{-1})(gx) \tag{4.36}$$

$$g_*^{-1}\xi(f) = \xi(f \circ g^{-1}) \circ g \tag{4.37}$$

Then:

$$
\begin{aligned}
[g_*^{-1}\xi, g_*^{-1}\eta](f) &= g_*^{-1}\xi\big(g_*^{-1}\eta(f)\big) - g_*^{-1}\eta\big(g_*^{-1}\xi(f)\big) \\
&= g_*^{-1}\xi\big(\eta(f\circ g^{-1})\circ g\big) - g_*^{-1}\eta\big(\xi(f\circ g^{-1})\circ g\big) \\
&= \xi\big(\eta(f\circ g^{-1})\big)\circ g - \eta\big(\xi(f\circ g^{-1})\big)\circ g \\
&= [\xi,\eta](f\circ g^{-1})\circ g \\
&= g_*^{-1}[\xi,\eta](f)
\end{aligned}
\tag{4.38}
$$

In all:

$$
g_*^{-1}[\xi,\eta] = [g_*^{-1}\xi, g_*^{-1}\eta] = [\xi,\eta] \tag{4.39}
$$

Lie algebra g .

定义 4.3 The Lie algebra ***g*** of a group G is the space of left-invariant vector fields with the Lie bracket as product.

Some Properties of flow Define integral curve of vector field ξ

$$
\alpha_\xi(t) = \phi(t,e) \tag{4.40}
$$

If ξ is a left invariant field,

$$
\phi(t,x) = x\alpha_\xi(t) \tag{4.41}
$$

证明 Obviously:

$$
\phi(0,x) = x\alpha_\xi(0) = x \tag{4.42}
$$

Consider:

$$
\begin{aligned}
\partial_t(x\alpha_\xi) &= T_t(L_x\circ\alpha_\xi)(\partial_t) \\
&= T_{\alpha_\xi(t)}L_x\circ T_t\alpha_\xi(\partial_t) = T_{\alpha_\xi(t)}L_x(\xi_{\alpha_\xi(t)}) \\
&= \xi_{L_x\alpha_\xi(t)}
\end{aligned}
\tag{4.43}
$$

End of Proof

推论 4.1

$$
\alpha_\xi(s+t) = \alpha_\xi(s)\alpha_\xi(t) \tag{4.44}
$$

证明

$$
\alpha_\xi(s+t) = \phi_{s+t}(e) = \phi(t,\phi_s(e)) = \phi_s(e)\alpha_\xi(t) = \alpha_\xi(s)\alpha_\xi(t) \tag{4.45}
$$

End of Proof

4.3 Lie Group Representation

We want to represent Lie group on the vector space of Lie Algebra:

Adjoint is a Group Homomorphism

Consider conjugacy

$$C_g: \ G \to G \ \ C_g(x) = gxg^{-1} \tag{4.46}$$

Aut(G) is a Group of all invertible endomorphisms(Hom(G,G), Aut means automorphisms)

$$C_g \in Aut(G) \tag{4.47}$$

There is a Group homomorphism:

$$C : G \to Aut(G) \quad g \mapsto C_g \tag{4.48}$$

C is indeed a group homomorphism:

$$C_{g_1 g_2}(x) = g_a g_2 x (g_1 g_2)^{-1} = g_1 g_2 x g_2^{-1} g_1^{-1} = C_{g_1} \circ C_{g_2}(x) \tag{4.49}$$

Representation of Lie Group

Consider:

$$C_g(e) = e \tag{4.50}$$

Tangent Map:

$$T_e C_g : T_e G \to T_e G \tag{4.51}$$

定义 4.4 Adjoint Representatioin of Lie Group G

$$Ad : G \to GL(\mathcal{L}(G)) \tag{4.52}$$

$$Ad(g) = T_e C_g : \ T_e G \to T_e G \tag{4.53}$$

Ad is indeed a representation:

$$Ad(g_1 g_2) = T_e C_{g_1 g_2} = T_E(C_{g_1} \circ C_{g_2}) = T_e C_{g_1} \circ T_e C_{g_2} = Ad(g_1) \circ Ad(g_2) \tag{4.54}$$

4.4 Lie algebra representation

Adjoint Representation of the Lie algebra

定义 4.5 Adjoint Representation of Lie algebra The adjoint representation of Lie algebra on itself is defined by:

$$ad = T_e Ad \tag{4.55}$$

$$ad : \mathcal{L}(G) \to End\big(\mathcal{L}(G)\big) \tag{4.56}$$

$$Ad(e) = id_{\mathcal{L}(G)} \tag{4.57}$$

∘ $End\big(\mathcal{L}(G)\big)$ Contains all $n \times n$ matrixes.

∘ The image of Ad, $Img(Ad) \subset GL(\mathcal{L}(G))$ is a n dimensional manifold,It's tangent space at $Ad(e) =$

$id_{\mathcal{L}(G)}$ is a n dimensional vector space.

∘ T_eG is isomorphic with $\mathcal{L}(G)$

The vector at the space $Img(Ad) \subset GL(\mathcal{L}(G))$ is some what:

$$\partial_t \alpha(t) \tag{4.58}$$

In which:

$$\alpha(t) \in GL(\mathcal{L}(G)) \tag{4.59}$$

We consider:

$$\partial_t\,(\alpha(t)(\xi_e)) \quad \xi_e \in T_eG \to \partial_t \eta_e(t) \quad \eta_e \in T_eG \tag{4.60}$$

We will find $\partial_t\,(\alpha(t)(\xi)) \in T_eG$

推论 4.2

$$ad(\xi_e)(\eta_e) = [\xi, \eta]_e \tag{4.61}$$

证明 Consider:

$$C_{\alpha_\xi(s)}\big(\alpha_\eta(t)\big) = \alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(s)^{-1} = \alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s) \tag{4.62}$$

Then:

$$\begin{aligned} Ad\big(\alpha_\xi(s)\big)(\eta_e) = T_e\big(C_{\alpha_\xi(s)}\big)(\eta_e) = T_e(C_{\alpha_\xi(s)})T_0(\alpha_\eta)(\partial t) = T_0(C_{\alpha_\xi(s)} \circ \alpha_\eta)(\partial t) \\ = \partial_t|_0 \alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s) \end{aligned} \tag{4.63}$$

$$\begin{aligned} ad(\xi_e)(\eta_e) &= T_eAd(\xi_e)(\eta_e) = T_eAd\; T_0\alpha_\xi(\partial_s)(\eta_e) \\ &= T_0(Ad \circ \alpha_\xi)(\partial_s)\eta_e \\ &= \partial_s|_0 Ad(\alpha_\xi(s))\eta_e \\ &= \partial_s\partial_t|_0 \alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s) \\ ad(\xi_e)(\eta_e)(f) &= \partial_s\partial_t|_0 f(\alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s)) \quad \text{这一步当作定义} \\ &= \partial_s\partial_t|_0 f(\alpha_\xi(s)\alpha_\eta(t)) - \partial_s\partial_t|_0 f(\alpha_\eta(t)\alpha_\xi(s)) \\ &= \xi \circ \eta(f)_e - \eta \circ \xi(f)_e \end{aligned} \tag{4.64}$$

End of Proof

推论 4.3 The adjoint Representation is a Lie algebra representation

证明

$$\begin{aligned} ad([\xi, \eta])(\lambda) &= [[\xi, \eta], \lambda] = [\xi, [\eta, \lambda]] - [\eta, [\xi, \eta]] \\ &= ad(\xi) \circ ad(\eta)(\lambda) - ad(\eta) \circ ad(\xi)(\eta) \\ &= [ad(\xi), ad(\eta)](\lambda) \end{aligned} \tag{4.65}$$

End of Proof

Suppose we choose a base, Then:

$$ad(\xi_i)(\xi_j) = [\xi_i, \xi_j] = f_{ij}{}^k \xi_k \tag{4.66}$$

denote:

$$[ad(\xi_i)]^k_{\ j} = f_{ij}^{\ \ k} \tag{4.67}$$

Commutation Relation

Will introduce some commutation relation

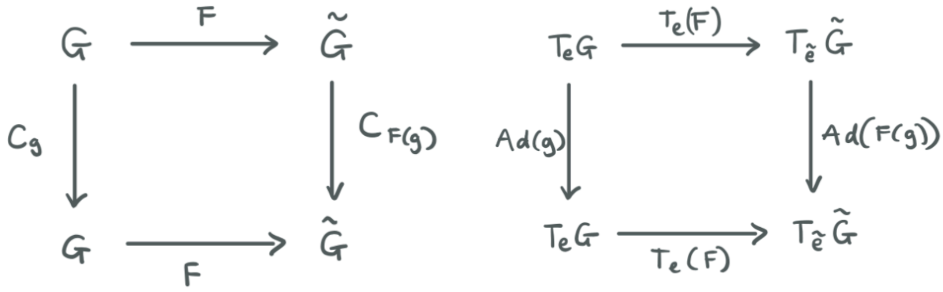

图 **4.1:** Commute Diagrams,Lie Group

∘ Left Diagram:

$$F \circ C_g\ (x) = F\ (gxg^{-1}) = F(g)F(x)F(g)^{-1} = C_{F(g)} \circ F\ (x) \tag{4.68}$$

∘ Right Diagram

$$T_e(F) \circ Ad(g) = T_e(F) \circ T_e(C_g) = T_e(F \circ C_g) = T_{\tilde{e}}(C_{F(g)}) \circ T_e(F) = Ad(F(g)) \circ T_e(F) \tag{4.69}$$

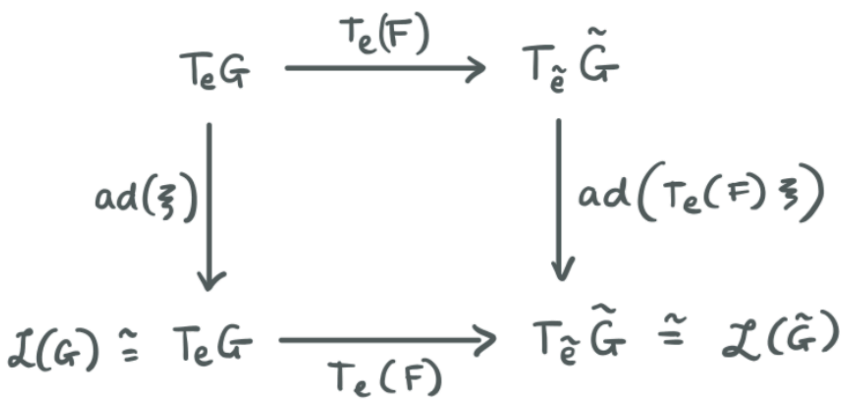

图 **4.2:** Commute Diagram,Lie Algebra

认为:

$$T_eF(\xi_e) = \tilde{\xi}_{\tilde{e}} \tag{4.70}$$

假设在 G 和 G' 中可以定义 Integral Curve:

$$\alpha_\xi(s) \quad \alpha_{\tilde{\xi}}(s) \tag{4.71}$$

$\tilde{G}$ 中的 Integral curve $\alpha_{\tilde{\xi}}(s)$ 需要满足的条件:

$$\tilde{\xi}_{\tilde{e}}(f) = T_0\alpha_{\tilde{\xi}}(\partial_s)(f) \quad f : \tilde{G} \to \mathbb{R} \tag{4.72}$$

考虑到:

$$\begin{cases} T_eF(\xi_e) = \tilde{\xi}_{\tilde{e}} \end{cases} \tag{4.73}$$

于是:

$$\tilde{\xi}_{\tilde{e}}(f) = T_eF(\xi_e)(f) = T_0\alpha_{\tilde{\xi}}(\partial_s)(f) \tag{4.74}$$

G 中的 integral curve $\alpha_\xi(s)$ 需要满足的条件:

$$\xi_e(f') = T_0\alpha_\xi(\partial_s)(f') \quad f' : G \to \mathbb{R} \tag{4.75}$$

下面证明:

$$\alpha_{\tilde{\xi}}(s) = F\alpha_\xi(s) \tag{4.76}$$

证明 直接将假设带入 $\tilde{G}$ 中 Integral curve $\alpha_{\tilde{\xi}}(s)$ 需要满足的条件:

$$\begin{cases} T_eF(\xi_e)(f) = T_0\alpha_{\tilde{\xi}}(\partial_s)(f) \end{cases} \tag{4.77}$$

注意到, 右式:

$$\begin{aligned} T_0\alpha_{\tilde{\xi}}(\partial_s)(f) &= T_0F\alpha_\xi(\partial_s)(f) \\ &= \partial_s|_0 f\left(F\alpha_\xi(s)\right) \end{aligned} \tag{4.78}$$

定义:

$$f' = f \circ F \quad G \to \mathbb{R} \tag{4.79}$$

于是:

$$\begin{aligned} \text{上式} &= \partial_s|_0 f'\left(\alpha_\xi(s)\right) \\ &= T_0\alpha_\xi(\partial_s)(f \circ F) \end{aligned} \tag{4.80}$$

由于 G 中 Integral Curve 满足条件:

$$\begin{cases} \xi_e(f') = T_0\alpha_\xi(\partial_s)(f') \quad f' : G \to \mathbb{R} \end{cases} \tag{4.81}$$

于是:

$$\begin{aligned} \text{上式} &= \xi_e(f \circ F) \\ &= T_eF(\xi_e)(f) \end{aligned} \tag{4.82}$$

□

接下来证明:

$$T_e(F) \circ ad(\xi_e)(\eta_e) = ad\left(T_eF(\xi_e)\right)\left(T_eF\eta_e\right) \tag{4.83}$$

证明 考虑到 adjoint representation 的性质:

$$\begin{cases} ad(\xi_e)(\eta_e) = \partial_s\partial_t|_0\alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s) \end{cases} \tag{4.84}$$

于是，左式:

$$\begin{aligned} T_eF\left(ad(\xi_e)(\eta_e)\right)(f) &= T_eF\left(\partial_s\partial_t|_0\alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s)\right)(f) \\ &= \partial_s\partial_t|_0 f\left(F\alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s)\right) \end{aligned} \tag{4.85}$$

右式:

$$
\begin{aligned}
ad\left(T_eF(\xi_e)\right)\left(T_eF(\eta_e)\right)(f) &= ad(\tilde{\xi}_{\tilde{e}}, \tilde{\eta}_{\tilde{e}})(f) \\
&= \partial_s\partial_t|_0 f\left(\alpha_{\tilde{\xi}}(s)\alpha_{\tilde{\eta}}(t)\alpha_{\tilde{\xi}}(-s)\right)
\end{aligned}
\tag{4.86}
$$

由于前面得到的对易关系:

$$
\left\{F \circ C_g\ (x) = F\ (gxg^{-1}) = F(g)F(x)F(g)^{-1} = C_{F(g)} \circ F\ (x)\right.
\tag{4.87}
$$

于是:

$$
\begin{aligned}
F\alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s) &= F \circ C_{\alpha_\xi(s)}\left(\alpha_\eta(t)\right) \\
&= C_{F(\alpha_\xi(s))} \circ F\left(\alpha_\eta(t)\right) \\
&= F\left(\alpha_\xi(s)\right) F\left(\alpha_\eta(t)\right) F\left(\alpha_\xi(-s)\right) \\
&= \alpha_{\tilde{\xi}}(s)\alpha_{\tilde{\eta}}(t)\alpha_{\tilde{\xi}}(-s)
\end{aligned}
\tag{4.88}
$$

也就是:

$$
\partial_s\partial_t|_0 f\left(F\alpha_\xi(s)\alpha_\eta(t)\alpha_\xi(-s)\right) = \partial_s\partial_t|_0 f\left(\alpha_{\tilde{\xi}}(s)\alpha_{\tilde{\eta}}(t)\alpha_{\tilde{\xi}}(-s)\right)
\tag{4.89}
$$

$$
T_e(F) \circ ad(\xi_e)(\eta_e) = ad\left(T_eF(\xi_e)\right)\left(T_eF\eta_e\right)
\tag{4.90}
$$

□

更近一步:

$$
T_eF\left([\xi, \eta]_e\right) = [T_eF(\xi_e), T_eF(\eta_e)]
\tag{4.91}
$$

States $T_e(F)$ is a Lie Algebra morphism

定理 4.1 A linear map $f\ :\ \mathcal{L}(G) \to \mathcal{L}(\tilde{G})$ is the tangent map of a group homomorphism F iff f is the tangent map of a group homomorphism F.

4.5 Exponential map

Exponential map constructs the Lie Group From its Lie Algebra.

定义 4.6 Exponential map For a left-invariant vector field ξ on group G, with $v = \xi_e$. we have an integral curve α_v. with condition $\alpha_v(0) = e$. Then, the exponential map: $Exp : T_eG \to G$ is defined by $Exp(v) = \alpha_v(1)$

The exponential map is:

∘(1) differentiable at the origin and $T_0(Exp) = id_{\mathcal{L}(G)}$

∘(2) maps $\mathcal{L}(G) \cong T_e(G)$ diffeomorphically into a neighbourhood of $e \in G$.

∘(3) satisfies: $F \circ Exp = \tilde{Exp} \circ T_e(F)$ for a group homomorphism $F :\ G \to \tilde{G}$

For the (1) statement,

证明 denote: $m_v(t) = tv$.

$$
Exp \circ m_v(t) = Exp(vt) = \alpha_{vt}(1) = \alpha_v(t)
\tag{4.92}
$$

Then:

$$\partial_t|_0 Exp \circ m_v(t) = T_0(Exp \circ m_v)(\partial_t) = T_0(Exp) \circ T_0(m_v)(\partial t) \tag{4.93}$$

State that:

$$T_0(m_v)(\partial t) = \partial_t|_0 m_v(t) = v \tag{4.94}$$

Then:

$$T_0(Exp)v = \partial_t|_0 \alpha_v(t) = v \tag{4.95}$$

End of Proof

For the (3) statement:

start with a lemma

引理 4.1 Group homomorphism maps left-invariant field into left invariant field If $F : G \to \tilde{G}$ is a group homomorphism and ξ is a left invariant vector field on G. then $\tilde{\xi} = F_*(\xi)$ is a Left-invariant vector field on $\tilde{G}$.

For this lemma, denote $\tilde{x} = F(x)$, $\tilde{g} = F(g)$, $\tilde{\xi} = F_*(\xi)$.

$$\begin{aligned} L_{\tilde{g}} \circ F(x) = F(g)F(x) = F(gx) = F \circ L_g \\ L_{\tilde{g}} \circ F(x) = F \circ L_g \end{aligned} \tag{4.96}$$

Then consider Lefi-invariant condition on $\tilde{x}$.

$$\begin{aligned} T_{\tilde{x}} L_{\tilde{g}}(\tilde{\xi}_{\tilde{x}}) = T_{\tilde{x}} L_{\tilde{g}} \circ T_x F(\xi_x) &= T_x(L_{\tilde{g}} \circ F)(\xi_x) \\ &= T_x(F \circ L_g)(\xi_x) \\ &= T_{gx} F \circ T_x L_g(\xi_x) \\ &= T_{gx} F(\xi_{gx}) = \tilde{\xi}_{F(gx)} = \tilde{\xi}_{\tilde{g}\tilde{x}} \end{aligned} \tag{4.97}$$

Then, back to (3)

证明 define $\beta_w = F \circ \alpha_v$ is an integral curve on $\tilde{G}$. associate to Left invariant vector field.

Then,

$$w = \partial_t|_0 \beta_w(t) = \partial_t|_0 F \circ \alpha_v = T_0(F \circ \alpha_v)(\partial_t) = T_e F \circ (\dot{\alpha}_v(0)) = T_e F(v) \tag{4.98}$$

We prooved that:

$$w = T_e F(v) \tag{4.99}$$

Hence,

$$\begin{aligned} \tilde{Exp} \circ T_e F(v) = \tilde{Exp}\big(T_e F(v)\big) = \tilde{Exp}(w) = \beta_w(1) \\ = F \circ \alpha_v(1) = F \circ Exp(v) \end{aligned} \tag{4.100}$$

End of proof

4.6 Matrix Lie Group

Parametrices

Entries: Work in a chart around the group identity $\mathbb{I}_d$. parametrices:

$$g = g(t) \quad t = (t^1 \cdots t^n) \in \mathbb{R}^n \tag{4.101}$$

Where:

$$g(0) = \mathbb{I}_d \tag{4.102}$$

and g has entries $g_\mu^{\ \nu}$, where $\mu\nu = 1 \cdots d$.

Vector fields: $\xi_t = \xi^i(t)\frac{\partial}{\partial t^i} = \xi^i(t)\frac{\partial g_\mu^{\ \nu}}{\partial t^i}(t)\frac{\partial}{\partial g_\mu^{\ \nu}} = \xi^i(t) tr\left(\frac{\partial g}{\partial t^i}(t)\frac{\partial}{\partial g^T}\right)$

Generators: $T_i = \frac{\partial g}{\partial t^i}(0)$

Tangent space at identity

$$\begin{aligned} \xi_{t=0} &= \xi^i(0) tr(T_i \frac{\partial}{\partial g^T}) \\ T_{\mathbb{I}}G &= \left\{v^i tr(T_i \frac{\partial}{\partial g^T}) | v \in \mathbb{R}^n\right\} \cong \left\{v^i T_i | v \in \mathbb{R}^n\right\} \cong \mathcal{L}(G) \end{aligned} \tag{4.103}$$

Left-invariant condition

Consider a tangent map that maps the vector field

$$T_x L_{g_0}(\xi_x)(f) = \xi_x(f \circ L_{g_0}) \tag{4.104}$$

The vectorfield ξ at the point x is:

$$\xi^i \frac{\partial g_\rho^{\ \sigma}}{\partial t^i}(x)\frac{\partial}{\partial g_\rho^{\ \sigma}} \tag{4.105}$$

denote that:

$$f \circ L_{g_0}(g) = f(g_0 g) = f'(g) \tag{4.106}$$

Then:

$$\xi_x(f') = \xi^i \frac{\partial g_\rho^{\ \sigma}}{\partial t^i}(x)\frac{\partial}{\partial g_\rho^{\ \sigma}} f(g_0 g)|_{g=x} \tag{4.107}$$

Consider the last part: $(g_1 = g_0 g)$

$$\begin{aligned}
\frac{\partial}{\partial g_{\rho}^{\ \sigma}} &= \frac{\partial g_{1\mu}^{\ \ \nu}}{\partial g_{\rho}^{\ \sigma}} \frac{\partial}{\partial g_{1\mu}^{\ \ \nu}} \\
&= \frac{\partial (g_0 g)_{\mu}^{\ \nu}}{\partial g_{\rho}^{\ \sigma}} \frac{\partial}{\partial g_{1\mu}^{\ \ \nu}} \\
&= \frac{\partial (g_{0\mu}^{\ \ \tau} g_{\tau}^{\ \nu})_{\mu}^{\ \nu}}{\partial g_{\rho}^{\ \sigma}} \frac{\partial}{\partial g_{1\mu}^{\ \ \nu}} \\
&= g_{0\mu}^{\ \ \rho} \delta_{\sigma}^{\nu} \frac{\partial}{\partial g_{1\mu}^{\ \ \nu}} \\
&= (T_x L_{g_0})_{\mu\sigma}^{\nu\rho} \frac{\partial}{\partial g_{1\mu}^{\ \ \nu}}
\end{aligned} \tag{4.108}$$

Then:

$$\begin{aligned}
T_x L_{g_0}(\xi_x)(f) &= \xi^i \frac{\partial g_{\rho}^{\ \sigma}}{\partial t^i}(x) (T_x L_{g_0})_{\mu\sigma}^{\nu\rho} \frac{\partial}{\partial g_{\mu}^{\ \nu}} |_{g=g_0 x} f \\
&= \xi^i \frac{\partial g_{\rho}^{\ \sigma}}{\partial t^i}(x) g_{0\mu}^{\ \ \rho} \delta_{\sigma}^{\nu} \frac{\partial}{\partial g_{\mu}^{\ \nu}} |_{g=g_0 x} f
\end{aligned} \tag{4.109}$$

However:

$$\xi_{g_0 x} = \xi^i \frac{\partial g_{\mu}^{\ \nu}}{\partial t^i} \frac{\partial}{\partial g_{\mu}^{\nu}} (g_0 x) \tag{4.110}$$

The constrain would be:

$$\xi^i \frac{\partial g_{\mu}^{\ \nu}}{\partial t^i}(g_0 x) = \xi^i \frac{\partial g_{\rho}^{\ \sigma}}{\partial t^i}(x) g_{0\mu}^{\ \ \rho} \delta_{\sigma}^{\nu} \tag{4.111}$$

Write as:

$$\xi^i(gx) \frac{\partial g}{\partial t^i}(gx) = \xi^i(x) g \ \frac{\partial g}{\partial t^i}(x) \tag{4.112}$$

Consider: $x = \mathbb{I}$ and $t = 0$

$$\xi^i(g) \frac{\partial g}{\partial t^i}(g) = \xi^i(0) g \ T_i \tag{4.113}$$

Consider a basis of Left invariant vector field:

$$L_i = \xi_i^j \partial_{t^j} \tag{4.114}$$

satisfies:

$$\xi_i^j(0) = \delta_i^j \tag{4.115}$$

Then:

$$\xi_i^j \frac{\partial g}{\partial t^j} = g T_i \tag{4.116}$$

The Left-invariant vector field:

$$\begin{aligned} L_i = \xi_i^j \frac{\partial}{\partial t^j} &= \xi_i^j tr(\frac{\partial g}{\partial t^j} \frac{\partial}{\partial g^T}) \\ &= tr(gT_i \frac{\partial}{\partial g^T}) \\ L_{i,\mathbb{I}} &= tr(T_i \frac{\partial}{\partial g^T}) \end{aligned} \tag{4.117}$$

We can regard Lie Algebra as a span of the generators.

Commutator

$$\begin{aligned} [L_i, L_j] &= tr(gT_i \frac{\partial}{\partial g^T}) \circ tr(gT_j \frac{\partial}{\partial g^T}) - \ (i \leftrightarrow j) \\ &= tr(g[T_i, T_j] \frac{\partial}{\partial g^T}) \in \mathcal{L}(G) \end{aligned} \tag{4.118}$$

In this case:

$$[T_i, T_j] = f_{ij}{}^k L_k \tag{4.119}$$

Exponential map

Left-invariant vector field can be written as $\xi^i = v^j \xi^i_j$. Associated with generator: $T = v^i T_i$. Integral curve: $t_i = t^i(s)$, $\alpha_v(s) = g(t(s))$ Satisfies:

$$\frac{dt^i}{ds} \partial_{t_i} = v^j \xi^i_j \partial_{t_i} \tag{4.120}$$

$$v^j \xi^i_j \partial_{t_i} = T_s(t(s))(\partial_s) \tag{4.121}$$

Then:

$$\frac{d\alpha_v}{ds} = \frac{\partial g}{\partial t^i} \frac{dt^i}{ds} = \frac{\partial g}{\partial t^i} v^j \xi^i_j = v^j g T_j = \alpha_v(s) T \tag{4.122}$$

Where used:

$$\xi_i^j \frac{\partial g}{\partial t^j} = g T_i \tag{4.123}$$

and

$$T = v^j T_j \tag{4.124}$$

Then:

$$Exp(T) = \alpha_v(1) = exp(T) \tag{4.125}$$

Ad representation

Consider the adjoint:

$$C_g(x) = gxg^{-1} \tag{4.126}$$

Then the adjoint representation would be:

$$\begin{aligned} Ad(g)(T) = T_e C_g(T) &= exp^{-1} \circ C_g \circ exp(T) \\ &= exp^{-1}(g e^T g^{-1}) \\ &= g exp^{-1}(e^T) g^{-1} \\ &= g T g^{-1} \end{aligned} \tag{4.127}$$

Which means Adjoint representation amounts to conjugation. Where we used:

$$F \circ Exp = \tilde{Exp} \circ T_e(F) \tag{4.128}$$

ad representation

$$ad(T)(S) = [T, S] \tag{4.129}$$

denote

$$\left[ad(T_i)\right]^k_{\ j} = f_{ij}^{\ \ k} \tag{4.130}$$

第五章 Lie algebras

The algebra is a **vector space**, we choose a **basis** T_i of Lie algebra $\boldsymbol{g}$.

Then, we introduce the **structure constants**

$$[T_i, T_j] = f_{ij}{}^{k} T_k \tag{5.1}$$

For which, we introduce the conventions:

$$[A, B]^{\dagger} = [B^{\dagger}, A^{\dagger}] \tag{5.2}$$

If $T_i^{\dagger} = \pm T_i$ (T_i is hermitean or anti-Hermitean), it can be deduced that:

$$(f_{ij}^{k})^{\dagger} = -f_{ij}^{k} \tag{5.3}$$

which means the structure constant is purely imaginary, for this, we usually denote:

$$[T_i, T_j] = i f_{ij}^{k} T_k \tag{5.4}$$

定义 5.1 (Representation of Lie algebra) A representation of Lie algebra is a **Lie algebra homomorphism** D from $\boldsymbol{g}$ to a **Lie algebra of matrices** with the matrix commutator as Lie bracket. The dimension of the representation is the dimension of the vector space.

Notice:

- A representation is reducible if there is an invariant subspace.
- Representations which are related by a similarity transformation are called equivalent.

5.1 Structure of Lie Algebras

For a Lie Algebra $\mathcal{L}$, have the definition:

(i) Sub Liealgebra: $\mathcal{A} \subset \mathcal{L}$ is called Lie sub-Algebra iff $\mathcal{A}$ $\mathcal{A}$ is a linear space and $[\mathcal{A}, \mathcal{A}] \subset \mathcal{A}$. denote $\mathcal{A} \leq \mathcal{L}$

(ii) Abelian: $\mathcal{A} \leq \mathcal{L}$, $\mathcal{A}$ is called Abelian iff: $[\mathcal{A}, \mathcal{A}] = 0$

(iii) Ideal: $\mathcal{A} \leq \mathcal{L}$, $\mathcal{A}$ is ideal iff $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$.

(iv) Derived series: The derived series $\{\mathcal{D}^k \mathcal{L}\}$ of $\mathcal{L}$ is defined by: $\mathcal{D}^1 \mathcal{L} = [\mathcal{L}, \mathcal{L}]$ and $\mathcal{D}^k \mathcal{L} = [\mathcal{D}^{k-1} \mathcal{L}, \mathcal{D}^{k-1} \mathcal{L}]$

(v) Solvable: $\mathcal{L}$ is called solvable iff $\mathcal{D}^k \mathcal{L} = 0$ for some k.

A Lie Algebra is called:

(i) Simple (no non-trival ideals), if: $\nexists \mathcal{A}$ s.t. $\mathcal{A} \leq \mathcal{L}$; $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$; $[\mathcal{A}, \mathcal{A}] = 0$; $\mathcal{A} \neq 0/\mathcal{L}$

(ii) Semi-simple (no non-zero solvable ideals), if: $\nexists \mathcal{A}$ s.t. $\mathcal{A} \leq \mathcal{L}$; $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$; $\mathcal{D}^k \mathcal{A} = 0$; $\mathcal{A} \neq 0$

引理 5.1 Semi-simple and nonzero abelian ideal

$$\text{Semi-simple} \Leftrightarrow \mathcal{L}\text{has no nonzero abelian ideal} \tag{5.5}$$

证明 $\Rightarrow$

$$\mathcal{L} \text{ has no nonzero solvable ideals} \Leftrightarrow \text{Semi-simple} \to \mathcal{L} \text{ has no nonzero abelian ideal} \tag{5.6}$$

逆否命题:

$$\mathcal{L} \text{ has nonzero solvable ideals} \leftarrow \mathcal{L} \text{ has nonzero abelian ideal} \tag{5.7}$$

If $\mathcal{A} \le \mathcal{L}$, $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$, $[\mathcal{A}, \mathcal{A}] = 0$, $\mathcal{A} \ne 0$, then $\mathcal{D}^1\mathcal{A} = 0$, which means $\mathcal{A}$ is non-zero solvable ideals

$\Leftarrow$

$$\mathcal{L} \text{ has no nonzero solvable ideals} \Leftrightarrow \text{Semi-simple} \leftarrow \mathcal{L} \text{ has no nonzero abelian ideal} \tag{5.8}$$

逆否命题:

$$\mathcal{L} \text{ has nonzero solvable ideals} \to \mathcal{L} \text{ has nonzero abelian ideal} \tag{5.9}$$

Derivative of ideal is ideal, this is because

$$if : [\mathcal{L}, \mathcal{A}] \subset \mathcal{A} \Rightarrow [\mathcal{L}, \mathcal{D}^k\mathcal{A}] \subset \mathcal{D}^k\mathcal{A} \tag{5.10}$$

用数学归纳法，假设

$$[\mathcal{L}, \mathcal{D}^{k-1}\mathcal{A}] \subset \mathcal{D}^{k-1}\mathcal{A} \tag{5.11}$$

于是:

$$\begin{aligned}
[\mathcal{L}, \mathcal{D}^k\mathcal{A}] &= \left[\mathcal{L}, [\mathcal{D}^{k-1}\mathcal{A}, \mathcal{D}^{k-1}\mathcal{A}]\right] \\
&= -\left[\mathcal{D}^{k-1}\mathcal{A}, [\mathcal{D}^{k-1}\mathcal{A}, \mathcal{L}]\right] - \left[\mathcal{D}^{k-1}\mathcal{A}, [\mathcal{L}, \mathcal{D}^{k-1}\mathcal{A}]\right] \\
&\subset \left[\mathcal{D}^{k-1}\mathcal{A}, \mathcal{D}^{k-1}\mathcal{A}\right] + \left[\mathcal{D}^{k-1}\mathcal{A}, \mathcal{D}^{k-1}\mathcal{A}\right] \\
&\subset \mathcal{D}^k\mathcal{A}
\end{aligned} \tag{5.12}$$

Then if $\mathcal{A}$ is nonzero solvable ideal, then $\mathcal{D}^{k-1}\mathcal{A}$ is nonzero abelian ideal.

□

5.2 Killing Form

定义 5.2 Killing Form The symmetric bilinear form: $\Gamma: \ \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ defined by:

$$\Gamma(T, S) = tr\big(ad(T)ad(S)\big) \tag{5.13}$$

Denote killing form of basis(γ_{ij} is symmetry):

$$\begin{aligned}
\gamma_{ij} = \Gamma(T_i, T_j) &= tr\big(ad(T_i)ad(T_j)\big) \\
&= f_{ik}{}^l f_{jl}{}^k.
\end{aligned} \tag{5.14}$$

In this case:

$$\Gamma(T, S) = \Gamma(t^i T_i, s^j T_j) = t^i s^j \gamma_{ij}. \tag{5.15}$$

Killing form under basis transformation Basis transformation under invertible matrix (A)

$$(T'_1, T'_2 \cdots) = (T_1, T_2 \cdots)A \Rightarrow T'_i = T_j A^j_{\ i}. \tag{5.16}$$

Notice that, if field $\mathbb{K}$ of Lie algebra $\mathcal{L}$ is $\mathbb{C}$, A is complex invertible matrix, if field $\mathbb{K} = \mathbb{R}$, A is real invertible matrix.

by doing this transformation, the invariant of algebra element requires transformation of coordinates:

$$(T'_1, T'_2 \cdots) \begin{pmatrix} t'^1 \\ t'^2 \\ \vdots \end{pmatrix} = (T_1, T_2 \cdots)A \begin{pmatrix} t'^1 \\ t'^2 \\ \vdots \end{pmatrix} = (T_1, T_2 \cdots) \begin{pmatrix} t^1 \\ t^2 \\ \vdots \end{pmatrix} \Rightarrow t'^i = (A^{-1})^i_{\ j} t^j. \tag{5.17}$$

Consider the transformation of structure constant:

$$\begin{aligned} [T'_i, T'_j] &= [T_\alpha A^\alpha_{\ i}, T_\beta A^\beta_{\ j}] \\ &= A^\alpha_{\ i} A^\beta_{\ j} f_{\alpha\beta}^{\ \ \gamma} T_\gamma \\ &= A^\alpha_{\ i} A^\beta_{\ j} f_{\alpha\beta}^{\ \ \gamma} T'_k (A^{-1})^k_{\ \gamma}, \\ f'^{\ k}_{ij} &= A^\alpha_{\ i} A^\beta_{\ j} (A^{-1})^k_{\ \gamma} f_{\alpha\beta}^{\ \ \gamma}. \end{aligned} \tag{5.18}$$

Killing Form is invariant under basis transformation is

$$\begin{aligned} \Gamma(T, S) &= Tr\left(ad(T)ad(S)\right) \\ &= t'^i s'^j Tr\left(ad(T'_i)ad(S'_j)\right) \\ &= t'^i s'^j f'^{\ m}_{il} f'^{\ l}_{jm}. \end{aligned} \tag{5.19}$$

Using basis transformation for structure constant and coefficients:

$$\begin{cases} f'^{\ k}_{ij} &= A^\alpha_{\ i} A^\beta_{\ j} (A^{-1})^k_{\ \gamma} f_{\alpha\beta}^{\ \ \gamma}, \\ t'^i &= (A^{-1})^i_{\ j} t^j. \end{cases} \tag{5.20}$$

By insertion:

$$\begin{aligned} \Gamma(T, S) &= t'^i s'^j A^\alpha_{\ i} A^\beta_{\ l} A^\epsilon_{\ j} A^\lambda_{\ m} (A^{-1})^m_{\ m'} (A^{-1})^l_{\ l'} f_{\alpha\beta}^{\ \ m'} f_{\epsilon\lambda}^{\ \ l'} \\ &= t^a s^b (A^{-1})^i_{\ a} (A^{-1})^j_{\ b} A^\alpha_{\ i} A^\beta_{\ l} A^\epsilon_{\ j} A^\lambda_{\ m} (A^{-1})^m_{\ m'} (A^{-1})^l_{\ l'} f_{\alpha\beta}^{\ \ m'} f_{\epsilon\lambda}^{\ \ l'} \\ &= t^a s^b \delta^\alpha_a \delta^\epsilon_b \delta^\beta_{l'} \delta^\lambda_{m'} f_{\alpha\beta}^{\ \ m'} f_{\epsilon\lambda}^{\ \ l'} \\ &= t^\alpha s^\epsilon f_{al'}^{\ \ \lambda} f_{\epsilon m'}^{\ \ \beta} \\ &= t^i s^j \Gamma(T_i, T_j). \end{aligned} \tag{5.21}$$

In all,

$$\Gamma(T, S) = t'^i s'^j Tr\left(ad(T'_i)ad(S'_j)\right) = t^i s^j \Gamma(T_i, T_j). \tag{5.22}$$

Killing Form and Lie-Algebra Structure

- killing Form Γ is called non-degenerate iff $\forall T \in \mathcal{L}\ \Gamma(T, S) = 0$ implies $S = 0$.
- killing Form Γ is called non-degenerate iff matrix γ_{ij} is invertible.

定理 5.1 (semi-simple and non-degenerate) A Lie Algebra $\mathcal{L}$ is semi-simple iff Γ is non-degenerate

证明 $\Leftarrow$If Γ is non-degenerate, Consider Lie Algebra $\mathcal{L}$ has an abelian ideal $\mathcal{A}$.(含有 nonzero abelian ideal 等价于不是 semi-simple) Consider basis:(T_a, T_α), T_a is basis of $\mathcal{A}$, T_α the basis of the remainder. Consider $\forall T \in \mathcal{L}, \forall S \in \mathcal{A}$.

$$\begin{aligned} ad(T) \circ ad(S)(T_a) &= [T, [S, T_a]] = [T, 0] = 0 \\ ad(T) \circ ad(S)(T_\alpha) &= [T, [S, T_\alpha]] \in \mathcal{A} \end{aligned} \tag{5.23}$$

For a matrix of $ad(T) \circ ad(S)$

$$ad(T) \circ ad(S) = \begin{bmatrix} 0_1 & * \\ 0_2 & 0_3 \end{bmatrix} \tag{5.24}$$

For this, $0_1 = 0_{a,a}$ $0_3 = 0_{\alpha,\alpha}$

Killing Form:

$$\Gamma(T, S) = tr(ad(T)ad(S)) = 0 \tag{5.25}$$

Since Γ is non-degenerate. Then, $S = 0$, $\mathcal{A} = 0$, which means $\mathcal{L}$ is semi-simple.

$\Rightarrow$ ...

□

引理 5.2 (Negative semi-definite and Compact) If G is compact, then the Killing form Γ on $\mathcal{L}(G)$ is negative semi-definite...

Jacobi identity for structure constants Consider the relation:

$$[[T_i, T_j], T_k] + [[T_j, T_k], T_i] + [[T_k, T_i], T_j] = 0, \tag{5.26}$$

As for:

$$[T_i, T_j] = f_{ij}{}^l T_l \tag{5.27}$$

By insertion:

$$f_{ij}{}^l f_{lk}{}^n + f_{jk}{}^l f_{li}{}^n + f_{ki}{}^l f_{lj}{}^n = 0 \tag{5.28}$$

Totally anti-symmetric structure constant introduce the structure constants

$$f_{ijk} = \gamma_{kl} f_{ij}{}^l \Rightarrow f_{ij}{}^l = \gamma^{lk} f_{ijk} \tag{5.29}$$

Insert the difinition of γ, and use the Jacobi indentity for structure constants talked before, we find that

$$f_{ijk} = tr(f_k f_j f_i) + tr(\tilde{f}_k \tilde{f}_i \tilde{f}_j) \tag{5.30}$$

for the tilde term:

$$(\tilde{f}_i)_j{}^k = f_{ji}{}^k \tag{5.31}$$

Cyclicity of the trace shows that the structure constants f_{ijk} are unchanged under cycle permutation, Consider the anti-symmetric for the first two term. We Conclude that f_{ijk} is totally anti-symmetric.

Quadratic Casimir If the algebra $\mathcal{L}$ is semi-simple(γ is invertible), we define the quadratic Casimir operator:

$$C^{(r)} = \gamma^{ij} T_i^r T_j^r. \tag{5.32}$$

γ^{ij} is inverse of γ_{ij}. T_i^r is representation of T_i in $dim(r) \times dim(r)$ matrix space.

引理 5.3 (commutation of Casimir operator) The Casimir operator satisfies $\forall T \in \mathcal{L} \quad [C^{(r)}, T] = 0$

证明

$$\begin{aligned}
[C, T_l^{(r)}] &= \gamma^{ij}\left[T_i^{(r)}T_j^{(r)}, T_l^{(r)}\right] \\
&= \gamma^{ij}\left(T_i^{(r)}T_j^{(r)}T_l^{(r)} - T_l^{(r)}T_i^{(r)}T_j^{(r)}\right) \\
&= \gamma^{ij}\left(T_i^{(r)}\left[T_j^{(r)}, T_l^{(r)}\right] + \left[T_i^{(r)}, T_l^{(r)}\right]T_j^{(r)}\right) \\
&= \gamma^{ij}\left(f_{jl}{}^m T_i^{(r)}T_m^{(r)} + f_{il}{}^m T_m^{(r)}T_j^{(r)}\right) \\
&\gamma_{ij} \text{ is symmetric, so does its inverse} \\
&= \gamma^{ij}\left(f_{jl}{}^m T_i^{(r)}T_m^{(r)} + f_{jl}{}^m T_m^{(r)}T_i^{(r)}\right) \\
&= \gamma^{ij} f_{jl}{}^m\left(T_i^{(r)}T_m^{(r)} + T_m^{(r)}T_i^{(r)}\right) \\
&= \gamma^{ij}\gamma^{mn} f_{jln}\left(T_i^{(r)}T_m^{(r)} + T_m^{(r)}T_i^{(r)}\right)
\end{aligned} \tag{5.33}$$

Noticed, f_{jln} is anti-symmetric of j,n while others are symmetric, Then:

$$[C, T_l^{(r)}] = 0 \tag{5.34}$$

□

By using Shur's Lemma, if the Lie Algebra is irreduciable, Then

$$C^{(r)} = C(r)\mathbb{I}_{dim(r)\times dim(r)} \tag{5.35}$$

Physics Convention Transformation of basis in Lie algebra, leads to transformation of Killing form of basis:

$$\begin{aligned}
\gamma'_{ij} &= A^\alpha_{\ i} A^\beta_{\ l} A^\epsilon_{\ j} A^\lambda_{\ m} (A^{-1})^m_{\ m'} (A^{-1})^l_{\ l'} f_{\alpha\beta}{}^{m'} f_{\epsilon\lambda}{}^{l'} \\
&= A^\alpha_{\ i} \delta^\beta_{l'} \delta^\lambda_{m'} A^\epsilon_{\ j} f_{\alpha\beta}{}^{m'} f_{\epsilon\lambda}{}^{l'} \\
&= A^\alpha_{\ i} A^\epsilon_{\ j} f_{\alpha\beta}{}^{\lambda} f_{\epsilon\lambda}{}^{\beta} \\
&= A^\alpha_{\ i} \gamma_{\alpha\epsilon} A^\epsilon_{\ j}, \\
\gamma' &= A^T \gamma A
\end{aligned} \tag{5.36}$$

Take Lie algebra Semi-simple, and compact, from 5.1 and 5.2. Killing form of basis is invertible and negative definite.

Physics conventions requires:

$$\gamma'_{ij} = -\delta_{ij}. \tag{5.37}$$

Under this convention, noticed that

$$f_{ij}{}^{l} = \gamma^{lk} f_{ijk} = -f_{ijl}, \ f_{ijk} \text{ is antisymmetric.} \tag{5.38}$$

Which means $f_{ij}{}^{l}$ is totally anti-symmetric.

Index Define M matrix (the basis is taken under the convention mentioned above):

$$M_{jk} = tr\left(T_j^{(r)} T_k^{(r)}\right). \tag{5.39}$$

it commutes all element in $ad(\mathcal{L})$.

证明

$$\begin{aligned}
\left([T_i^{(ad)}, M]\right)_{jk} &= \left(T_i^{(ad)}\right)_{jl} tr\left(T_l^{(r)} T_k^{(r)}\right) - tr\left(T_j^{(r)} T_l^{(r)}\right)\left(T_i^{(ad)}\right)_{lk} \\
&= f_{ij}{}^{l} tr\left(T_l^{(r)} T_k^{(r)}\right) - f_{il}{}^{k} tr\left(T_j^{(r)} T_l^{(r)}\right) \\
&= tr\left(f_{ij}{}^{l} T_l^{(r)} T_k^{(r)} - f_{il}{}^{k} T_j^{(r)} T_l^{(r)}\right) \\
&= tr\left(f_{ij}{}^{l} T_l^{(r)} T_k^{(r)} + f_{ilk} T_j^{(r)} T_l^{(r)}\right) \\
&= tr\left(f_{ij}{}^{l} T_l^{(r)} T_k^{(r)} - f_{ikl} T_l^{(r)} T_j^{(r)}\right) \\
&= tr\left(f_{ij}{}^{l} T_l^{(r)} T_k^{(r)} + f_{ik}{}^{l} T_l^{(r)} T_j^{(r)}\right) \\
&= tr\left([T_i^{(r)}, T_j^{(r)}] T_k^{(r)} + [T_i^{(r)}, T_k^{(r)}] T_j^{(r)}\right) \\
&= tr\left(T_i^{(r)} T_j^{(r)} T_k^{(r)} - T_j^{(r)} T_i^{(r)} T_k^{(r)} + T_i^{(r)} T_k^{(r)} T_j^{(r)} - T_k^{(r)} T_i^{(r)} T_j^{(r)}\right) \\
&= 0.
\end{aligned} \tag{5.40}$$

□

Shur's Lemma proves:

$$M_{jk} = tr\left(T_j^{(r)} T_k^{(r)}\right) = -c(r)\delta_{ij}. \tag{5.41}$$

c(r) is called index of representation.

Relation between index and Casimier Under the convention mentioned above:

$$tr\left(C^{(r)}\right) = dim(r) C^{(r)} = tr\left(\gamma^{ij} T_i^{(r)} T_j^{(r)}\right) = -\sum_i tr\left(T_i^{(r)} T_i^{(r)}\right) = c(r) dim(ad), \tag{5.42}$$

$$c(r) = \frac{dim(r)}{dim(ad)} C^{(r)} \tag{5.43}$$

5.3 Cartan-Weyl basis

定义 5.3 (Cartan Algebra) A maximal, diagonalisable **Abelian** subalgebra $H \subset \mathcal{L}$ is called a Cartan subalgebra. The dimension of H is called the rank of $\mathcal{L}$ denoted by $rk(\mathcal{L}) = dim(H)$. The lie algebra here is complex algebra.

定理 5.2 (Commute Matrix and their mutual eigen values) Consider two commute matrices A and B.

(a) We can always decompose the eigen space with eigen value α of operator A into direct sum of eigen space of operator B.

(b) We can find the common eigen space of operators A and B.

证明

$$AB|\alpha\rangle = BA|\alpha\rangle = \alpha(B|\alpha\rangle) \tag{5.44}$$

Which means $B|\alpha\rangle \in$ subspace of eigen value α

(i) If subspace of eigen value α is non degenerate:

$$B|\alpha\rangle = \beta|\alpha\rangle \tag{5.45}$$

(ii) If subspace of eigen value α is degenerate:

$$\begin{aligned} B|\alpha_1\rangle &= C_{11}|\alpha_1\rangle + C_{12}|\alpha_2\rangle \cdots \\ B|\alpha_2\rangle &= C_{21}|\alpha_1\rangle + C_{22}|\alpha_2\rangle \cdots \\ &\vdots \end{aligned} \tag{5.46}$$

we can decompose the subspace spaned by $|\alpha_1\rangle, |\alpha_2\rangle \cdots$ into eigenspace of operator B. by studying the eigen value of matrix C^T.

This means, (a) We can always decompose the eigen space with eigen value α of operator A into direct sum of eigen space of operator B. (b) We can find the common eigen space of operators A and B.

□

Cartan Decomposition We can study the simultaneous eigenvectors $T \in \mathcal{L}(G)$ Satisfies the equation:

$$ad(H)(T) = \alpha(H)T. \tag{5.47}$$

For which $\alpha(H)$ depends on elements in H linearly. $\alpha \in \mathcal{H}'$, where $\mathcal{H}'$ is the dual space of $\mathcal{H}'$.

Denote mutual eigen-space $\mathcal{L}_\alpha \subset \mathcal{L}$ as the eigen space of root α. The Lie Algebra can be written as:

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_\alpha \mathcal{L}_\alpha \tag{5.48}$$

This is called the **Cartan Decomposition**. Sometimes, we write: $\mathcal{H} = \mathcal{L}_0$

The dimension of $\mathcal{H}$ is called the rank of $\mathcal{L}$, $rk(\mathcal{L}) = dim(\mathcal{H})$.

Root A non-zero linear functional $\alpha \in \mathcal{H}'$ is called a root of the Lie Algebra $\mathcal{L}$ if there is a non-zero $T \in \mathcal{L}$ such that $ad(H)(T) = \alpha(H)T$. The set $\Delta = \{\alpha \in \mathcal{H}' | \alpha \text{ is a root}\}$ is called **roots**, the lattice generated by roots are called root lattice Λ_R

Structure of Cartan decomposition

1-The Cartan decomposition is consistent with the commutator if $T \in \mathcal{L}_\alpha$ & $S \in \mathcal{L}_\beta$. $\Rightarrow$ $[T, S] \in \mathcal{L}_{\alpha+\beta}$

证明

$$[H, T] = \alpha(H)T \quad [H, S] = \beta(H)S \tag{5.49}$$

Then:

$$\begin{aligned}\big[H, [T, S]\big] &= \big[T, [H, S]\big] - \big[S, [H, T]\big] \\ &= \beta(H)[T, S] - \alpha(H)[S, T] = (\alpha(H) + \beta(H))[T, S]\end{aligned} \tag{5.50}$$

Sometimes, written as:

$$[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subset \mathcal{L}_{\alpha+\beta} \tag{5.51}$$

□

2-Relation between cartan decomposition and killing form

$$T \in \mathcal{L}_\alpha \;;\; S \in \mathcal{L}_\beta \;;\; \alpha + \beta \neq 0 \Rightarrow \Gamma(T, S) = tr\big(ad(T) \circ ad(S)\big) = 0 \quad \big(\mathcal{L}_\alpha \perp \mathcal{L}_\beta\big) \tag{5.52}$$

证明 首先寻找:

$$T \in \mathcal{L}_\alpha \quad S \in \mathcal{L}_\beta \quad U \in \mathcal{L}_\gamma \tag{5.53}$$

将其作用于 U:

$$ad(T) \circ ad(S)(U) = \big[T, [S, U]\big] \in \mathcal{L}_{\alpha+\beta+\gamma} \left([\mathcal{L}_\alpha, \mathcal{L}_\beta] \subset \mathcal{L}_{\alpha+\beta}\right) \tag{5.54}$$

This means the $ad(T) \circ ad(S)$ has vanishing diagonal elements.

$$\Gamma(T, S) = tr\big(ad(T) \circ ad(S)\big) = 0 \quad \mathcal{L}_\alpha \perp \mathcal{L}_\beta \quad \text{for } \alpha + \beta \neq 0 \tag{5.55}$$

□

3-$\Gamma_{\mathcal{H}\times\mathcal{H}}$ is non-degenerate.

$$\nexists H \in \mathcal{H} \;;\; H \neq 0 \; s.t. \; \forall \tilde{H} \in \mathcal{H} \; \Gamma(H, \tilde{H}) = 0 \tag{5.56}$$

And

$$\forall \alpha \in \mathcal{H}' \;,\; \exists H_\alpha \in \mathcal{H} \; s.t. \; \Gamma(H, H_\alpha) = \alpha(H) \tag{5.57}$$

Define:

$$(\alpha, \beta) = \Gamma(H_\alpha, H_\beta) \tag{5.58}$$

证明 $\mathcal{L}$ is semi-simple $\Rightarrow$ Killing form is non degenerate. on $\mathcal{L} \times \mathcal{L}$.
$\exists H \in \mathcal{H}$, s.t. $\forall \tilde{H} \in \mathcal{H}, \Gamma(H, \tilde{H}) = 0$. We need to show that $H = 0$.
$\forall S \in \mathcal{L}$

$$S = \tilde{H} + \sum_{\alpha\in\Delta} S_\alpha \quad S_\alpha \in \mathcal{L}_\alpha \quad \tilde{H} \in \mathcal{H} \tag{5.59}$$

if $\exists H \in \mathcal{H}$, s.t. $\forall \tilde{H} \in \mathcal{H}, \Gamma(H, \tilde{H}) = 0$

Noticed that:

$$\forall S_\alpha \quad \Gamma(H, S_\alpha) = 0 \tag{5.60}$$

Then:

$$\forall S_\alpha \in \mathcal{L}_\alpha \ \ \forall \tilde{H} \in \mathcal{H} \quad \sum_\alpha \Gamma(H, S_\alpha) + \Gamma(H, \tilde{H}) = 0 \tag{5.61}$$

Which means:

$$\forall S \in \mathcal{L} \quad \Gamma(H, S) = 0 \tag{5.62}$$

From theorem about relation of semi-simple and non-degenerate 5.1.

$$H = 0 \tag{5.63}$$

Which means $\Gamma_{\mathcal{H}\times\mathcal{H}}$ is non-degenerate.
To find an $H_\alpha \in \mathcal{H}$,

$$\Gamma(H, H_\alpha) = \alpha(H) \Rightarrow \Gamma(H_i, H_\alpha) = \alpha(H_i) \tag{5.64}$$

Consider:

$$H_\alpha = H_\alpha^j H_j \tag{5.65}$$

This relation means:

$$\gamma_{ij} H_\alpha^j = \alpha(H_i) \tag{5.66}$$

if γ matrix is invertible, the exact H_α can be recovered from this equation.
□

4- If Δ Contains α, it contains $-\alpha$.

证明 Assume that $-\alpha \notin \Delta$, Then $\mathcal{L}_\alpha \perp \mathcal{L}_\beta \forall \beta \in \Delta$. As well as $\mathcal{L}_\alpha \perp \mathcal{H}$ So $\mathcal{L}_\alpha \perp \mathcal{L}$. however, this is contradiction since Γ is non-degenerate.
□

5- For $T \in \mathcal{L}_\alpha$, $S \in \mathcal{L}_{-\alpha}$. we have $[T, S] = \Gamma(T, S) H_\alpha$. One can normalise as: $\Gamma(T, S) = 1$.

证明 Let $H \in \mathcal{H}, T \in \mathcal{L}_\alpha, S \in \mathcal{L}_{-\alpha}$

$$\begin{aligned} \Gamma(H, [T, S]) &= \Gamma([H, T], S) = \alpha(H)\Gamma(T, S) \\ &= \Gamma(H, H_\alpha)\Gamma(T, S) = \Gamma(H, \Gamma(T, S) H_\alpha) \end{aligned} \tag{5.67}$$

with non-degeneracy of Γ, $[T, S] = \Gamma(T, S) H_\alpha$. There must be a $T \in \mathcal{L}_\alpha$ and $S \in \mathcal{L}_{-\alpha}$. with $\Gamma(T, S) \neq 0$. (Γ is non-degenerate in $\mathcal{L}_\alpha \oplus \mathcal{L}_{-\alpha}$). by suitable normalising T, S . $\Gamma(T, S) = 1$
□

6- $dim(\mathcal{L}_\alpha) = 1$ for all $\alpha \in \Delta$

7- Let $\alpha \in \Delta$, from $\{k\alpha | k \in \mathbb{Z}\}$, only α and $-\alpha$ are roots .

证明 We proof these two statement together.
Choose: $T \in \mathcal{L}_\alpha$, $S \in \mathcal{L}_{-\alpha}$, $H_\alpha \in \mathcal{H}$,such that $[T, S] = H_\alpha$.
Define space:

$$V = \mathbb{C}S + \mathbb{C}H_\alpha + \sum_{k\geq 1} \mathcal{L}_{k\alpha} \tag{5.68}$$

Which is invariant under $ad(H)$. We proof this two statement by computin trace of $ad(H_\alpha)|_V$ in two different ways.
(1)

$$tr\big(ad(H_\alpha)|_V\big) = tr\big(ad([T, S])|_V\big) = tr\big([ad(T), ad(S)]|_V\big) = 0 \tag{5.69}$$

(2) evaluating it on a basis of V.

$$\begin{aligned} ad(H_\alpha)(S) &= [H_\alpha, S] = -\alpha(H_\alpha)S = -\Gamma(H_\alpha, H_\alpha)S = -(\alpha,\alpha)S \\ ad(H_\alpha)(H_\alpha) &= [H_\alpha, H_\alpha] = 0 \\ \underbrace{ad(H_\alpha)(U)}_{\text{U is basis of } \mathcal{L}_{k\alpha}} &= [H_\alpha, U] = k\alpha(H_\alpha)U = k(\alpha,\alpha)U \end{aligned} \tag{5.70}$$

Which implies:

$$tr\big(ad(H_\alpha)|_V\big) = (\alpha,\alpha)\Big(-1 + \sum_{k\geq 1} k\,dim(\mathcal{L}_{k\alpha})\Big) \tag{5.71}$$

Which means $dim(\mathcal{L}_{k\alpha}) = 1$ for $k = 1$, others: $dim(\mathcal{L}_{k\alpha}) = 0$
□

8- For $H, \tilde{H} \in \mathcal{H}$, we have $\Gamma(H, \tilde{H}) = \sum_{\alpha\in\Delta} \alpha(H)\alpha(\tilde{H})$.

证明 Start with: $T \in \mathcal{L}_\alpha$

$$ad(H) \circ ad(\tilde{H})(T) = \big[H, [\tilde{H}, T]\big] = \alpha(H)\alpha(\tilde{H})(T) \tag{5.72}$$

In this case:

$$\Gamma(H, \tilde{H}) = tr\big(ad(H) \circ ad(\tilde{H})\big) = \sum_{\alpha\in\Delta} \alpha(H)\alpha(\tilde{H}) \tag{5.73}$$

□

9- Δ contains a basis of $\mathcal{H}'$ (roots span root space).

证明 Assuming $Span(\Delta) \neq \mathcal{H}'$. choose basis of $Span(\Delta)$ as $(\alpha_1, \cdots \alpha_n)$. Complete it to $(\alpha_1 \cdots \alpha_N)$. Introducing its dual basis: $(H_1, \cdots H_N)$ of $\mathcal{H}$. This means:

$$\alpha_i(H_j) = \delta_{ij} \Rightarrow \alpha(H_N) = 0 \text{ for all } \alpha \in \Delta \tag{5.74}$$

This means:

$$\Gamma(H_N, H) = 0 \text{ for all } H \in \mathcal{H} \tag{5.75}$$

Which is contradict with non-degeneracy of Γ.

□

Cartan Weyl basis The basis would be: (H_i, E_α) where $i = 1 \cdots r = rk(\mathcal{L})$, $\mathcal{L}_\alpha = Span(E_\alpha)$, and $\Gamma(E_\alpha, E_{-\alpha}) = 1$

Relative to basis of Cartan, elements $\alpha \in \mathcal{H}'$ can be described by $(\alpha_1 \cdots \alpha_r)$ where $\alpha_i = \alpha(H_i)$

– Killing form can be used to upper or lower the index Consider $\alpha \in \mathcal{H}'$, $H_\alpha = \lambda^i H_i$

$$\alpha_j \equiv \alpha(H_j) = \Gamma(H_j, H_\alpha) = \lambda^i \Gamma(H_i, H_j) = \lambda^i \gamma_{ij} \tag{5.76}$$

take inverse and denote λ as upper index α

$$\alpha^i = \gamma^{ij} \alpha_j \tag{5.77}$$

– Commutation relations for the Cartan-Weyl basis

$$\begin{aligned} &[H_i, H_j] = 0 \quad [H_i, E_\alpha] = \alpha_i E_\alpha \\ &[E_\alpha, E_{-\alpha}] = H_\alpha = \alpha^i H_i \quad [E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & 0 \neq \alpha+\beta \in \Delta \\ 0 & 0 \neq \alpha+\beta \notin \Delta \end{cases} \end{aligned} \tag{5.78}$$

– Subalgebra of Cartan-Wel basis For any root $\alpha \in \Delta$, the three generators $(H_\alpha, E_\alpha, E_{-\alpha})$ form a subalgebra, with commutation relation:

$$\begin{aligned} &[H_\alpha, E_{\pm\alpha}] = \pm\alpha(H_\alpha) E_{\pm\alpha} = \pm(\alpha, \alpha) E_{\pm\alpha} \\ &[E_\alpha, E_{-\alpha}] = H_\alpha \end{aligned} \tag{5.79}$$

Weights

– Basic of weights For a representation $r : \mathcal{L} \to End(V)$, we call $w \in \mathcal{H}'$ a weight of r if there is a non-zero vector $v \in V$ s.t.

$$r(H)(v) = w(H)v \ \forall H \in \mathcal{H} \tag{5.80}$$

The eigen space of weight w denoted by V_w consists of all $v \in V$ satisfies above equation. Representation vector space:

$$V = \oplus_w V_w \tag{5.81}$$

A weight $w \in \mathcal{H}'$ is represented by:

$$(w(H_1) \cdots w(H_r)) \quad r = rk(\mathcal{L}) \tag{5.82}$$

– Raising and Lowering operator

$$\begin{aligned} r(H)\big(r(E_\alpha)v\big) &= \Big(r(E_\alpha)r(H) + \big[r(H), r(E_\alpha)\big]\Big)v \\ &= \Big(r(E_\alpha)w(H) + r\big([H, E_\alpha]\big)\Big)v \\ &= \Big(r(E_\alpha)w(H) + r\big(\alpha(H)E_\alpha\big)\Big)v \\ &= \big(w(H) + \alpha(H)\big)r(E_\alpha)v \end{aligned} \tag{5.83}$$

Shows that:

$$r(E_\alpha)v \begin{cases} \in V_{w+\alpha} & w+\alpha \text{ is a weight} \\ = 0 & w+\alpha \text{ is not a weight} \end{cases} \tag{5.84}$$

命题 5.1 Weights in representation differ by roots. If representation $r : \mathcal{L} \to End(V)$ is irreducible, Then any two weights w_1, w_2 of r satisfy $w_1 - w_2 \in \Lambda_R$.

证明 If $w_1 - w_2 \notin \Lambda_R$, $\oplus_{\alpha\in\Lambda_R} V_{w_1+\alpha} \subset V$ is invariant under r, but it dose not contains V_{w_2}. which is contradict with irreducible condition of r.

□

–Weight Lattice Consider the commutation relation of $(H_\alpha, E_\alpha, E_{-\alpha})$

$$\begin{aligned} [H_\alpha, E_{\pm\alpha}] &= \pm\alpha(H_\alpha)E_{\pm\alpha} = \pm(\alpha,\alpha)E_{\pm\alpha} \\ [E_\alpha, E_{-\alpha}] &= H_\alpha \end{aligned} \tag{5.85}$$

Eigen values of H_α,$w(H_\alpha)$ has to be $\frac{(\alpha,\alpha)}{2}\mathbb{Z}$

$$(w,\alpha) \triangleq w(H_\alpha) \in \frac{(\alpha,\alpha)}{2}\mathbb{Z} \tag{5.86}$$

remember: $\Gamma(H, H_\alpha) = \alpha(H)$, $(\alpha, \beta) = \Gamma(H_\alpha, H_\beta)$ The weight lattice:

$$\Lambda_w = \{w \in \mathcal{H}' | \frac{2(w,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}\ \forall\alpha \in \Delta\} \tag{5.87}$$

It follows that all weights of a representation of $\mathcal{L}$ lie in Λ_w

–Positive and negative roots Choose a direction $l \in \mathcal{H}'$ in root space. Define two subset of roots:

$$\begin{aligned} \Delta_+ &= \{\alpha \in \Delta | l(\alpha) > 0\} \\ \Delta_- &= \{\alpha \in \Delta | l(\alpha) < 0\} \\ l(\alpha) &\triangleq \alpha(H_l) = \Gamma(H_l, H_\alpha) \end{aligned} \tag{5.88}$$

Need to be careful that $l(\alpha) \neq 0$ for all $\alpha \in \Delta$ Since Δ is a **finite site** of roots, this is always possible. In this case $\Delta = \Delta_+ \cup \Delta_-$. E_α with $\alpha \in \Delta_+$ are raising operator, E_α with ... are lowering operator.

–Highest weight vector Let $r : \mathcal{L} \to End(V)$ be a representation. A non-zero vector $v \in V$ is called the highest weight vector if $E_\alpha(v) = 0\ \forall\alpha \in \Delta_+$. (Definition of Highest weight vector). The weight λ of highest weight vector is called highest weight.

Properties of Cartan-Weyl decomposition For a semi-simple complex Lie Algebra $\mathcal{L}$ with representation $r : \mathcal{L} \to End(V)$.

1- r has a highest weight vector .

证明 Choose a weight λ as highest weight which means $l(\lambda) = \lambda(H_l)$ is a highest over all weights. Consider

$$\forall \alpha \in \Delta_+ \quad (\lambda + \alpha)(H_l) = \lambda(H_l) + \alpha(H_l) > \lambda(H_l) \tag{5.89}$$

This means $\lambda + \alpha$ can not be weight.

$$E_\alpha(v) = 0 \tag{5.90}$$

□

2- Construct irrep successive application of E_α, where $\alpha \in \Delta_-$ on v gives a sub-representation of r. If r is irreducible representation, it is obtained in this way.

证明 Define

$$W_k = Span\{E_{\alpha_1} \cdots E_{\alpha_k} v | \alpha_i \in \Delta_-\} \tag{5.91}$$

$$W = \oplus_k W_k \in V \tag{5.92}$$

We can show that W is invariant under all generators.$\cdots$

□

3- If r is an irrep, the highest weight vector is unique up to re-scaling .

证明 Suppose there are two linearly indenpendent highest weight vector, $v_1, v_2 \in V_\lambda$, Then v_1 generates irrep which does not contain v_2, So V is not irreducible, so $dim(V_\lambda) = 1$

□

Simple roots A positive (or negative) roots is called simple if it cannot be written as a sum of other two positive (negative) roots.

推论 5.1 (Statement about simple roots) (i) An irrep r can be obtained by successively applying lowering operators of simple roots E_α

(ii) The simple positive roots form a basis of $\mathcal{H}'$

– Dynkin labels and Cartan Matrix Choose a basis formed by positive simple roots $(\alpha_1 \cdots \alpha_r)$. For a weight w, construct:

$$a_i = \frac{2(w, \alpha_i)}{(\alpha_i, \alpha_i)} \tag{5.93}$$

The vector $(a_1 \cdots a_r)$ Carateries weight w. (Dynkin label)

Dynkin labels for positive simple roots:

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_j)} \tag{5.94}$$

This is called Cartan Matrix.

第六章 例子

6.1 $SU(2)$

Group $SU(2)$ SU(2) 群定义为

$$SU(2) = \left\{U \in End(\mathbb{C}^2) | det(U) = 1,\ U^\dagger U = \mathbb{I}\right\}. \tag{6.1}$$

Lie algebra $su(2)$ 对 $SU(2)$ group 在恒元附近做线形展开 $U = \mathbb{I} + T$，需要满足的群条件变为对 Lie algebra 的约束条件

$$det(\mathbb{I} + T) = 0 \Rightarrow tr(T) = 0, \quad (\mathbb{I} + T)^\dagger (\mathbb{I} + T) = \mathbb{I} \Rightarrow T^\dagger + T = 0. \tag{6.2}$$

用语言来表述就是 Traceless and Anti-Hermitian. 利用 Pauli-matrix 展开，其中 Pauli matrix 定义为

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.3}$$

利用 Pauli matrix 展开的 $su(2)$ 写为 (ϵ 是 levi-civita 符号)

$$su(2) = Span(\tau_i = -i\sigma_i/2)_\mathbb{R}, \quad [\tau_i, \tau_j] = \epsilon_{ij}{}^k \tau_k. \tag{6.4}$$

Lie algebra $su(2)_\mathbb{C}$ 将 $su(2)$ algebra 扩充

$$su(2)_\mathbb{C} = su(2)_\mathbb{R} + isu(2)_\mathbb{R} = sl(2)_C. \tag{6.5}$$

扩充后的 Algebra 实际上有 6 个基底。它实际上是 $sl(2)_\mathcal{R}$。

$$su(2)_\mathbb{C} = sl(2)_\mathbb{R} = \{T \in End(\mathbb{C}) | tr(T) = 0\}. \tag{6.6}$$

Representation of $su(2)_\mathbb{C}$ 此时的 Lie algebra 写为基底展开的形式可以写为

$$su(2)_\mathbb{C} = Span(\tau_i,\ i\tau_i)_\mathbb{R}. \tag{6.7}$$

如果有 $su(2)_\mathbb{C}$ 的李代数表示，记为 $T_i = r(\tau_i)$。定义新的基底为:

$$E_\pm = \frac{1}{2}(T_1 \pm iT_2), \quad H = iT_3, \quad r\left(su(2)\right)_\mathbb{C} = Span(H,\ E_\pm)_\mathbb{C}. \tag{6.8}$$

李代数表示新的基底满足对易关系

$$[H, H] = 0, \quad [H, E_\pm] = \pm E_\pm, \quad [E_+, E_-] = \frac{1}{2}H. \tag{6.9}$$

量子力学的结论是，对于表示 r_j，$j \in \mathbb{Z}/2$，表示空间的矢量记为 $|j, m\rangle$, $m = -j \cdots + j$。表示矩阵写为 $(H, T_\pm)$，对表示空间向量的作用满足

$$H|j, m\rangle = m|j, m\rangle, \quad T_\pm|j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}|j, m \pm 1\rangle. \tag{6.10}$$

虽然在推导 $su(2)$ 的表示时候将它推广到了复数域，但是利用基底变换公式 (6.8) 可以反解出可以反解出 $su(2)$ 表示的基底 — T_1, T_2, T_3 。可以验证他们也是满足 $su(2)$ 的对易关系的。

也就是说，按照量子力学的方法可以找到 $su(2)$ 的表示。

6.2 Lorentz Group

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{6.11}$$

定义 6.1 (Lorentz 变换) Lorentz 变换定义为:

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} \tag{6.12}$$

$$x'_{\mu} = \Lambda_{\mu}^{\ \nu} x_{\nu} \tag{6.13}$$

满足

$$x'^{\mu} x'_{\mu} = x^{\mu} x_{\mu} \tag{6.14}$$

也就是

$$\Lambda^{\mu}_{\ \nu} \Lambda_{\mu}^{\ \rho} = \delta^{\rho}_{\nu} \tag{6.15}$$

考虑到度规算符。

$$\Lambda^{\mu}_{\ \nu}\, g_{\mu\lambda}\, \Lambda^{\lambda}_{\ \sigma}\, g^{\sigma\rho} = \delta^{\rho}_{\nu} \tag{6.16}$$

总的来说，Lorentz 变换:

$$\left\{\Lambda^{\mu}_{\ \nu} |\ \Lambda^{\mu}_{\ \nu}\, g_{\mu\lambda}\, \Lambda^{\lambda}_{\ \sigma}\, g^{\sigma\rho} = \delta^{\rho}_{\nu}\right\} \tag{6.17}$$

或者写为

$$\left\{\Lambda^{\mu}_{\ \nu} |\ \Lambda^{\mu}_{\ \nu}\, g_{\mu\lambda}\, \Lambda^{\lambda}_{\ \sigma} = g_{\nu\sigma}\right\}. \tag{6.18}$$

□

引理 6.1 (Lorentz 变换矩阵的行列式)

$$\det \Lambda = \pm 1 \tag{6.19}$$

证明 由6.1:

$$\left\{\Lambda^{\mu}_{\ \nu}\, g_{\mu\lambda}\, \Lambda^{\lambda}_{\ \sigma}\, g^{\sigma\rho} = \delta^{\rho}_{\nu}\right. \tag{6.20}$$

对上式取行列式:

$$\begin{aligned} \det(\Lambda^{T}) \det(g_{\mu\lambda}) \det(\Lambda) \det(g^{\sigma\rho}) = \det(\mathbb{I}) = 1 \\ \det(\Lambda)^2 = 1 \det\Lambda \quad = \pm 1 \end{aligned} \tag{6.21}$$

□

Lorentz 群的参数 当 Lorentz 变换在恒等变换附近时，可以假设:

$$\Lambda^{\nu}{}_{\mu} = \delta^{\nu}{}_{\mu} + \Delta w^{\nu}{}_{\mu} \tag{6.22}$$

考虑 Lorentz 变换的约束条件:

$$\left\{\Lambda^{\mu}{}_{\nu}\Lambda_{\mu}^{\rho} = \delta_{\nu}^{\rho}\right. \tag{6.23}$$

于是:

$$\left(\delta_{\nu}^{\mu} + \Delta w^{\mu}{}_{\nu}\right)\left(\delta_{\mu}^{\rho} + \Delta w_{\mu}{}^{\rho}\right) = \delta_{\nu}^{\rho} \tag{6.24}$$

忽略二阶小量:

$$\Delta w^{\rho}{}_{\nu} + \Delta w_{\nu}{}^{\rho} = 0 \tag{6.25}$$

升指标后得到关系:

$$\Delta w^{\rho\nu} + \Delta w^{\nu\rho} = 0 \tag{6.26}$$

也就是说 Δw 矩阵中只有 $(4 \times 4 - 4)/2 = 6$ 个独立变量

x 方向运动 Lorentz 变换 此时

$$\Delta w^{10} = -\Delta w^{01} = -\Delta\beta \tag{6.27}$$

考虑 Covariant vector x 的变换。

$$\Delta w^{1}{}_{0} = -\Delta\beta \quad \Delta w^{0}{}_{1} = -\Delta\beta \tag{6.28}$$

也就是:

$$\begin{aligned}
(x')^0 &= x^0 - \Delta\beta x_1 \\
(x')^1 &= -\Delta\beta x^0 + x^1 \\
(x')^2 &= x^2 \\
(x')^3 &= x^3
\end{aligned} \tag{6.29}$$

之后要用 infinitesimal transformation 构造 finite transformation。考虑矩阵:

$$I_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad I_1^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (I_1)^3 = I_1 \tag{6.30}$$

于是这种微小的 Lorentz 变换写为:

$$\Lambda^{\mu}{}_{\nu} = \delta_{\nu}^{\mu} + \Delta\beta (I_1)^{\mu}{}_{\nu} \tag{6.31}$$

把 $\Delta\beta$ 写为 $\frac{w}{N}$, N 是一个比较大的数，所以它表示着微元 Lorentz 变换之后, 考虑变换:

$$\lim_{N\to+\infty}\left(\mathbb{I} + \frac{w}{N} I_1\right)^N = e^{wI_1} \tag{6.32}$$

下面是一些化简

$$\begin{aligned} e^{wI_1} &= \cosh(wI_1) + \sinh(wI_1) \\ &= \left[\left(\mathbb{I} + \frac{(wI_1)^2}{2!} + \frac{(wI_1)^4}{4!} + \cdots\right) + \left(wI_1 + \frac{(wI_1)^3}{3!}\cdots\right)\right] \\ &= \left[\left(\mathbb{I} + \frac{(wI_1)^2}{2!} + \frac{(wI_1)^2}{4!} + \cdots\right) + \left(wI_1 + \frac{(wI_1)}{3!}\cdots\right)\right] \\ &= \mathbb{I} - (I_1)^2 + \cosh(w)(I_1)^2 + \sinh(w)I_1 \end{aligned} \tag{6.33}$$

具体来说, 对于 Covariant vector，变换可以写为:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{6.34}$$

为了把它和传统的侠义相对论知识联系在一起。首先考虑 $x'^0 = 0$ 时, 原始坐标系中的坐标

$$\cosh w(x^1 - x^0 \tanh w) = 0 \quad \frac{x^1}{x^0} = \frac{x^1}{ct^1} = \frac{v_x}{c} = \tanh w = \beta \tag{6.35}$$

考虑到关系 $\cosh^2 w - \sinh^2 w = 1$。$\cosh w = \frac{\cosh w}{\sqrt{\cosh^2 w - \sinh^2 w}} = \frac{1}{\sqrt{1-\tanh^2 w}} = \frac{1}{\sqrt{1-\beta^2}}$ 那么:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ -\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{6.36}$$

绕 z 轴旋转 (坐标系绕 z 轴旋转) 此时:

$$\Delta w^{21} = -\Delta w^{12} = \Delta\varphi \quad others = 0 \tag{6.37}$$

考虑 Covariant vector x 的变换:

$$\Delta w^2{}_1 = -\Delta\varphi \quad \Delta w^1{}_2 = \Delta\varphi \tag{6.38}$$

也就是:

$$\begin{aligned} (x')^0 &= x^0 \\ (x')^1 &= x^1 + \Delta\varphi x^2 \\ (x')^2 &= -\Delta\varphi x^1 + x^2 \\ (x')^3 &= x^3 \end{aligned} \tag{6.39}$$

接下来，想用 infinitesmial Lorentz 变换构造 finite Lorentz 变换, 先定义矩阵 I_6

$$I_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (I_6)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (I_6)^3 = -I_6 \tag{6.40}$$

这样，这种微小 Lorentz 变换可以写为:

$$\Lambda = \mathbb{I} + \Delta\varphi I_6 \tag{6.41}$$

考虑:

$$\lim_{N\to+\infty} (\mathbb{I} + \frac{w}{N} I_6)^N = e^{wI_6} \tag{6.42}$$

$$\begin{aligned} e^{wI_6} &= \cosh(wI_6) + \sinh(wI_6) \\ &= \left(\mathbb{I} + \frac{(wI_6)^2}{2!} + \frac{(wI_6)^4}{4!} + \cdots\right) + \left(wI_6 + \frac{(wI_6)^3}{3!} + \cdots\right) \\ &= \mathbb{I} + (I_6)^2 + \left(-\mathbb{I} + \frac{w^2}{2!} - \frac{w^4}{4!} + \cdots\right)(I_6)^2 + \left(w - \frac{w^3}{3!} + \cdots\right)(I_6) \\ &= \mathbb{I} + (I_6)^2 - \cos(w)(I_6)^2 + \sin(w) I_6 \end{aligned} \tag{6.43}$$

具体来说，对于 Covariant vector, 它的变换是:

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos w & \sin w & 0 \\ 0 & -\sin w & \cos w & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{6.44}$$

相当于坐标系绕着 z 轴转动

6.3 Lorentz 群 LieAlgebra 表示

Lorentz 群的 Lie Algebra Lorentz 变换条件是

$$\Lambda^b{}_a \eta_{bc} \Lambda^c{}_d = \eta_{ad}. \tag{6.45}$$

忽略上下指标，简单的将他写为

$$\Lambda^T \eta \Lambda = \eta. \tag{6.46}$$

其中 η 是度归矩阵。将 Lorentz 变换矩阵在单位矩阵附近展开 $\Lambda = \mathbb{I} + T$。展开的线性项满足条件

$$\Lambda^T \eta \Lambda = \eta \Rightarrow \left(\mathbb{I} + T^T\right) \eta \left(\mathbb{I} + T\right) = \eta \Rightarrow T^T \eta + \eta T = 0 \Rightarrow T = -\eta T^T \eta. \tag{6.47}$$

于是 Lorentz Group 的 Lie algebra 是

$$\mathcal{L}(L) = \left\{T \in End(\mathbb{R}^4) | T = -\eta T^T \eta\right\}. \tag{6.48}$$

对于李代数的维度。注意到 T 的对角项是 0，$T_{ij} = -T_{ji}$，$T_{0i} = T_{i0}$。也就是 Antisymmetry in space-space entries, Symmetry in space-time entries。所以他的维度是 $1+2+3=6$。

$$dim\left(\mathcal{L}(L)\right)=6 \tag{6.49}$$

Lorentz 群的 Lie algebra 可以用两种基底展开

$$\mathcal{L}(L)=Span(\sigma_{\mu\nu})_{\mathbb{R}}=Span(\tilde{J}_i,\tilde{K}_i)_{\mathbb{R}}. \tag{6.50}$$

第一种基底不加证明的写为

$$(\sigma_{\mu\nu})^{\rho}{}_{\sigma}=\eta^{\rho}_{\mu}\eta_{\nu\sigma}-\eta_{\mu\sigma}\eta^{\rho}_{\nu}. \tag{6.51}$$

同样不加证明的发现，它满足对易关系

$$[\sigma_{\alpha\beta},\sigma_{\gamma\delta}]=\eta_{\alpha\delta}\sigma_{\beta\gamma}+\eta_{\alpha\gamma}\sigma_{\delta\beta}+\eta_{\beta\delta}\sigma_{\gamma\alpha}+\eta_{\beta\gamma}\sigma_{\alpha\delta}. \tag{6.52}$$

另一种基底写为

$$\tilde{J}_i=-\frac{1}{2}\epsilon_{ijk}\sigma_{jk},\quad \tilde{K}_i=-\sigma_{0i}. \tag{6.53}$$

具体的矩阵形式是

$$\tilde{J}_i=\begin{pmatrix}0&0\\0&T_i\end{pmatrix},\quad \tilde{K}_i|_{i=1}=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \tag{6.54}$$

其中的 T 是 SO(3) 群的生成元，具体的矩阵形式是 (被动观点)

$$T_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\quad T_2=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},\quad T_3=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}. \tag{6.55}$$

第二种 Lorentz Lie Algebra 生成元矩阵满足对易关系

$$[\tilde{J}_i,\tilde{J}_j]=-\epsilon_{ij}{}^{k}\tilde{J}_k,\quad [\tilde{K}_i,\tilde{K}_j]=\epsilon_{ij}{}^{k}\tilde{J}_k,\quad [\tilde{J}_i,\tilde{K}_j]=-\epsilon_{ij}{}^{k}\tilde{K}_k. \tag{6.56}$$

当然，如果采用主动的观点 (坐标系不变，给粒子一个 Boost)，相当于给所有的生成元矩阵都乘以-1。此时，对易关系是（一般采用下面这个）

$$[\tilde{J}_i,\tilde{J}_j]=\epsilon_{ij}{}^{k}\tilde{J}_k,\quad [\tilde{K}_i,\tilde{K}_j]=-\epsilon_{ij}{}^{k}\tilde{J}_k,\quad [\tilde{J}_i,\tilde{K}_j]=\epsilon_{ij}{}^{k}\tilde{K}_k. \tag{6.57}$$

其实有一个 bug 就是在量子力学 Sakurai 引出角动量算符作为转动的生成元算符时，将转动理解为主动转动（粒子在坐标系中转动）是比较自然的。但是在 Lorentz 变换时候，一般考虑的是有相对运动的坐标系对同一个粒子的坐标变换（它是被动的观点）。在考虑 Lorentz 群时，最好将主动和被动统一，于是变为了都是主动的观点。不过主动观点下的 Lorentz 变换不好理解，就先当作是一个数学变换好了。

Group homomorphism $SL(2,\mathbb{C})\to L$ $\quad SL(2,C)$ 群定义为

$$SL(2,\mathbb{C})=\left\{M\in End((C)^2)|det(M)=1\right\}. \tag{6.58}$$

Lorentz 群定义为

$$L = \left\{\Lambda \in GL(\mathbb{R}^4) | \forall X \in \mathbb{R}^4,\ (\Lambda X)^T \eta (\Lambda X) = X^T \eta X\right\}. \tag{6.59}$$

定义 4 维线性空间

$$\mathcal{S} = Span\left(\sigma_\mu = (\mathbb{I}, \sigma_i)\right) = \left\{S \in End(\mathbb{C}^2) | S = S^\dagger\right\}. \tag{6.60}$$

可以将这个四维线性空间和 4×1 矩阵一一对应起来，

$$\mathcal{T}:\ \mathbb{R}^4 \Leftrightarrow \mathcal{S}, \quad \mathcal{T}(X) = X^\mu \sigma_\mu = \begin{pmatrix} X^0 + X^3 & X^1 - iX^2 \\ X^1 + iX^2 & X^0 - X^3 \end{pmatrix}. \tag{6.61}$$

有一个比较好的性质是

$$X^T \eta X = -det\left(\mathcal{T}(X)\right). \tag{6.62}$$

现在考虑同态映射 $R_V:\ SL(2,\mathbb{C}) \to GL(\mathbb{R}^4)$

$$R_V(M)(X) = \mathcal{T}^{-1}\left(M\mathcal{T}(X)M^\dagger\right) \Leftrightarrow \sigma_\mu R_V(M)^\mu{}_\nu X^\nu = M(X^\mu \sigma_\mu)M^\dagger. \tag{6.63}$$

验证映射的像确实是 Lorentz group 中的元素

$$\begin{aligned}(R_V(M)(X))^T \eta\left(R_V(M)(X)\right) = -det\Big(\mathcal{T}\left(R_V(M)(X)\right)\Big) = -det\Big(M\mathcal{T}(X)M^\dagger\Big) = -det\Big(\mathcal{T}(X)\Big) \\ = X^T \eta X.\end{aligned} \tag{6.64}$$

参考 Lorentz Group 的定义 $R_V(M) \in L$。

发现 $Ker\{R_V\} = \{\pm\mathbb{I}_2\}$，并且像空间是是整个固有保时 Lorentz group，于是根据同态核定理

$$L_+^\uparrow = SL(2,\mathbb{C})/\mathbb{Z}_2. \tag{6.65}$$

Lie algebra Hom $SL(2,\mathbb{C})$ 的李代数生成元是 (SU(2) 部分有讲)

$$J_i = -i\frac{\sigma_i}{2}, \quad K_i = \frac{\sigma_i}{2}. \tag{6.66}$$

满足对易关系

$$[J_i, J_j] = \epsilon_{ij}{}^k J_k, \quad [K_i, K_j] = -\epsilon_{ij}{}^k J_k, \quad [J_i, K_j] = \epsilon_{ij}{}^k K_k. \tag{6.67}$$

这个对易关系和 Lorentz 群李代数对易关系一样，李代数同构写为

$$J_i \mapsto \tilde{J}_i, \quad K_i \mapsto \tilde{K}_i \tag{6.68}$$

Lorentz Lie alg Rep 也相当于考虑 $sl(2,\mathbb{C})$ 的表示。改变 $sl(2,\mathbb{C})$ 的基底为

$$J_i^\pm = \frac{1}{2}\left(J_i \pm iK_i\right). \tag{6.69}$$

变换后的基底满足对易关系

$$[J_i^\pm, J_j^\pm] = \epsilon_{ij}{}^k J_k^\pm, \quad [J_i^+, J_j^-] = 0 \Rightarrow sl(2,\mathbb{C}) \sim su(2) \oplus su(2). \tag{6.70}$$

所以它的表示可以用 $(j_+, j_-) \in \mathbb{Z}/2 \times \mathbb{Z}/2$ 来表示。

在改变 $sl(2,\mathbb{C})$ 的基底时，引入了虚数，这一步非常奇怪，因为 Lie algebra 的数域本来

应该是实数。这个事情应该这样理解。如果找到了两个合适的 $su(2)$ 的表示, 用基底变换公式 (6.69) 反写出 $sl(2)$ 的表示的基底。可以证明这些基底是满足 $sl(2)$ 的对易关系的。

接下来，将会讨论 $sl(2,\mathbb{C})$ 的表示，利用 $sl(2,\mathbb{C})$ 的表示以及 Exp 函数得到 $SL(2,C)$ 的群表示。

将两个 $su(2)$ 表示分别记为 r_+ 以及 r_-。于是将两个 $su(2)$ 表示直和起来的办法是

$$r_{(j_+,j_-)}(J_i^+) = r_+(J_i^+) \times \mathbb{I}_{2j_-+1}, \quad r_{(j_+,j_-)}(J_i^-) = \mathbb{I}_{2j_i^-+1} \times r_-(J_i^-). \tag{6.71}$$

上面提到的利用两个 $su(2)$ 表示可以找到 $sl(2,\mathbb{C})$ 表示的构造方法是（为什么不是直和呢? –因为要找不可约表示）

$$\begin{aligned} r_{(j_+,j_-)}(J_i) &= r_+(J_i^+) \times \mathbb{I}_{2j_-+1} + \mathbb{I}_{2j_++1} \times r_-(J_i^-), \\ r_{(j_+,j_-)}(K_i) &= -i\left(r_+(J_i^+) \times \mathbb{I}_{2j_-+1} - \mathbb{I}_{2j_++1} \times r_-(J_i^-)\right). \end{aligned} \tag{6.72}$$

可以通过直接计算验证确实满足 $sl(2,\mathbb{C})$ 对易关系。得到的表示矩阵是 $(2j_++1)(2j_-+1)$ 空间中的复数矩阵。表示空间是 $V_{(j_+,j_-)} = \mathbb{C}^{(2j_++1)(2j_-+1)}$，表示是 $r_{(j_+,j_-)}:\ sl(2,\mathbb{C}) \to End\left(V_{(j_+,j_-)}\right)$。

利用 Lie algebra 表示得到群表示，群表示的映射关系是

$$M = \exp\left(t^i J_i + s^i K_i\right) \mapsto R(M) = \exp\left(t^i r(J_i) + s^i r(K_i)\right). \tag{6.73}$$

注意，如果群 G 有表示 R_1 与 R_2。群的表示等价于生成元的表示 $r_1(T)$, $r_2(T)$ 。群的表示的直积表示 $R_{1\otimes 2}(g) = R_1(g) \otimes R_2(g)$ 等价于生成元的直积表示 $r_{1\otimes 2}(T) = r_1(T) \otimes \mathbb{I}_2 + \mathbb{I}_1 \otimes r_2(T)$ 。群表示的直和表示 $R_{1\oplus 2}(g) = R_1(g) \oplus R_2(g)$ 等价于生成元的直积表示 $r_{1\oplus 2}(T) = r_1(T) \oplus r_2(T)$ 。

(0,0) Scalar 此时两个李代数 $su(2)$ 的表示都是一维表示。回顾量子力学，中 $su(2)$ 的 $j=0$ 表示相当于总角动量为 0。此时，只有一个态 $|0,0\rangle$，并且满足 $J_x|0,0\rangle = J_y|0,0\rangle = J_z|0,0\rangle = 0$。

在这个问题中，$sl(2,\mathbb{C})$ 的 $(0,0)$ 表示是

$$r_+(J_i^+) = 0, \quad r_-(J_i^-) = 0. \tag{6.74}$$

也就是 0 表示

$$r(J_i) = r(K_i) = 0. \tag{6.75}$$

Lie 代数的 0 表示等价于 Lie 群的恒等表示

$$M = \exp\left(t^i J_i + s^i K_i\right) \mapsto R(M) = 1. \tag{6.76}$$

(1/2,0) Left hand Weyl spinor 对于第一个 $su(2)$ 表示 $j=1/2$ ，第二个表示 $j=0$。

$$r_+(J_i^+) = -i\sigma_i/2, \quad r_-(J_i^-) = 0. \tag{6.77}$$

通过 $sl(2,\mathbb{C})$ 表示的构造方法 (6.72)

$$r_{(1/2,0)}(J_i) = -i\sigma_i/2, \quad r_{(1/2,0)}(K_i) = -\sigma_i/2. \tag{6.78}$$

通过 Lie algebra 的表示构造群的表示

$$R_L(M) = R_{(1/2,0)}(M) = \exp\left[\frac{1}{2}(-s^i - it^i)\sigma_i\right]. \tag{6.79}$$

叫做 Left-handed weyl spinor representation, 表示空间 $V_{(1/2,0)}$ 中的矢量叫做 Left-handed Weyl spinors.

(0,1/2) Right hand Weyl spinor 对于第一个 $su(2)$ 表示 $j=0$，第二个表示 $j=1/2$。

$$r_{+}(J_i^{+})=0,\quad r_{-}(J_i^{-})=-i\sigma_i/2. \tag{6.80}$$

通过 $sl(2,\mathbb{C})$ 表示的构造方法 (6.72)

$$r_{(0,1/2)}(J_i)=-i\sigma_i/2,\quad r_{(0,1/2)}(K_i)=\sigma_i/2. \tag{6.81}$$

通过 Lie algebra 的表示构造群的表示

$$R_R(M)=R_{(0,1/2)}(M)=\exp\left[\frac{1}{2}(s^i-it^i)\sigma_i\right]. \tag{6.82}$$

叫做 Right-handed weyl spinor representation, 表示空间 $V_{(0,1/2)}$ 中的矢量叫做 Right-handed Weyl spinors.

它就是 $SL(2,\mathbb{C})$ 的自身表示。

(1/2,1/2) Vector 不加证明的将 Lie 代数表示写为

$$r(J_i)=\tilde{J}_i,\quad r(K_i)=\tilde{K}_i. \tag{6.83}$$

相应的群表示是 (Lorentz group)

$$R_V(M)=\exp\left[t^i\tilde{J}_i+s^i\tilde{K}_i\right]. \tag{6.84}$$

(1/2,0) ⊕ (0,1/2) Dirac spinor 李代数表示写为

$$r(J_i)=\begin{pmatrix} r_{(1/2,0)}(J_i) & 0 \\ 0 & r_{(0,1/2)}(J_i)\end{pmatrix},\quad r(K_i)=\begin{pmatrix} r_{(1/2,0)}(K_i) & 0 \\ 0 & r_{(0,1/2)}(K_i)\end{pmatrix}. \tag{6.85}$$

相应的群表示写为

$$R_D(M)=\begin{pmatrix} R_L(M) & 0 \\ 0 & R_R(M)\end{pmatrix}. \tag{6.86}$$

Parity 考虑 Lorentz 变换中的空间反射变换 Parity $\mathcal{P}=diag(1,-1,-1,-1)$。它作用在 Lorentz 变换的生成元上有性质

$$\mathcal{P}\tilde{J}_i\mathcal{P}=\tilde{J}_i,\quad \mathcal{P}\tilde{K}_i\mathcal{P}=-\tilde{K}_i. \tag{6.87}$$

物理上的解释: 先空间反射，再转动，再空间反射等价于直接转动；先空间反射，再 Boost，再空间反射等价反向 Boost。

赝矢量定义为空间反射下不变的矢量，转动生成元满足这个条件。矢量定义为空间反射下反向的矢量，Boost 生成元满足这个条件。

对于 Lorentz Group 的表示，由群表示乘法规律不变性

$$R(\mathcal{P})r(\tilde{J}_i)R(\mathcal{P})=r(\tilde{J}_i),\quad R(\mathcal{P})r(\tilde{K}_i)R(\mathcal{P})=-r(\tilde{K}_i). \tag{6.88}$$

Left-handed spinor rep 和 Right-handed Spinor rep 的 $r(J)$ 相同，$r(K)$ 相差负号，所以

$$R_L(\mathcal{P})R_L(\Lambda)R_L(\mathcal{P}) = R_R(\Lambda), \quad R_R(\mathcal{P})R_L(\Lambda)R_R(\mathcal{P}) = R_L(\Lambda) \tag{6.89}$$

也就是 Parity exchange Left and Right handed Weyl Spinor rep.

更进一步，Lorentz 变换中的空间反射变换 Parity 作用在 Lorentz 变换的生成元上有性质:

$$\mathcal{P}\tilde{J}_i\mathcal{P} = \tilde{J}_i, \quad \mathcal{P}\tilde{K}_i\mathcal{P} = -\tilde{K}_i, \Rightarrow P\tilde{J}_i^{\pm}P = J_i^{\mp}. \tag{6.90}$$

在表示中

$$R(\mathcal{P})r(\tilde{J}_i^{\pm})R(\mathcal{P}) = r(\tilde{J}_i^{\mp}). \tag{6.91}$$

注意到

$$\begin{cases} r_{(j_+,j_-)}(J_i) = r_+(J_i^+) \times \mathbb{I}_{2j_-+1} + \mathbb{I}_{2j_++1} \times r_-(J_i^-), \\ r_{((j_+,j_-))}(K_i) = -i\left(r_+(J_i^+) \times \mathbb{I}_{2j_-+1} - \mathbb{I}_{2j_++1} \times r_-(J_i^-)\right). \end{cases} \tag{6.92}$$

变换表示的 1, 2 空间, 表示也可以写为 (加了个下指标 2)

$$\begin{cases} r_{2(j_+,j_-)}(J_i) = \mathbb{I}_{2j_-+1} \times r_+(J_i^+) + r_-(J_i^-) \times \mathbb{I}_{2j_++1}, \\ r_{2(j_+,j_-)}(K_i) = -i\left(\mathbb{I}_{2j_-+1} \times r_+(J_i^+) - r_-(J_i^-) \times r_+(J_i^+)\right). \end{cases} \tag{6.93}$$

我觉得直接将 $sl(2)$ 写为 $sl(2) = su(2) \oplus su(2)$ 是有问题的，因为如果直接这么写，上面提到的第二种写法应该满足 $r_{2((j_-,j_+))}(K_i) = r_{1(j_+,j_-)}(K_i)$. (同时交换两个空间以及两个空间中承载的表示，得到的结果本应该是相同的，但是这里差了一个负号)。

回到 Parity 对表示的作用，按照 Parity 对本来的 Lie algebra 的表示

$$\begin{aligned} r_{(j_+,j_-)}(J_i) &= r_+(J_i^+) \times \mathbb{I}_{2j_-+1} + \mathbb{I}_{2j_++1} \times r_-(J_i^-) \Rightarrow r_+(J_i^+) \times \mathbb{I}_{2j_-+1} + \mathbb{I}_{2j_++1} \times r_-(J_i^-) \\ r_{(j_+,j_-)}(K_i) &= -i\left(r_+(J_i^+) \times \mathbb{I}_{2j_-+1} - \mathbb{I}_{2j_++1} \times r_-(J_i^-)\right) \Rightarrow -i\left(\mathbb{I}_{2j_++1} \times r_-(J_i^-) - r_+(J_i^+) \times \mathbb{I}_{2j_-+1}\right) \end{aligned} \tag{6.94}$$

考虑到刚才提到的交换表示空间的操作，发现

$$\begin{aligned} R(\mathcal{P})r_{(j_+,j_-)}(J_i)R(\mathcal{P}) &= r_{(j_+,j_-)}(J_i) = r_{2(j_-,j_+)}(J_i), \\ R(\mathcal{P})r_{(j_+,j_-)}(K_i)R(\mathcal{P}) &= -r_{(j_+,j_-)}(K_i) = r_{2(j_-,j_+)}(K_i) \end{aligned} \tag{6.95}$$

因此，可以理解为 Parity 交换了 $(j_+, j_-) \to (j_-, j_+)$.

Dual of spinor representations 注意到 Left-handed Spinor rep 和 Right-handed Spinor rep 分别是

$$\begin{cases} R_L(M) = R_{(1/2,0)}(M) = \exp\left[\frac{1}{2}(-s^i - it^i)\sigma_i\right], \\ R_R(M) = R_{(0,1/2)}(M) = \exp\left[\frac{1}{2}(s^i - it^i)\sigma_i\right]. \end{cases} \tag{6.96}$$

Left-handed rep 的对偶 dual 是

$$\left(R_L(M)^{-1}\right)^{\dagger} = \exp\left[\frac{1}{2}(s^i - it^i)\sigma_i\right] = R_R(M). \tag{6.97}$$

Complex conjugate of spinor rep Pauli 矩阵性质

$$\left\{\sigma_2\sigma_i\sigma_2^{-1} = -\sigma_i^*.\right. \tag{6.98}$$

对 Left-handed Spinor rep 取 Complex Conjugate

$$R_L(M)^* = \exp\left[\frac{1}{2}(-s^i + it^i)\sigma_i^*\right] = \sigma_2 \exp\left[\frac{1}{2}(s^i - it^i)\sigma_i\right]\sigma_2^{-1} = \sigma_2 R_R(M)\sigma_2. \tag{6.99}$$

等价于 ($\sigma_2^* = -\sigma_2$)

$$R_R(M)^* = \sigma_2 R_L(M)\sigma_2. \tag{6.100}$$

Spinor rep and Lorentz trans 可以用 spinor rep 构造出一个 Lorentz trans。在求 $SL(2,\mathbb{C})$ 群和 Lorentz 群的同态关系时，得到了式 (6.63)

$$R_V(M)(X) = \mathcal{T}^{-1}\left(M\mathcal{T}(X)M^\dagger\right) \Leftrightarrow \sigma_\mu R_V(M)^\mu{}_\nu X^\nu = M(X^\mu\sigma_\mu)M^\dagger. \tag{6.101}$$

注意到 (0, 1/2) 是 $SL(2,\mathbb{C})$ 的自身表示，可以直接将 M 写为 $R_R(M)$。将对应的 Lorentz 群元写为 $\Lambda^\mu{}_\nu = R_V(M)^\mu{}_\nu$. 忽略 X , 上式写为

$$\begin{aligned}
\sigma_\mu\Lambda^\mu{}_\nu &= R_R(M)\sigma_\nu R_R(M)^\dagger, \\
\sigma_\mu\Lambda^\mu{}_\nu\Lambda_\alpha{}^\nu &= R_R(M)\Lambda_\alpha{}^\nu\sigma_\nu R_R(M)^\dagger, \\
R_R(M)^{-1}\sigma_\alpha\left(R_R(M)^\dagger\right)^{-1} &= \Lambda_\alpha{}^\nu\sigma_\nu.
\end{aligned} \tag{6.102}$$

考虑到 Dual of spinor rep 的性质

$$R_L(M)^\dagger\sigma_\alpha R_L(M) = \Lambda_\alpha{}^\nu\sigma_\nu. \tag{6.103}$$

如果对上式取 Complex conjugate, 利用前面提到的 Complex conjugate of spinor rep 的性质

$$\begin{aligned}
(R_L(M)^*)^\dagger\,\sigma_\mu^*\,(R_L(M))^* &= \Lambda_\mu{}^\nu\sigma_\nu^*, \\
(\sigma_2 R_R(M)\sigma_2)^\dagger\,\sigma_\mu^*\sigma_2 R_R(M)\sigma_2 &= \Lambda_\mu{}^\nu\sigma_\nu^*, \\
\sigma_2 R_R(M)^\dagger\sigma_2\sigma_\mu^*\sigma_2 R_R(M)\sigma_2 &= \Lambda_\mu{}^\nu\sigma_\nu^*, \\
R_R(M)^\dagger\sigma_2\sigma_\mu^*\sigma_2 R_R(M) &= \sigma_2\Lambda_\mu{}^\nu\sigma_\nu^*\sigma_2, \\
R_R(M)^\dagger\bar{\sigma}_\mu R_R(M) &= \Lambda_\mu{}^\nu\bar{\sigma}_\nu.
\end{aligned} \tag{6.104}$$

其中，定义

$$\bar{\sigma}_\mu = \sigma_2\sigma_\mu^*\sigma_2 = (\mathbb{I}, -\sigma_i). \tag{6.105}$$

Lorentz trans of Left and Right spinor 在经典场论中，场 Lorentz 不变的意思是在下述变换下，$S = S'$

$$\begin{cases}
S = \int d^4x\mathcal{L}\left(\phi_a(x), \partial^\mu\phi_a(x)\right) \\
S' = \int d^4x'\mathcal{L}\left(\phi'_a(x'), \partial^\mu\phi'_a\circ(x')\right)
\end{cases} \tag{6.106}$$

其中，场满足变换条件

$$\phi'_a(x') = R(\Lambda)_a{}^b\phi_b(x), \Rightarrow \partial'^\mu\phi'_a(x') = \Lambda^\mu{}_\nu R(\Lambda)_a{}^b\partial^\nu\phi_b(x). \tag{6.107}$$

其中的 R 是 Lorentz 群的表示。

在 Lorentz 群的 Spin rep 下，场是表示空间中的向量，对于 Left, Right 表示，场分别写为 χ_L，χ_R。场的变化写为

$$\chi_{L,R} \mapsto R_{L,R}(M)\chi_{L,R}. \tag{6.108}$$

Conjugation of Weyl Spinonr 定义 Left-hand Weyl Spinor 的 Conjugation 为

$$\chi_L^c = \sigma_2\chi_L^*. \tag{6.109}$$

Left-handed Spinor 的 Conjugation 按照 Right-handed Weyl Spinor representation 变换，具体来说

$$\chi_L^c \mapsto \sigma_2\left(R_L(M)\chi_L\right)^* = \sigma_2 R_L(M)^*\chi_L^*. \tag{6.110}$$

注意到 Complex Conjugate of spinor rep 的性质

$$\begin{cases} R_L(M)^* = \sigma_2 R_R(M)\sigma_2. \end{cases} \tag{6.111}$$

于是 Left-handed Spinor 的 Conjugation 的 Lorentz 变换

$$\chi_L^c \mapsto R_R(M)\sigma_2\chi_L^* = R_R(M)\chi_L^c. \tag{6.112}$$

类似的，定义 Right-handed Weyl Spinor 的 Conjugation 为

$$\chi_R^c = \sigma_2\chi_R^*. \tag{6.113}$$

Right-hand Spinor Conjugation 是 Left-handed Spinor.

$$\begin{aligned} \chi_R^c \mapsto \sigma_2 R_R(M)^*\chi_R^* =& \sigma_2\left(\sigma_2 R_L(M)\sigma_2\right)\chi_R^*, \\ =& R_L(M)\sigma_2\chi_R^* = R_L(M)\chi_R^c. \end{aligned} \tag{6.114}$$

Weyl mass term 考虑两个 Left-handed Weyl spinor rep 表示空间中的矢量 χ_L，ψ_L. 由量子力学角动量耦合性质，两个 Left-handed Weyl spinor rep 表示张量积为表示 $(1/2,0)\otimes(1/2,0) = (0,0)\oplus(1,0)$。也就是张量表示空间中一定有一个一维不变子空间。

构造一个 Lorentz 不变的量为。

$$\chi_L^T\sigma_2\psi_L. \tag{6.115}$$

可以验证它确实是不变的。

$$\chi_L^T\sigma_2\psi_L \mapsto \left(R_L(M)\chi_L\right)^T\sigma_2 R_L(M)\psi_L = \chi_L^T R_L(M)^T\sigma_2 R_L(M)\psi_L. \tag{6.116}$$

考虑到 Dual of spinor rep 的性质以及 Complex Conjugate of spinor rep 的性质

$$\begin{cases} \left(R_L(M)^{-1}\right)^\dagger = R_R(M) \Rightarrow & R_L(M)^T = \left(R_R(M)^*\right)^{-1}, \\ R_R(M)^* = \sigma_2 R_L(M)\sigma_2 \Rightarrow & R_L(M)^T = \sigma_2 R_L(M)^{-1}\sigma_2. \end{cases} \tag{6.117}$$

于是

$$\chi_L^T\sigma_2\psi_L \mapsto \chi_L^T R_L(M)^T\sigma_2 R_L(M)\psi_L = \chi_L^T\sigma_2\psi_L \tag{6.118}$$

如果考虑到前面提到的 Conjugate of Left-hand Spinor, 这个 Lorentz 不变量是

$$\left(\chi_L^c\right)^\dagger\psi_L. \tag{6.119}$$

同理，张量表示 $(0,1/2)\otimes(0,1/2) = (0,0)\oplus(0,1)$ 空间中也有一维不变子空间。于是可以

构造一个 Lorentz 不变的量

$$\chi_R^T \sigma_2 \psi_R. \tag{6.120}$$

如果考虑到前面提到的 Conjugate of Right-hand Spinor, 这个 Lorentz 不变量是

$$(\chi_R^c)^\dagger \psi_R. \tag{6.121}$$

Lorentz vector 利用 Spinor rep 构造出的 Lorentz trans

$$\begin{cases} R_R(M)^\dagger \bar{\sigma}_\alpha R_R(M) = \Lambda_\alpha^{\ \nu} \bar{\sigma}_\nu, \\ R_L(M)^\dagger \sigma_\mu R_L(M) = \Lambda_\mu^{\ \nu} \sigma_\nu. \end{cases} \tag{6.122}$$

发现

$$\chi_R^\dagger \bar{\sigma}_\mu \psi_R, \quad \chi_L^\dagger \sigma_\mu \psi_L, \tag{6.123}$$

是按照 Lorentz transformation 变化的。

另一种参数方法 从 Lorentz 变换矩阵性质出发

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0. \tag{6.124}$$

矩阵具体写为

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ -\omega_{01} & 0 & \omega_{12} & \omega_{13} \\ -\omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ -\omega_{03} & -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}. \tag{6.125}$$

相当于 6 个参数，生成 Lorentz 变换

$$\omega^\mu_{\ \nu} = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{01} & 0 & -\omega_{12} & -\omega_{13} \\ \omega_{02} & \omega_{12} & 0 & -\omega_{23} \\ \omega_{03} & \omega_{13} & \omega_{23} & 0 \end{pmatrix}. \tag{6.126}$$

对比原来的生成元算符和群参数引起的微小 Lorentz 变化

$$\tilde{J}_i = \begin{pmatrix} 0 & 0 \\ 0 & T_i \end{pmatrix}, \quad \tilde{K}_i|_{i=1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.127}$$

其中的 T 是 SO(3) 群的生成元，具体的矩阵形式是 (主动观点)

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.128}$$

群参数和群元的关系

$$\omega^\mu_{\ \nu} = t^j \tilde{J}_i + s^i \tilde{K}_i. \tag{6.129}$$

对比发现，群参数的关系

$$t^1 = \omega_{23},\ t^2 = -\omega_{13},\ t^3 = \omega_{12},\ s^i = \omega_{0i}. \tag{6.130}$$

群元写为

$$\begin{aligned}\omega^\mu_{\ \nu} =& \omega_{23}\tilde{J}_1 - \omega_{13}\tilde{J}_2 + \omega_{12}\tilde{J}_3 + \sum_i \omega_{0i}\tilde{K}_i, \\ =& \frac{1}{2}\epsilon^{ijk}\omega_{ij}\tilde{J}_k + \sum_i \frac{1}{2}\left(\omega_{0i} - \omega_{i0}\right)\tilde{K}_i.\end{aligned} \tag{6.131}$$

用下指标的 ω 当作群参数，变化元写为

$$\Lambda = \mathbb{I} + \omega_{\mu\nu}\tilde{J}^{\mu\nu}. \tag{6.132}$$

其中

$$\tilde{J}^{ij} = \frac{1}{2}\epsilon^{ijk}\tilde{J}_k,\quad \tilde{J}^{0i} = \frac{1}{2}\tilde{K}_i,\quad \tilde{J}^{i0} = -\frac{1}{2}\tilde{K}_i. \tag{6.133}$$

对于 $(1/2,0)$ 表示，

$$\begin{aligned}r(\tilde{J})^{ij} &= \frac{1}{2}\epsilon^{ijk}r(\tilde{J})_k = \frac{1}{2}\epsilon^{ijk}(-i)\sigma_k/2 = -\frac{i}{4}\epsilon^{ijk}\sigma_k, \\ r(\tilde{J})^{0i} &= \frac{1}{2}r(\tilde{K})_i = \frac{1}{2}(-\sigma_i/2) = -\frac{1}{4}\sigma_i, \\ r(\tilde{J})^{i0} &= -\frac{1}{2}r(\tilde{K})_i = -\frac{1}{2}(-\sigma_i/2) = \frac{1}{4}\sigma_i.\end{aligned} \tag{6.134}$$

对于 $(0,1/2)$ 表示，

$$\begin{aligned}r(\tilde{J})^{ij} &= \frac{1}{2}\epsilon^{ijk}r(\tilde{J})_k = \frac{1}{2}\epsilon^{ijk}(-i)\sigma_k/2 = -\frac{i}{4}\epsilon^{ijk}\sigma_k, \\ r(\tilde{J})^{0i} &= \frac{1}{2}r(\tilde{K})_i = \frac{1}{2}(\sigma_i/2) = \frac{1}{4}\sigma_i, \\ r(\tilde{J})^{i0} &= -\frac{1}{2}r(\tilde{K})_i = -\frac{1}{2}(\sigma_i/2) = -\frac{1}{4}\sigma_i.\end{aligned} \tag{6.135}$$

Left/Right Spinor rep Generator 为了和物理里面的约定相符合

$$\begin{aligned}R_L(M) &= \mathbb{I} - i\frac{\omega_{\mu\nu}}{2}r_L(\tilde{J})^{\mu\nu}, \\ R_R(M) &= \mathbb{I} - i\frac{\omega_{\mu\nu}}{2}r_R(\tilde{J})^{\mu\nu}, \\ r_L(\tilde{J})^{ij} &= \frac{1}{2}\epsilon^{ijk}\sigma_k, \\ r_L(\tilde{J})^{0i} &= -\frac{i}{2}\sigma_i, r_L(\tilde{J})^{i0} = \frac{i}{2}\sigma_i, \\ r_R(\tilde{J})^{ij} &= \frac{1}{2}\epsilon^{ijk}\sigma_k, \\ r_R(\tilde{J})^{0i} &= \frac{i}{2}\sigma_i, r_R(\tilde{J})^{i0} = -\frac{i}{2}\sigma_i.\end{aligned} \tag{6.136}$$

注意到 Pauli 矩阵有性质

$$[\sigma_a, \sigma_b] = 2i\epsilon^{abc}\sigma_c. \tag{6.137}$$

可以将生成元算符写为

$$\begin{aligned} r_L(\tilde{J})^{ij} &= -\frac{i}{4}\left[\sigma_i, \sigma_j\right], \\ r_L(\tilde{J})^{0i} &= -\frac{i}{2}\sigma_i, r_L(\tilde{J})^{i0} = \frac{i}{2}\sigma_i, \\ r_R(\tilde{J})^{ij} &= -\frac{i}{4}\left[\sigma_i, \sigma_j\right], \\ r_R(\tilde{J})^{0i} &= \frac{i}{2}\sigma_i, r_R(\tilde{J})^{i0} = -\frac{i}{2}\sigma_i. \end{aligned} \tag{6.138}$$

对于 Dirac Spinor, 它的 minimal 变换写为

$$R_D = \mathbb{I} - \frac{i}{2}\omega_{\mu\nu} r_D(J)^{\mu\nu}. \tag{6.139}$$

生成元算符应当是 Left-spinor rep 和 Right-spinor rep 生成元的直和形式

$$r_D(J)^{ij} = \begin{pmatrix} -\frac{i}{4}\left[\sigma_i, \sigma_j\right] & 0 \\ 0 & -\frac{i}{4}\left[\sigma_i, \sigma_j\right] \end{pmatrix}. \tag{6.140}$$

$$r_D(J)^{0i} = \begin{pmatrix} -\frac{i}{2}\sigma_i & 0 \\ 0 & \frac{i}{2}\sigma_i \end{pmatrix}. \tag{6.141}$$

定义

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}. \tag{6.142}$$

发现

$$\left[\gamma^i, \gamma^j\right] = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = -\begin{pmatrix} \left[\sigma_i, \sigma_j\right] & 0 \\ 0 & \left[\sigma_i, \sigma_j\right] \end{pmatrix}. \tag{6.143}$$

$$\left[\gamma^0, \gamma^i\right] = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} = (-2)\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}. \tag{6.144}$$

于是可以理所当然的将 Dirac Rep 生成元矩阵写为

$$r_D(J)^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \tag{6.145}$$

再讨论下 Lorentz 变换，在这种参数方法下，Lorentz 变化可以写为

$$\begin{aligned} \Lambda^\alpha_{\ \beta} &= \delta^\alpha_{\ \beta} - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha_{\ \beta}, \\ (\mathcal{J}^{\mu\nu})^\alpha_{\ \beta} &= i(g^{\mu\alpha}\delta^\nu_{\ \beta} - g^{\nu\alpha}\delta^\mu_{\ \beta}). \end{aligned} \tag{6.146}$$

直接相乘可以验证

$$\Lambda^\alpha_{\ \beta} = \delta^\alpha_{\ \beta} + w^\alpha_{\ \beta}. \tag{6.147}$$

Levi-civita symbol invariant　在 Left-handed spinor rep 中，有两个指标的 Levi-civita 符号 ϵ_{ab} 是 Lorentz 不变的。(没有证明...)

$$\epsilon_{12} = -1,\ \epsilon_{21} = 1,\ \epsilon^{12} = 1,\ \epsilon^{21} = -1. \tag{6.148}$$

上指标的 Levi-Civita 符号是下指标的逆（理解为它的定义）

$$\epsilon_{ab}\epsilon^{bc} = \delta_a^{\ c}, \ \epsilon^{ab}\epsilon_{bc} = \delta^a_{\ c}. \tag{6.149}$$

可以用 Levi-civita 符号升降指标，比如之前都是下指标描述表示空间中的向量，现在定义上指标的向量。

$$\psi^a \equiv \epsilon^{ab}\psi_b. \tag{6.150}$$

用 Levi-civita 符号升降指标是自洽的因为

$$\epsilon_{ba}\psi^a = \epsilon_{ba}\epsilon^{ac}\psi_c = \psi_b. \tag{6.151}$$

升指标后的向量在 Lorentz 变换下的表现应该是 (这里为了确定升指标的 $R_L(M)$)

$$\psi^a \mapsto R_L(M)^a_{\ c}\psi^c \Rightarrow \epsilon_{ba}R_L(M)^a_{\ c}\psi^c = R_L(M)_b^{\ d}\psi_d. \text{ (下指标按照原来的方式变换)} \tag{6.152}$$

也就是

$$\begin{aligned} \epsilon^{eb}\epsilon_{ba}R_L(M)^a_{\ c}\psi^c &= \epsilon^{eb}R_L(M)_b^{\ d}\psi_d, \\ R_L(M)^e_{\ c}\psi^c &= \epsilon^{eb}R_L(M)_b^{\ d}\epsilon_{dc}\psi^c, \\ R_L(M)^e_{\ c} &\equiv \epsilon^{eb}R_L(M)_b^{\ d}\epsilon_{dc}. \end{aligned} \tag{6.153}$$

也就是，可以利用 Levi-civita 符号定义上指标表示向量的变换矩阵。并且恰好和升降指标的性质是自洽的。

对于 Right-handed spinor rep, 也有 Lorentz 不变的张量 Levi-civita 张量。

$$\epsilon_{\dot{1}\dot{2}} = -1, \ \epsilon_{\dot{2}\dot{1}} = 1, \ \epsilon^{\dot{1}\dot{2}} = 1, \ \epsilon^{\dot{2}\dot{1}} = -1. \tag{6.154}$$

指标带了点是为了和 Left-handed representation 区分开。类似于 Left-hand rep 中的情况，Right-hand 情况中也可以升降指标。

Conjugate of Left-spinor Levi-Civita 符号可以将 Conjugate of Left-spinor 写的更简洁。首先定义 Left-spinor。

$$\psi_a = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{6.155}$$

它的 Conjugate 用矩阵表示

$$\psi^{\dagger a} \equiv \left(\psi_1^*, \psi_2^*\right). \tag{6.156}$$

用 Levi-Civita 符号降指标

$$\psi_a^\dagger = \epsilon_{ab}\psi^{\dagger b} = \begin{pmatrix} -\psi_2^* \\ \psi_1^* \end{pmatrix}. \tag{6.157}$$

下面说这样定义的好处，之前定义的 Conjugation of Left-spinor 会按照 Right-spinor rep 变化。按照之前 Conjugation 的定义

$$\psi^c = \sigma_2\psi^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = i\begin{pmatrix} -\psi_2^* \\ \psi_1^* \end{pmatrix}. \tag{6.158}$$

和利用了升降指标算符定义的普通 Hermitian Conjugate 是相同的。也就是说 $\psi_a^\dagger$ 确实会按照右

手 Spinor 变化。