Linear chain.

Classical treatment

$$\overset{\circ}{1}\ \overset{\circ}{2}\ \overset{\circ}{3}\ \cdots\ \overset{\circ}{N}$$

$$L = T - V = \sum_n \frac{1}{2} m \dot{q}_n^2 - \sum_n \frac{k}{2}(q_{n+1}-q_n)^2$$

$$\ddot{q}_n(t) = K(q_{n+1}+q_{n-1}-2q_n)$$

▽ Normal coordinates

$$q_n(t) = \sum_k a_k(t)\cdot u_n^k \qquad u_n^k = \frac{1}{\sqrt{N}}\exp(ikan)$$

▽ 周期边界条件：确定 $k=\frac{2\pi}{aN}l$，$l\in(-\frac{N}{2},\frac{N}{2}]$ $l\in Z$

$$u_{n+N}^k = \frac{1}{\sqrt{N}}\cdot\exp(ika(n+N)) = u_n^k = \frac{1}{\sqrt{N}}\exp(ikan)$$

$ikaN = i\cdot 2\pi\cdot l$ （l 是一个整数）

$\downarrow$

$k = \frac{2\pi}{aN}\cdot l$

$\downarrow$

ka 应当是在 $[-\pi,\pi]$ 中的一个值.（$u_{n+1}^k = e^{ika}u_n^k$）

$\downarrow$

$\frac{2\pi}{N}\cdot l \in (-\pi,\pi]$

$\downarrow$

$l\in(-\frac{N}{2},\frac{N}{2}]$

- N 为偶数：$l = \underbrace{-\frac{N}{2}+1 \cdots \frac{N}{2}}_{\text{共N个值}}$ $\quad -N/2,\ -N/2+1,\ \cdots\ 0\ \cdots\ N/2$ → k的取值在0+和0-不对称！
  共N个值 ← $\underbrace{N/2}_{k正} + \underbrace{(N/2-1)}_{k为负} + \underbrace{1}_{k=0} = N$
- N为奇数：$l = \underbrace{-\frac{N}{2}+\frac{1}{2},\cdots,\frac{N}{2}-\frac{1}{2}}_{\text{共N个值}}$ $\quad -N/2,\ -N/2+1\ \cdots\cdots\ N/2$ → k的取值在0+和0-对称
  $[-N/2,N/2]$ — 共N+1个半整数点，有N个整数点. $\Rightarrow \frac{N+1}{2}\times 2$.

▽ 由于 $q_n(t)$ 是实数 $a_{-k}(t) = a_k^*(t) \Leftarrow q_n^* = \sum_k a_k^* u_n^{k*} = \sum_k a_k^* u_n^{-k} = q_n$

Q：u_n^{-k} 不一定 $=u_n^{k*}$，因为当 N为偶，$l=N/2$，不存在 $u_n^{l=-N/2}$

▽ u_n^k 满足正交性关系

$$\sum_n u_n^{k*}u_n^{k'} = \delta_{kk'} \qquad \text{for } u_n^k = \frac{1}{\sqrt{N}}\exp(ikan)$$

proof：

$$\sum_{n=1}^{N}\frac{1}{N}\exp(ian(k-k')) = \frac{1}{N}\cdot\exp(ia(k-k'))\frac{\overbrace{\exp(ia(k-k')N)-1}^{\exp(i2\pi(l-l'))-1\ =0\ \text{for } l\neq l'}}{\exp(ia(k-k'))-1} = 0 \quad (\text{for } k\neq k')$$

$$= 1 \quad \text{for } k=k'$$

▽ u_n^k 满足完备性关系.

$$\sum_k u_n^{k*}u_{n'}^{k} = \delta_{n,n'} \qquad \text{for } u_n^k = \frac{1}{\sqrt{N}}\cdot\exp(ikan)$$

proof：

$$\frac{1}{N}\sum_k \exp(ika(n'-n)) = \frac{1}{N}\exp(ik_{min}a(n'-n))\frac{\exp(ia(n'-n)\Delta k\cdot N)-1}{\exp(i\Delta k a(n'-n))-1} \overset{\Delta k=\frac{2\pi}{aN}}{=} \frac{1}{N}\exp(ik_{min}a(n'-n))\frac{\exp(i2\pi(n'-n))-1}{\exp(i\frac{2\pi}{N}(n'-n))-1} = 0 \quad n'\neq n$$

$$= 1 \quad n'=n$$

为什么叫作完备性：

$$f_n = \sum_{n'} f_{n'}\delta_{n'n} = \sum_{n'} f_{n'}\sum_k u_{n'}^{k*}u_n^k = \sum_{n'k} f_{n'}u_{n'}^{k*}u_n^k = \sum_k\left(\sum_{n'} f_{n'}u_{n'}^{k*}\right)u_n^k$$

↑ 说明 f_n 可表为 u_n^k 的线性叠加！

▽ 利用 Normal coordinate，求 $a_k(t)$ 满足的动力学方程，最终得到色散关系！

$$q_n(t) = \sum_k a_k(t)u_k^n$$

$$m\sum_{k'}\ddot{a}_{k'}(t)u_n^{k'} = \sum_{k'} K(a_{k'}(t)u_{n+1}^{k'} + a_{k'}(t)u_{n-1}^{k'} - 2a_{k'}(t)u_n^{k'})$$

$$\sum_n u_n^{k*} m\sum_{k'}\ddot{a}_{k'}(t)u_n^{k'} = \sum_{k'}\sum_n K(a_{k'}(t)u_n^{k*}u_{n+1}^{k'} + a_{k'}(t)u_n^{k*}u_{n-1}^{k'} - 2a_{k'}(t)u_n^{k*}u_n^{k'})$$

$$\ddot{a}_k(t) = \sum_{n,k'}\frac{K}{m}a_{k'}(t)u_n^{k*}(u_{n+1}^{k'} + u_{n-1}^{k'} - 2u_n^{k'})$$

利用：

$$u_{n+1}^{k'} = u_n^{k'}\exp(ik'a)$$

$$u_{n-1}^{k'} = u_n^{k'}\exp(-ik'a)$$

$$\ddot{a}_k(t) = \sum_{nk'} \frac{k}{m} a_{k'}(t) \cdot u_n^{k*} u_n^{k'} \cdot (2\cos(k'a)-2)$$
$$= \frac{k}{m} a_k(t) (2\cos(ka)-2)$$
$$= -\frac{k}{m}(2-2\cos(ka)) a_k(t)$$

色散关系： $\omega_k^2 = \frac{k}{m}(2-2\cos(ka)) = \frac{4k}{m}\sin^2(\frac{ka}{2})$ dispersion-relation!

▽ 求解动力学方程.

$$a_k(t) = b_k \exp(-i\omega_k t) + b_{-k}^* \exp(i\omega_k t) \leftarrow \text{考虑到了 } a_{-k}(t) = a_k^*(t)$$
$$g_n(t) = \sum_k a_k(t) u_n^k$$
$$= \sum_k (b_k \exp(-i\omega_k t) + b_{-k}^* \exp(i\omega_k t)) u_n^k$$
$$= \sum_k (b_k e^{-i\omega_k t} u_n^k + b_k^* e^{i\omega_k t} u_n^{k*})$$
$$= \sum_k \frac{1}{\sqrt{N}} (b_k e^{-i(\omega_k t - kan)} + b_k^* e^{i(\omega_k t - kan)})$$
$$P_n(t) = m\dot{g}_n = \frac{\partial L}{\partial \dot{g}_n}$$
$$= m\sum_k (-i\omega_k)(b_k e^{-i\omega_k t} u_n^k - b_k^* e^{i\omega_k t} u_n^{k*})$$

▽通过初态 $g_n(0)$ 与 $\dot{g}_n(0)$ 来确定任意时刻的 $g_n(t)$ 与 $\dot{g}_n(t)$

$$\begin{cases} g_{n'}(0) = \sum_{k'} (b_{k'} + b_{-k'}^*) u_{n'}^{k'} \\ \dot{g}_{n'}(0) = \sum_{k'} (b_{k'} - b_{-k'}^*) u_{n'}^{k'} (-i\omega_{k'}) \end{cases} \quad (\text{利用 } g_n(t) = \sum_k (b_k \cdot e^{-i\omega_k t} + b_{-k}^* \exp(i\omega_k t)) u_n^k$$

$$\begin{cases} \sum_{n'} u_{n'}^{k*} g_{n'}(0) = \sum_{k'} (b_{k'} + b_{-k'}^*) \delta_{k,k'} = b_k + b_{-k}^* \\ \sum_{n'} u_{n'}^{k*} \dot{g}_{n'}(0) = \sum_{k'} (b_{k'} - b_{-k'}^*) \delta_{k,k'} (-i\omega_{k'}) = (b_k - b_{-k}^*)(-i\omega_k) \end{cases}$$

$$\begin{cases} b_k = \frac{1}{2} \sum_{n'} u_{n'}^{k*} (g_{n'}(0) + \frac{i}{\omega_k} \dot{g}_{n'}(0)) \\ b_k^* = \frac{1}{2} \sum_{n'} u_{n'}^{k*} (g_{n'}(0) - \frac{i}{\omega_k} \dot{g}_{n'}(0)) \end{cases}$$

$$g_n(t) = \sum_{k,n'} \frac{1}{2} [u_{n'}^{k*} (g_{n'}(0) + \frac{i}{\omega_k} \dot{g}_{n'}(0)) \cdot e^{-i\omega_k t} + (u_{n'}^k)^* (g_{n'}(0) - \frac{i}{\omega_k} \dot{g}_{n'}(0)) e^{i\omega_k t}] u_n^k$$
$$= \sum_{k,n'} \frac{1}{2} \frac{1}{N} [(g_{n'}(0) + \frac{i}{\omega_k} \dot{g}_{n'}(0)) e^{-i\omega_k t} e^{ik(n-n')a} + (g_{n'}(0) - \frac{i}{\omega_k} \dot{g}_{n'}(0)) e^{i\omega_k t} e^{-ik(n-n')a})$$
$$\sum_k \rightarrow \sum_{-k}$$
$$= \frac{1}{N} \sum_{k,n'} [g_{n'}(0)\cos(ka(n-n') - \omega_k t) - \frac{1}{\omega_k} \dot{g}_n(0) \sin(ka(n-n') - \omega_k t)]$$

Q：N为偶时 k可=$\frac{\pi}{a}$ 但不可等于$-\frac{\pi}{a}$

↓后面都有这个问题.

▽证明 $H = T + V = \sum_k m\omega_k^2 (b_k b_k^* + b_k^* b_k) = \sum_k 2m\omega_k^2 b_k^* b_k$

由于 $P_n(t) = \sum_k m(-i\omega_k)(b_k e^{-i\omega_k t} u_n^k - b_k^* e^{i\omega_k t} u_n^{k*})$

$$T = \sum_n \frac{1}{2} m \dot{g}_n^2(t)$$
$$= \sum_n \frac{1}{2} m \sum_k (-i\omega_k)(b_k e^{-i\omega_k t} u_n^k - b_k^* e^{i\omega_k t} u_n^{k*})$$
$$\sum_{k'} (-i\omega_{k'})(b_{k'} e^{-i\omega_{k'} t} u_n^{k'} - b_{k'}^* e^{i\omega_{k'} t} u_n^{k'*})$$

$$= \frac{m}{2} \sum_{n,k,k'} (-i\omega_k)(-i\omega_{k'}) \cdot (b_{k'} b_k e^{-i(\omega_k + \omega_{k'})t} u_n^k u_n^{k'} - b_{k'} b_k^* e^{-i(\omega_{k'} - \omega_k)t} u_n^{k'*} u_n^k$$
$$- b_{k'}^* b_k e^{i(\omega_{k'} - \omega_k)t} u_n^{k'} u_n^{k*} + b_{k'}^* b_k^* e^{i(\omega_{k'} + \omega_k)t} u_n^{k'*} u_n^{k*})$$

由于：

$$\sum_n u_n^{k'*} u_n^k = \delta_{k',k} \qquad \sum_n u_n^{k'} u_n^k = \sum_n u_n^{k'} u_n^{-k*} = \delta_{k',-k} \qquad \sum_n u_n^{k'*} u_n^{k*} = \sum_n u_n^{k'*} u_n^{-k} = \delta_{k',-k}$$

$$T = \frac{m}{2} \sum_{n,k,k'} (-i\omega_k)(-i\omega_{k'}) (b_{k'} b_k e^{-i(\omega_k + \omega_{k'})t} \delta_{k',-k} - b_{k'} b_k^* e^{-i(\omega_{k'} - \omega_k)t} \delta_{k',k}$$
$$- b_{k'}^* b_k e^{i(\omega_{k'} - \omega_k)t} \delta_{k',k} + b_{k'}^* b_k^* e^{i(\omega_{k'} + \omega_k)t} \delta_{k',-k})$$

$$= \frac{m}{2}\sum_{n,k}(-\omega_k^2)\left(b_{-k} b_k e^{-i2\omega_k t} - b_k b_k^* - b_k^* b_k + b_{-k}^* b_k e^{i2\omega_k t} \right)$$

$$V = \frac{K}{2}\sum_n (q_{n+1} - q_n)^2$$

$$= \frac{K}{2}\sum_n \left[\sum_k \left(b_k e^{-i\omega_k t} u_{n+1}^k + b_k^* e^{i\omega_k t} u_{n+1}^{k*}\right) - \sum_{k'}\left(b_{k'} e^{-i\omega_{k'} t} u_{n+1}^{k'} + b_{k'}^* e^{-i\omega_{k'} t} u_{n+1}^{k'*}\right)\right]^2$$

$$= \frac{K}{2}\sum_n \left[\sum_k b_k e^{-i\omega_k t} u_n^k (e^{ika}-1) + \sum_{k'} b_{k'}^* e^{i\omega_{k'} t} u_n^{k'*} (e^{-ik'a}-1) \right]$$

$$= \frac{K}{2}\sum_{n,k,k'} \Big[b_{k'} b_{k'} e^{-i(\omega_k+\omega_{k'})t} (e^{ik'a}-1)(e^{ika}-1) u_n^{k'} u_n^k + b_{k'} b_k^* e^{-i(\omega_{k'}-\omega_k)t} (e^{ik'a}-1)(e^{-ika}-1) u_n^{k'} u_n^{k*}$$
$$+ b_{k'}^* b_k e^{i(\omega_{k'}-\omega_k)t} (e^{-ik'a}-1)(e^{ika}-1) u_n^{k'*} u_n^k + b_{k'}^* b_k^* e^{i(\omega_{k'}+\omega_k)t} (e^{-ik'a}-1)(e^{-ika}-1) u_n^{k'*} u_n^{k*} \Big]$$

由于 $(e^{-ika}-1)(e^{ika}-1) = 4\sin^2(\frac{ka}{2})$, $\omega_k = 2\sqrt{\frac{K}{m}}\,|\sin(\frac{ka}{2})| = \sqrt{2\frac{K}{m}(1-\cos ka)}$

$$V = \frac{m}{2}\sum_k \frac{4K}{m}\sin^2(\frac{ka}{2}) \left[b_{-k} b_k e^{-2i\omega_k t} + b_k b_k^* + b_k^* b_k + b_{-k}^* b_k^* e^{2i\omega_k t} \right]$$

$$H = T + V = \sum_k m\omega_k^2 (b_k b_k^* + b_k^* b_k) = \sum_k 2m\omega_k^2 b_k^* b_k$$

▽ 证明 $\{b_k, b_{k'}^*\}_{P.B} = \frac{-i}{2m\omega_k}\delta_{k,k'}$

由于: $b_k = \frac{1}{2}\sum_n u_n^{k*}\left(q_n(0) + \frac{i}{m\omega_k}\dot{q}_n(0)\right)$; $q_n(t) = \sum_k \left(b_k e^{-i\omega_k t} u_n^k + b_k^* e^{i\omega_k t} u_n^{k*}\right)$

$$b_k(t) = \frac{1}{2}\sum_n u_n^{k*} e^{i\omega_k t}\left(q_n(t) + \frac{i}{m\omega_k}\dot{q}_n(t)\right) \longrightarrow q_n(t_0+t) = \sum_k \left(b_k(t_0) e^{-i\omega_k(t+t_0)} u_n^k + b_k^*(t_0) e^{i\omega_k(t+t_0)} u_n^{k*}\right)$$

$$b_k^* = \frac{1}{2}\sum_n u_n^k e^{-i\omega_k t}\left(q_n(t) - \frac{i}{m\omega_k}\dot{q}_n(t)\right)$$

$$\{b_k, b_{k'}^*\} = \sum_n \frac{\partial b_k}{\partial q_n}\frac{\partial b_{k'}^*}{\partial p_n} - \frac{\partial b_{k'}^*}{\partial q_n}\frac{\partial b_k}{\partial p_n}$$

$$= \sum_n \frac{1}{2}u_n^{k*} e^{i\omega_k t}\,\frac{1}{2}u_n^{k'} e^{-i\omega_{k'} t}\left(-\frac{i}{m\omega_{k'}}\right) - \sum_n \frac{1}{2}u_n^{k'} e^{-i\omega_{k'} t}\,\frac{1}{2}u_n^{k*} e^{i\omega_k t}\left(\frac{i}{m\omega_k}\right)$$

$$= -\frac{i}{2m\omega_k}\delta_{k,k'}$$

同理, $\{b_k, b_{k'}\} = \{b_k^*, b_{k'}^*\} = 0$.

Quantum treatment

Hamiltonian (All in Heisenberg Picture)

$$H = \sum_{n=1}^{N} \frac{1}{2m}\hat{p}_n^2(t) + \frac{k}{2}(\hat{q}_{n+1}(t) - \hat{q}_n(t))^2$$

Commutation relation

$$[\hat{q}_n(t), \hat{q}_m(t)] = 0 \quad [\hat{q}_n(t), \hat{p}_{n'}(t)] = i\hbar\delta_{n,n'}$$

Analogy:

$$\left.\begin{aligned}\hat{q}_n(t) &= \sum_k (\hat{b}_k(t) u_n^k + \hat{b}_k^\dagger u_n^{k*}) \\ \hat{p}_n(t) &= \sum_k (-im\omega_k)(\hat{b}_k(t) u_n^k - \hat{b}_k^\dagger u_n^{k*})\end{aligned}\right\} \longrightarrow \text{gurantees} \quad \begin{aligned}\hat{q}_n^\dagger &= \hat{q}_n \\ \hat{p}_n^\dagger &= \hat{p}_n\end{aligned}$$

$$\hat{b}_k(t) = \frac{1}{2}\sum_n u_n^{k*}\left(\hat{q}_n(t) + \frac{i}{\omega_k m}\hat{p}_n(t)\right)$$

- Heisenberg equation of motion for $\hat{b}_k(t)$

$$i\hbar\frac{d\,b_k(t)}{dt} = [\hat{b}_k(t), H] \longleftarrow \text{It (Greiner, (1.87)) says only need to use } \hat{b}_k(t) = \sim q \sim p \text{, and commutation relation between } \hat{q}_n \text{ and } \hat{p}_n \text{ but, how?)}$$

$$= \hbar\omega_k b_k(t)$$

Proof:
$$[\hat{b}_k, H] = \left[\frac{1}{2}\sum_n u_n^{k*}\left(\hat{q}_n + \frac{i}{\omega_k m}\hat{p}_n\right), \sum_n \frac{1}{2m}\hat{p}_n^2 + \frac{k}{2}(\hat{q}_{n+1} - \hat{q}_n)^2\right]$$
$$= \frac{1}{4m}\sum_{n,n'} u_n^{k*}[\hat{q}_n, \hat{p}_{n'}^2] + \frac{ik}{4\omega_k m}\sum_{n,n'} u_n^{k*}[\hat{p}_n, (\hat{q}_{n'+1} - \hat{q}_{n'})^2]$$
$$= \sum_{n,n'} \frac{1}{4m} u_n^{k*}[\hat{q}_n, \hat{p}_{n'}^2] + \frac{ik}{4\omega_k m} u_n^{k*}\left([\hat{p}_n, \hat{q}_{n'+1}^2] + [\hat{p}_n, \hat{q}_{n'}^2] - [\hat{p}_n, \hat{q}_{n'+1}\hat{q}_{n'}] - [\hat{p}_n, \hat{q}_{n'}\hat{q}_{n'+1}]\right)$$

using:
$$[\hat{q}_n, \hat{p}_{n'}^2] = \hat{q}_n\hat{p}_{n'}\hat{p}_{n'} + \hat{p}_{n'}\hat{q}_n\hat{p}_{n'} - \hat{p}_{n'}\hat{q}_n\hat{p}_{n'} - \hat{p}_{n'}\hat{p}_{n'}\hat{q}_n$$
$$= [\hat{q}_n, \hat{p}_{n'}]\hat{p}_{n'} + \hat{p}_{n'}[\hat{q}_n, \hat{p}_{n'}]$$
$$= i2\hbar\hat{p}_n\delta_{n,n'}$$
$$[\hat{p}_n, \hat{q}_{n'}^2] = -i2\hbar\hat{q}_n\delta_{n,n'}$$
$$[\hat{p}_n, \hat{q}_{n'+1}^2] = -i2\hbar\hat{q}_n\delta_{n-1,n'}$$
$$[\hat{p}_n, \hat{q}_{n'+1}\hat{q}_{n'}] = \hat{p}_n\hat{q}_{n'+1}\hat{q}_{n'} + \hat{q}_{n'+1}\hat{p}_n\hat{q}_{n'} - \hat{q}_{n'+1}\hat{p}_n\hat{q}_{n'} - \hat{q}_{n'+1}\hat{q}_{n'}\hat{p}_n$$
$$= -i\hbar\hat{q}_{n-1}\delta_{n-1,n'} - i\hbar\hat{q}_{n+1}\delta_{n,n'} = [\hat{p}_n, \hat{q}_{n'}\hat{q}_{n'+1}]$$

$$[\hat{b}_k, H] = \sum_{n,n'}\frac{1}{4m}u_n^k\, i2\hbar\hat{p}_n\delta_{n,n'} + \frac{ik}{4\omega_k m}u_n^{k*}\left(-i2\hbar\hat{q}_n\delta_{n-1,n'} - i2\hbar\hat{q}_n\delta_{n,n'}\right.$$
$$\left. + i2\hbar\hat{q}_{n-1}\delta_{n-1,n'} + i2\hbar\hat{q}_{n+1}\delta_{n,n'}\right)$$
$$= \sum_n \frac{i\hbar}{2m}u_n^{k*}\hat{p}_n + \frac{\hbar k}{2\omega_k m}u_n^{k*}(\hat{q}_n - \hat{q}_{n-1} + \hat{q}_n - \hat{q}_{n+1})$$
$$= \sum_n \frac{i\hbar}{2m}u_n^{k*}\hat{p}_n + \frac{\hbar k}{2\omega_k m}u_n^{k*}\underbrace{(2\hat{q}_n - \hat{q}_n e^{-ika} - \hat{q}_n e^{-ika})}_{= 2\hat{q}_n(1-\cos(ka)) = 4\sin^2(\frac{ka}{2})\hat{q}_n \cdot (\frac{m}{k}) = \omega_k^2\frac{m}{k}\hat{q}_n}$$
$$= \sum_n \hbar\omega_k \frac{1}{2} u_n^{k*}\left(\hat{q}_n + \frac{i}{m\omega_k}\hat{p}_n\right) = \hbar\omega_k\cdot\hat{b}_k$$

- Commutation relation for $\hat{b}_k(t)$

using

$$\left|\begin{aligned}\hat{b}_k(t) &= \frac{1}{2}\sum_n u_n^{k*}\left(\hat{q}_n(t) + \frac{i}{\omega_k m}\hat{p}_n(t)\right) \\ \hat{b}_k^\dagger(t) &= \frac{1}{2}\sum_n u_n^k\left(\hat{q}_n(t) - \frac{i}{\omega_k m}\hat{p}_n(t)\right)\end{aligned}\right.$$

insert:
$$[\hat{b}_k, \hat{b}_{k'}^\dagger] = \left[\frac{1}{2}\sum_n u_n^{k*}\left(\hat{q}_n(t) + \frac{i}{\omega_k m}\hat{p}_n(t)\right), \frac{1}{2}\sum_{n'} u_{n'}^{k'}\left(\hat{q}_{n'}(t) - \frac{i}{\omega_k m}\hat{p}_{n'}(t)\right)\right]$$
$$= \left[\frac{1}{2}\sum_n u_n^{k*}\hat{q}_n(t), -\frac{1}{2}\sum_{n'} u_{n'}^{k'}\frac{i}{\omega_k m}\hat{p}_{n'}(t)\right] + \left[\frac{1}{2}\sum_n u_n^{k*}\frac{i}{\omega_k m}\hat{p}_n(t), \frac{1}{2}\sum_{n'} u_{n'}^{k'}\hat{q}_{n'}(t)\right]$$
$$= -\frac{1}{4}\sum_n u_n^{k*}u_n^{k'}\frac{i}{\omega_k m}(i\hbar) - \frac{1}{4}\sum_n u_n^{k*}u_n^{k'*}\frac{i}{\omega_k m}(i\hbar)$$
$$= \frac{1}{2}\frac{\hbar}{\omega_k m}\delta_{k,k'}$$

$$[\hat{b}_k, \hat{b}_{k'}] = [\hat{b}_k^\dagger, \hat{b}_{k'}^\dagger] = 0$$

▿ introduce new dimensionless operator $\hat{C}_k$.

$$\hat{C}_k = \sqrt{\frac{2mW_k}{\hbar}}\ \hat{b}_k$$

$$\hat{q}_n(t) = \sum_k \sqrt{\frac{\hbar}{2mW_k}} \left(\hat{C}_k u_n^k + \hat{C}_k^\dagger u_n^{k*} \right)$$

$$[\hat{C}_k, \hat{C}_{k'}^\dagger] = \delta_{k,k'}$$

$$[\hat{C}_k, \hat{C}_{k'}] = [\hat{C}_k^\dagger, \hat{C}_{k'}^\dagger] = 0$$

$$\hat{H} = \sum_k mW_k^2 \left(\hat{b}_k \hat{b}_k^\dagger + \hat{b}_k^\dagger \hat{b}_k \right) = \sum_k \frac{\hbar w}{2} \left(C_k C_k^\dagger + C_k^\dagger C_k \right) = \sum_k \hbar w_k \left(\hat{C}_k^\dagger \hat{C}_k + \frac{1}{2} \right)$$

▿ Ground state:

$$\hat{C}_k |0\rangle = 0 \quad \forall k.$$

▿ Multi phonon state:

$$|n\rangle \equiv |n_1, n_2, \cdots\rangle = \prod_k |n_k\rangle = \prod_k \frac{1}{\sqrt{n_k!}} (\hat{C}_k^\dagger)^{n_k} |0_k\rangle$$

▿ Energy eigenvalue:

$$E_n = \sum_k \hbar w_k \left(n_k + \frac{1}{2} \right)$$

Quantum Linear chain Subjected to External Forces.

Hamiltonian

$$\hat{H}_0 = \sum_n \frac{\hat{P}_n^2}{2m} + \frac{K}{2}(\hat{q}_{n+1} - \hat{q}_n)^2$$

$$\hat{V}_1 = -\sum_n \mathcal{F}_n \hat{q}_n$$

$$\hat{H} = \hat{H}_0 + \hat{V}_1$$

▿ Define fourier transformation of force

$$\hat{V}_1 = -\sum_n \mathcal{F}_n \hat{q}_n$$

$$= -\sum_{n,k} \mathcal{F}_n \sqrt{\frac{\hbar}{2m\omega_k}} (\hat{C}_k u_n^k + \hat{C}_k^\dagger u_n^{k*})$$

$$= -\sum_k (F_k \hat{C}_k + F_k^* \hat{C}_k^\dagger)$$

$$\begin{cases} F_k = \sum_n \mathcal{F}_n \sqrt{\frac{\hbar}{2m\omega_k}} u_n^k \\ F_k^* = \sum_n \mathcal{F}_n \sqrt{\frac{\hbar}{2m\omega_k}} u_n^{k*} \end{cases}$$ (fourier transformation of force)

$$\hat{H} = \sum_k \hbar\omega_k C_k^\dagger C_k - F_k \hat{C}_k - F_k^* \hat{C}_k^\dagger$$

▿ New creation / annihilation operator

$$\hat{d}_k = \hat{C}_k - \alpha_k \qquad \hat{d}_k^\dagger = \hat{C}_k^\dagger - \alpha_k^* \qquad [d_k, d_{k'}^\dagger] = \delta_{k,k'}$$

$$\hat{H} = \sum_k \hbar\omega_k (\hat{d}_k^\dagger + \alpha_k^*)(\hat{d}_k + \alpha_k) - F_k(\hat{d}_k + \alpha_k) - F_k^*(\hat{d}_k^\dagger + \alpha_k^*)$$

$$= \sum_k \hbar\omega_k \hat{d}_k^\dagger \hat{d}_k + \hat{d}_k(\hbar\omega_k \alpha_k^* - F_k) + \hat{d}_k^\dagger(\hbar\omega_k \alpha_k - F_k^*) + \hbar\omega_k \alpha_k^* \alpha_k - F_k\alpha_k - F_k^*\alpha_k^*$$

$$\alpha_k = \frac{1}{\hbar\omega} F_k^*$$

$$\hat{H} = \sum_k \hbar\omega_k \hat{d}_k^\dagger \hat{d}_k - \frac{1}{\hbar\omega} F_k^* F_k$$

$$= \sum_k \hbar\omega_k \hat{d}_k^\dagger \hat{d}_k - \hbar\omega \alpha_k^* \alpha_k$$

▿ Eigenstate / Ground state

ground state

$$\hat{d}_k |0, \alpha\rangle = 0 \qquad \forall k$$

Eigen state

$$|n, \alpha\rangle = \prod_k \frac{1}{\sqrt{n_k!}} (C_k^\dagger)^{n_k} |0, \alpha\rangle$$

$$= \cdots \otimes \frac{1}{\sqrt{n_k!}} (C_k^\dagger)^{n_k} |0\rangle \otimes \cdots$$

Hilbert space $\mathcal{H} = V_1 \otimes V_2 \cdots \otimes V_N$; $V = \mathrm{span}(|0\rangle, |1\rangle, |2\rangle, \cdots)$

▿ Shift operator $\hat{S}(\alpha)$, unitary,

satisfy:

$$\hat{S}(\alpha)\hat{C}_k\hat{S}(\alpha)^\dagger = \hat{C}_k + \alpha_k \xrightarrow{\text{invert}} \hat{C}_k - \alpha_k = \hat{S}^\dagger(\alpha)\hat{C}_k\hat{S}(\alpha) = \hat{d}_k$$

$$\hat{S}(\alpha)\hat{C}_k^\dagger\hat{S}(\alpha)^\dagger = \hat{C}_k^\dagger + \alpha_k^* \xrightarrow{\text{invert}} \hat{C}_k^\dagger - \alpha_k^* = \hat{S}^\dagger(\alpha)\hat{C}_k^\dagger\hat{S}(\alpha) = \hat{d}_k^\dagger$$

Hamiltonian: $\hat{H}|\Psi\rangle = E|\Psi\rangle$

$(\hat{S}\hat{H}\hat{S}^{\dagger})\hat{S}|\Psi\rangle = E\,\hat{S}|\Psi\rangle$

shifted Hamiltonian $\hat{H}' = \hat{S}\hat{H}\hat{S}^{\dagger} = \hat{S}\left(\sum_k \hbar\omega_k(d_k^{\dagger}d_k + \frac{1}{2}) + \Delta E\right)\hat{S}^{\dagger}$ $\quad \hat{d}_k = \hat{C}_k - \alpha_k$

$= \sum_k \hbar\omega_k(C_k^{\dagger}C_k + \frac{1}{2}) + \Delta E$

Eigenstate of $\hat{H}'$

$|n',\alpha\rangle = \prod_k \frac{1}{\sqrt{n_k!}}(\hat{C}_k^{\dagger})^{n_k}|0',\alpha\rangle = |n'\rangle$ ← $\hat{H}_0$'s eigenstate!

Groundstate: (Ground state $|0\rangle$ after transition is ground state of H')

$|0',\alpha\rangle = \hat{S}|0,\alpha\rangle \longrightarrow \hat{C}_k|0',\alpha\rangle = \hat{C}_k\hat{S}|0,\alpha\rangle = \hat{S}(S^{\dagger}\hat{C}_k S)|0,\alpha\rangle = \hat{S}\,\hat{d}_k|0,\alpha\rangle = 0$

▽ Relation between Eigen state

$S^{\dagger}|n',\alpha\rangle = \prod_k S^{\dagger}\frac{1}{\sqrt{n_k}}(\hat{C}_k^{\dagger})^{n_k}|0',\alpha\rangle$

$= \prod_k \frac{1}{\sqrt{n_k!}}(\hat{S}^{\dagger}\hat{C}_k^{\dagger}\hat{S})^{n_k}\hat{S}^{\dagger}|0',\alpha\rangle$

$= \prod_k \frac{1}{\sqrt{n_k!}}(d_k^{\dagger})^{n_k}\hat{S}^{\dagger}|0',\alpha\rangle$

$= |n,\alpha\rangle$

▽ Explicit representation of Shift operator.

suppose:

$\hat{S}(\alpha) = e^{\hat{\Lambda}(\alpha)}$

unitary requirement

$\hat{S}(\alpha)^{\dagger} = e^{\hat{\Lambda}(\alpha)^{\dagger}} = \hat{S}(\alpha)^{-1} = e^{-\hat{\Lambda}(\alpha)}$

$\Downarrow$

$\Lambda^{\dagger}(\alpha) = -\hat{\Lambda}(\alpha)$

demand: $\hat{S}\hat{C}_k\hat{S}^{\dagger} = \hat{C}_k + \alpha_k$ $\Downarrow$

$e^{\hat{\Lambda}}\hat{C}_k e^{-\hat{\Lambda}} = \hat{C}_k + \alpha_k$

Noticed $e^{\hat{\Lambda}}\hat{C}_k e^{-\hat{\Lambda}} = (1 + \hat{\Lambda} + \frac{1}{2!}\hat{\Lambda}^2 + \frac{1}{3!}\hat{\Lambda}^3)\hat{C}_k(1 - \hat{\Lambda} + \frac{1}{2!}\hat{\Lambda}^2 - \frac{1}{3!}\hat{\Lambda}^3 \ldots)$

$= \hat{C}_k + [\hat{\Lambda},\hat{C}_k] + \frac{1}{2!}(\hat{\Lambda}^2\hat{C}_k + \hat{C}_k\hat{\Lambda}^2 - 2\hat{\Lambda}\hat{C}_k\hat{\Lambda}) + \cdots$

$= \hat{C}_k + [\hat{\Lambda},\hat{C}_k] + \frac{1}{2!}[\hat{\Lambda},[\hat{\Lambda},\hat{C}_k]] + \cdots$

$= \hat{C}_k + \alpha_k$

Solution:

$\hat{\Lambda} = -\sum_k \alpha_k\hat{C}_k^{\dagger}$

Require $-\hat{\Lambda} = \Lambda^{\dagger}$

$\hat{\Lambda} = \sum_k -\alpha_k\hat{C}_k^{\dagger} + \alpha_k^{*}\hat{C}_k$

In all $\hat{S} = e^{-\sum_k(\alpha_k\hat{C}_k^{\dagger} - \alpha_k^{*}\hat{C}_k)}$

▽ Using $\hat{H}_0$'s eigenstate to represent $\hat{H}$'s eigenstate

$$|n,\alpha\rangle = S^{\dagger}(\alpha)|n',\alpha\rangle = S^{\dagger}(\alpha)|0\rangle$$
$$= \exp\left(\sum_k (\alpha_k \hat{C}_k^{\dagger} - \alpha_k^* \hat{C}_k)\right)|0\rangle$$

$$|0,\alpha\rangle = \exp\left(\sum_k \alpha_k \hat{C}_k^{\dagger} + \sum_k (-\alpha_k^*)\cdot \hat{C}_k\right)|0\rangle$$

Noticed: Baker-Campbell-Hausdorff relation.

$$\exp(\hat{A}+\hat{B}) = \exp(-\tfrac{1}{2}[\hat{A},\hat{B}])\exp(\hat{A})\exp(\hat{B}) \qquad \text{when } [\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$$

$$[\hat{A},\hat{B}] = [\sum_k \alpha_k \hat{C}_k^{\dagger}, \sum_{k'} -\alpha_{k'}^* \hat{C}_{k'}] = \sum_k (\alpha_k \alpha_k^*)[\hat{C}_k, \hat{C}_k^{\dagger}] = \sum_k |\alpha_k|^2$$

$$|0,\alpha\rangle = \exp(-\tfrac{1}{2}\sum_k |\alpha_k|^2)\cdot\exp(\sum_k \alpha_k \hat{C}_k^{\dagger})\exp(-\sum_k \alpha_k^* \hat{C}_k)|0\rangle$$

only acts in k th Hilbert space!

$$= \prod_k \hat{S}(\alpha_k)|0\rangle$$

$$\hat{S}(\alpha_k) = \exp(-\tfrac{1}{2}|\alpha_k|^2)\exp(\alpha_k \hat{C}_k^{\dagger})\exp(-\alpha_k^* \hat{C}_k)$$

$$|0,\alpha\rangle = S(\alpha_1)|0\rangle_1 \otimes S(\alpha_2)|0\rangle_2 \cdots \otimes S(\alpha_N)|0\rangle_N$$
$$\equiv |0,\alpha_1\rangle \otimes |0,\alpha_2\rangle \cdots \otimes |0,\alpha_N\rangle$$

where

$$|0,\alpha_k\rangle = S(\alpha_k)|0\rangle_k$$
$$= \exp(-\tfrac{1}{2}|\alpha_k|^2)\exp(\alpha_k \hat{C}_k^{\dagger})\exp(-\alpha_k^* \hat{C}_k)\;|0\rangle_k$$

Noticed $\hat{C}_k|0\rangle_k = 0$.

$$|0,\alpha_k\rangle = \exp(-\tfrac{1}{2}|\alpha_k|^2)\sum_{n=0}^{+\infty}\frac{1}{n!}(\alpha_k)^n\cdot(\hat{C}_k^{\dagger})^n|0\rangle_k$$

$$= \sum_{n=0}^{+\infty}\exp(-\tfrac{1}{2}|\alpha_k|^2)(\alpha_k)^n\frac{1}{\sqrt{n!}}|n\rangle_k$$

Average particle per site

$$\langle 0,\alpha_k|C_k^{\dagger}C_k|0,\alpha_k\rangle = \sum_{n=0}^{+\infty}\exp(-|\alpha_k|^2)(\alpha_k)^{2n}\frac{1}{n!}\cdot n$$
$$= \sum_{n=1}^{+\infty}\exp(-|\alpha_k|^2)(\alpha_k)^2(\alpha_k)^{2(n-1)}\frac{1}{(n-1)!}$$
$$= (\alpha_k)^2.$$

Baker - Campbell-Hausdorff Relation

- Prove $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots$

  Auxiliary parameter x

  $$\hat{U}(x) = e^{x\hat{A}}\hat{B}e^{-x\hat{A}}$$

  derivative of x:

  $$\frac{d\hat{U}(x)}{dx} = \hat{A}e^{x\hat{A}}\hat{B}e^{-x\hat{A}} - e^{x\hat{A}}\hat{B}e^{-x\hat{A}}\hat{A}$$

  $$= [\hat{A}, \hat{U}(x)]$$

  $$\hat{U}(x=0) = \hat{B}$$

  $$\hat{U}(x) = \hat{B} + \int_0^x dy\,[\hat{A}, \hat{U}(y)]$$

  Neumann series

  $$\hat{U}^0(x) = \hat{B}$$

  $$\hat{U}^1(x) = \hat{B} + \int_0^x dy\,[\hat{A}, \hat{U}^{(0)}(y)] = \hat{B} + x[\hat{A}, \hat{B}]$$

  $$\hat{U}^2(x) = \hat{B} + \int_0^x dy\,[\hat{A}, \hat{U}^{(1)}(y)] = \hat{B} + x[\hat{A}, \hat{B}] + \frac{1}{2}x^2[\hat{A}, [\hat{A}, \hat{B}]]$$

  $$\vdots$$

  $$\hat{U}(x) = \hat{B} + x[A, B] + \frac{1}{2}x^2[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!}x^3[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] \cdots$$

  $$\hat{U}(1) = e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] \cdots$$

- Prove $e^{\hat{A}+\hat{B}} = e^{-\frac{1}{2}[\hat{A}, \hat{B}]}e^{\hat{A}}e^{\hat{B}}$ when $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$

  Auxiliary parameter x

  $$e^{x(\hat{A}+\hat{B})} = e^{x\hat{A}}\hat{Q}(x)e^{x\hat{B}}$$

  derivative of x

  $$(\hat{A}+\hat{B})e^{x(\hat{A}+\hat{B})} = \hat{A}e^{x\hat{A}}\hat{Q}(x)e^{x\hat{B}} + e^{x\hat{A}}\frac{d\hat{Q}(x)}{dx}e^{x\hat{B}} + e^{x\hat{A}}\hat{Q}(x)e^{x\hat{B}}\hat{B}$$

  $$(\hat{A}+\hat{B})e^{x\hat{A}}\hat{Q}(x)e^{x\hat{B}} = \sim$$

  $$\hat{B}e^{x\hat{A}}\hat{Q}(x)e^{x\hat{B}} = e^{x\hat{A}}\frac{d\hat{Q}(x)}{dx}e^{x\hat{B}} + e^{x\hat{A}}\hat{Q}(x)e^{x\hat{B}}\hat{B}$$

  Left $e^{-x\hat{A}}$, right $e^{-x\hat{B}}$

  $$(e^{-x\hat{A}}\hat{B}e^{x\hat{A}})\hat{Q}(x) = \frac{d\hat{Q}(x)}{dx} + \hat{Q}(x)e^{x\hat{B}}\hat{B}e^{-x\hat{B}} \qquad [x\hat{A}, [x\hat{A}, \hat{B}]] = 0$$

  $$(\hat{B} - x[\hat{A}, \hat{B}])\hat{Q}(x) = \frac{d\hat{Q}(x)}{dx} + \hat{Q}(x)\hat{B}$$

  $$\frac{d\hat{Q}(x)}{dx} = \hat{B}\hat{Q}(x) - \hat{Q}(x)\hat{B} + x[\hat{B}, \hat{A}]Q(x)$$

  Suppose $[\hat{Q}(x), \hat{B}] = 0$; satisfies!

  $$\hat{Q}(0) = 1$$

  $$\hat{Q}(x) = \exp(-\tfrac{1}{2}x^2[\hat{A}, \hat{B}])$$

  $x = 1$

  $$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{-\frac{1}{2}[\hat{A}, \hat{B}]}e^{\hat{B}} = e^{-\frac{1}{2}[\hat{A}, \hat{B}]}e^{\hat{A}}e^{\hat{B}}$$

Attempts To Relativistic Quantum Mechanics.

- 坐标表象 schrodinger 方程(自由粒子)

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x},t)$$

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \frac{\hat{p}^2}{2m}\psi(\vec{x},t)$$

简单认为 relativistic Trans 是:

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = +\sqrt{m^2c^4+\hat{p}^2c^2}\,\psi(\vec{x},t)$$

$$= +\sqrt{-\hbar^2c^2\nabla^2+m^2c^4}\,\psi(\vec{x},t)$$

问题:

1° Treat Time & Position differently

2° 展开 sqrt 生成许多 ∇^2 ⟶ Theory is not local (P4 srednicki)

- Klein-Gordan Equation.

为了解决上面 2 个问题. squaring differential operator

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi(\vec{x},t) = (-\hbar^2c^2\nabla^2+m^2c^4)\psi(\vec{x},t)$$

4 维指标形式: $x^\mu = (ct,\vec{x})$

$$-\hbar^2c^2\partial^\mu\partial_\mu\psi(\vec{x},t) - m^2c^4\psi(\vec{x},t) = 0$$

$$(\hbar^2\partial^\mu\partial_\mu + m^2c^2)\psi(\vec{x},t) = 0$$

$$(\hbar^2\partial^2 + m^2c^2)\psi(\vec{x},t) = 0$$

—— 优点: 方程形式在不同参考系中相同.

$x'^\mu = \Lambda^\mu{}_\nu x^\nu \longrightarrow x^\nu = \Lambda_\mu{}^\nu x'^\mu$ (Lorentz trans 性质)

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu\frac{\partial}{\partial x^\nu}$$

变量代换

$$\partial'^\mu = \frac{\partial}{\partial x'_\mu} = \frac{\partial x_\nu}{\partial x'_\mu}\frac{\partial}{\partial x_\nu} = \Lambda^\mu{}_\nu\frac{\partial}{\partial x_\nu}$$

$$\partial'^\mu\partial'_\mu = \Lambda^\mu{}_\alpha\Lambda_\mu{}^\beta\cdot\partial^\alpha\partial_\beta = \partial^\mu\partial_\mu$$

$$\psi'(x') = \psi(x)$$

$$(\hbar^2\partial'^2 + m^2c^2)\psi'(\vec{x}',t') = 0$$

—— 缺点: KG-Equation is second-order time derivative, 但 Schrodinger Eq 是 first order Time derivative.

Norm Is not Time independent

- 另起炉造的 Dirac Function.

$$\hat{P}_\mu = i\hbar\frac{\partial}{\partial x^\mu} \leftarrow \text{Imagine} \to \exp\left(\frac{1}{i\hbar}(P_\mu x^\mu)\right)$$

$$i\hbar\frac{\partial}{\partial t}\psi_a(x) = \left(+i\hbar c\,(\alpha_j)_{ab}\partial^j + mc^2(\beta)_{ab}\right)\cdot\psi_b(x)$$

$$H_{ab} = \left(c\cdot\hat{P}^j(\alpha_j)_{ab} + m^2c^2(\beta)_{ab}\right)\psi_b(x)$$

平方

$$(H^2)_{ab} = c^2\hat{P}^i\hat{P}^j(\alpha_i\alpha_j)_{ab} + mc^3\cdot\hat{P}^i\left((\alpha_i\beta)_{ab} + (\beta\alpha_i)_{ab}\right) + m^4c^4\cdot(\beta^2)_{ab}$$

$$\begin{cases}\{\alpha_i,\alpha_j\} = 2\delta_{ij}\cdot(\mathbb{I})\\ \{\alpha_i,\beta\} = 0\\ \beta^2 = \mathbb{I}\end{cases}$$

$$(H^2)_{ab} = \left(c^2\sum_i\hat{P}^{2}_i + m^2c^4\right)(\mathbb{I})_{ab}$$

—— 问题：

存在负能量粒子.

Dirac 海

问题

1° 有粒子不服从 Pauli 不相容性.

2° 从描述单一粒子变为了描述多个粒子.

○ 量子力学本身问题：

$$i\hbar\frac{\partial}{\partial t}|\alpha,t\rangle = H|\alpha,t\rangle$$

时间与空间并不统一.（时间是一种演化参数）

- Non-relativistic QM for fixed number particles $\longrightarrow$ QFT

多粒子 Schrodinger 方程.

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x}_1\cdots\vec{x}_n,t) = \left[\sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\vec{x}_j)\right) + \sum_{j=1}^{n}\sum_{k=1}^{j-1}V(\vec{x}_j-\vec{x}_k)\right]\psi$$

构建

$$H = \int d^3x\; a^\dagger(\vec{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\vec{x})\right)\cdot a(\vec{x}) + \frac{1}{2}\int d^3x\,d^3y\cdot V(\vec{x}-\vec{y})\cdot a^\dagger(\vec{x})a^\dagger(\vec{y})\cdot a(\vec{y})\cdot a(\vec{x})$$

$$\begin{cases}[a(\vec{x}),a^\dagger(\vec{x}')] = \delta^{(3)}(x-x')\\ \text{others} = 0\end{cases}$$

$$|\alpha,t\rangle = \int d^3x_1\cdots d^3x_n\cdot\psi(\vec{x}_1\cdots\vec{x}_n,t)\,a^\dagger(\vec{x}_1)a^\dagger(\vec{x}_2)\cdots a^\dagger(\vec{x}_n)|0\rangle$$

$$a(\vec{x})|0\rangle = 0$$

计算. $i\hbar\frac{\partial}{\partial t}|\alpha,t\rangle = H|\alpha,t\rangle$ 可得 $\psi(\vec{x}_1\cdots)$ 的方程.

Nonrelativistic QFT

From schrodinger equation to Hamiltonian

○ from schrödinger Equation to Hamiltonian

Schrodinger Equation

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(x,t)\,\psi$$

Lagrange density

$$\mathcal{L}(\psi,\nabla\psi,\dot\psi) = i\hbar\,\psi^*\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\nabla\psi^*\nabla\psi - V(x,t)\,\psi^*\psi$$

Lagrange-Euler equation leads to Schrodinger equation!

$$\frac{\partial \mathcal{L}}{\partial\psi} - \nabla\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\dot\psi}\right)=0$$

$$-V(x,t)\,\psi^* + \frac{\hbar^2}{2m}\nabla^2\psi^* - i\hbar\frac{\partial}{\partial t}(\psi^*)=0$$

正则共轭场 (canonical conjugate field)

$$\pi(x,t) = \frac{\partial\mathcal{L}}{\partial\dot\psi} = i\hbar\,\psi^*(x,t)$$

$$\mathcal{H} = \pi\dot\psi - \mathcal{L} = \frac{\hbar^2}{2m}\nabla\psi^*\nabla\psi + V(x,t)\,\psi^*\psi$$

$L = L[\psi,\psi^*,\dot\psi,\dot\psi^*]$?

$H = H[\pi,\pi^*,\psi,\psi^*]$

Hamiltonian

$$H = H[\psi,\psi^*] = \int d^3x\, \psi^*(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x,t)\right)\psi(x)$$

Bose Particles Quantization Rules.

- 引入场算符：

Schrodinger Equation对应的Hamiltonian.

$$H = H[\psi, \psi^*] = \int d^3x\, \psi^*(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x,t)\right)\psi(x)$$

认为：ψ^*是$\hat{\psi}^\dagger$算符，ψ是$\hat{\psi}$算符；

Hamiltonian：(量子化后的Hamiltonian)（且认为是在Hamilton绘景下）

$$H = \int d^3x\, \hat{\psi}^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x,t)\right)\hat{\psi}(x)$$

$$= \int d^3x\, \hat{\psi}^\dagger(x)\, D_x\, \hat{\psi}(x)$$

D_x是Schrodinger differential operator！

$$D_x = -\frac{\hbar^2}{2m}\nabla^2 + V(x,t)$$

- Quantization relation（Equal-Time-Commutation-Relation. ETCR）

$$[\hat{\psi}(x,t), \hat{\psi}^\dagger(x',t)] = \delta^{(3)}(x-x')$$

$$[\hat{\psi}(x,t), \hat{\psi}(x',t)] = [\hat{\psi}^\dagger(x,t), \hat{\psi}^\dagger(x',t)] = 0$$

- 场算符的Dynamic Equation：

$$\frac{1}{i\hbar}[\hat{\psi}, \hat{H}] = \frac{\partial}{\partial t}\psi \qquad \frac{1}{i\hbar}[\hat{\psi}^\dagger, \hat{H}] = \frac{\partial}{\partial t}\hat{\psi}^\dagger$$

对于$\hat{\psi}$的Dynamic Equation：

$$[\hat{\psi}(x), \hat{H}] = [\hat{\psi}(x), \int d^3x'\, \hat{\psi}^\dagger(x')D_{x'}\hat{\psi}(x')]$$

$$= \int d^3x'\, [\hat{\psi}(x), \hat{\psi}^\dagger(x')D_{x'}\hat{\psi}(x')]$$

$$= \int d^3x' \left(\hat{\psi}(x)\hat{\psi}^\dagger(x')D_{x'}\hat{\psi}(x') - \hat{\psi}^\dagger(x')D_{x'}\hat{\psi}(x')\hat{\psi}(x)\right)$$

$$= \int d^3x' \Big(\hat{\psi}(x)\hat{\psi}^\dagger(x')D_{x'}\hat{\psi}(x') - \hat{\psi}^\dagger(x')\hat{\psi}(x)D_{x'}\hat{\psi}(x') + \hat{\psi}^\dagger(x')\hat{\psi}(x)D_{x'}\hat{\psi}(x')$$
$$- \hat{\psi}^\dagger(x')D_{x'}\hat{\psi}(x')\hat{\psi}(x)\Big)$$

$$= \int d^3x' \left([\hat{\psi}(x), \hat{\psi}^\dagger(x')]D_{x'}\hat{\psi}(x') + \hat{\psi}^\dagger(x')D_{x'}\hat{\psi}(x)\hat{\psi}(x') - \hat{\psi}^\dagger(x')D_{x'}\hat{\psi}(x')\hat{\psi}(x)\right)$$

$$= \int d^3x'\, \delta^{(3)}(x-x')D_{x'}\hat{\psi}(x') + \int d^3x'\, \hat{\psi}^\dagger(x')D_{x'}[\hat{\psi}(x), \hat{\psi}(x')]$$

$$= D_x\hat{\psi}(x)$$

和Schrodinger方程自洽：

$$i\hbar\frac{\partial}{\partial t}\hat{\psi} = D_x\hat{\psi}$$

- 场算符的广义Fourier分解(Generlized Fourier Decomposition)

在坐标空间中有完全complete一正交orthogonal基底. u_i

$$\int d^3x\, u_i^*(\vec{x})u_j(\vec{x}) = \delta_{ij} \qquad (\text{orthogonal})$$

$$\sum_i u_i(\vec{x})u_i^*(\vec{x}') = \delta^{(3)}(x-x') \qquad (\text{complete}) \to f(\vec{x}) = \int d^3x'\, f(\vec{x}')\sum_i u_i(\vec{x})u_i^*(\vec{x}') = \sum_i\left(\int d^3x'\, f(\vec{x}')u_i^*(\vec{x}')\right)u_i(\vec{x})$$

认为：

$$\hat{\psi}(x) = \sum_i \hat{a}_i(t)u_i(\vec{x}) \qquad \hat{\psi}^\dagger(x) = \sum_i \hat{a}_i^\dagger(t)u_i^*(\vec{x})$$

○ $\hat{a}_i(t)$ 和 $\hat{a}_i^\dagger(t)$ 的对易关系.

由于： $\hat{a}_i(t)=\int d^3x'\,\hat{\psi}(\vec{x}')\,u_i^*(\vec{x}')$

$\hat{a}_i^\dagger(t)=\int d^3x'\,\hat{\psi}^\dagger(\vec{x}')\,u_i(\vec{x}')$

$$[\hat{a}_i(t),\hat{a}_j^\dagger(t)]=\int d^3x'\,d^3x''\,u_i^*(\vec{x}')u_j(\vec{x}'')\overbrace{[\hat{\psi}(x'),\hat{\psi}^\dagger(x'')]}^{\text{等时}}$$
$$=\int d^3x'\,d^3x''\,u_i^*(\vec{x}')u_j(\vec{x}'')\,\delta^{(3)}(x'-x'')$$
$$=\int d^3x'\,u_i^*(\vec{x}')u_j(\vec{x}')$$
$$=\delta_{ij}$$

$$[\hat{a}_i(t),\hat{a}_j(t)]=[\hat{a}_i^\dagger(t),\hat{a}_j^\dagger(t)]=0$$

○ Schrodinger方程 V 不含时间/Hamiltonian的简洁表示/ $\hat{n}_i$ operator/ $\hat{N}$ op

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V(\vec{x})\right)u_i(\vec{x})=\varepsilon_i u_i(\vec{x})$$

用 Sch 方程本征函数时 Hamiltonian

$$H=\int d^3x\,\hat{\psi}^\dagger(x)\,D_x\,\psi(x)$$
$$=\sum_{ij}\int d^3x\,u_i^*(\vec{x})\,\hat{a}_i^\dagger(t)\,D_x u_j(\vec{x})\hat{a}_j(t)$$
$$=\sum_{ij}\int d^3x\,\hat{a}_i^\dagger(t)\,u_i^*(\vec{x})u_j(\vec{x})\cdot\varepsilon_j\,\hat{a}_j(t)$$
$$=\sum_{ij}\hat{a}_i^\dagger(t)\,\delta_{ij}\,\varepsilon_j\,\hat{a}_j(t)$$
$$=\sum_i\varepsilon_i\,\hat{a}_i^\dagger(t)\hat{a}_i(t)=\sum_i\varepsilon_i\hat{n}_i,$$

$$[\hat{n}_i,\hat{n}_j]=[\hat{a}_i^\dagger(t)\hat{a}_i(t),\hat{a}_j^\dagger(t)\hat{a}_j(t)]=\hat{a}_i^\dagger(t)\hat{a}_i(t)\hat{a}_j^\dagger(t)\hat{a}_j(t)-\hat{a}_j^\dagger(t)\hat{a}_j(t)\hat{a}_i^\dagger(t)\hat{a}_i(t)$$
$$=a_i^\dagger a_j\delta_{ij}-a_j^\dagger a_i\delta_{ij}=0$$

$$[\hat{n}_j,\hat{a}_i^\dagger(t)]=[\hat{a}_j^\dagger(t)\hat{a}_j(t),\hat{a}_i^\dagger(t)]=\delta_{ij}\,\hat{a}_j^\dagger(t)$$

$$[\hat{n}_j,\hat{a}_i(t)]=[\hat{a}_j^\dagger(t)\hat{a}_j(t),\hat{a}_i(t)]=-\delta_{ij}\,\hat{a}_j(t)$$

$$\hat{N}=\sum_i\hat{n}_i=\sum_i\hat{a}_i^\dagger(t)\,a_i(t)\qquad \hat{N}=\int d^3x'\,\hat{\psi}^\dagger(x',t)\,\hat{\psi}(x',t)$$

○ $\hat{N}$是守恒量：

$$[\hat{N},\hat{H}]=\sum_{ij}[\hat{a}_i^\dagger(t)a_i(t),\varepsilon_j\hat{a}_j^\dagger(t)a_j(t)]=0\Rightarrow i\hbar\frac{\partial}{\partial t}\hat{N}(t)=0$$

○ 由 $\hat{n}_i$ 对易性，构造 能量本征态：

$|n_1,n_2\cdots n_i\cdots\rangle$，满足： $\hat{n}_i|\cdots\rangle=n_i|\cdots\rangle$ ； $\hat{H}|\cdots\rangle=\sum_i n_i\varepsilon_i|\cdots\rangle$

- $\hat{a}_i$，$\hat{a}_i^\dagger$ 的动力方程.

$$\frac{1}{i\hbar}[\hat{a}_i(t), \hat{H}] = \frac{\partial}{\partial t}\hat{a}_i(t)$$
$$\frac{1}{i\hbar}[\hat{a}_i^\dagger(t), \hat{H}] = \frac{\partial}{\partial t}\hat{a}_i^\dagger(t)$$

$$\sum_j \frac{1}{i\hbar}[\hat{a}_i(t), \varepsilon_j \hat{a}_j^\dagger(t)\hat{a}_j(t)] = \sum_j \frac{1}{i\hbar}\varepsilon_j\left(\hat{a}_i(t)\hat{a}_j^\dagger(t)a_j(t) - \hat{a}_j^\dagger(t)\hat{a}_j(t)\hat{a}_i(t)\right)$$

$$= \sum_j \frac{1}{i\hbar}\varepsilon_j\left(\hat{a}_i(t)\hat{a}_j^\dagger(t)a_j(t) - \hat{a}_j^\dagger(t)\hat{a}_i(t)\hat{a}_j(t)\right)$$
$$= \sum_j \frac{1}{i\hbar}\varepsilon_j[\hat{a}_i(t), \hat{a}_j^\dagger(t)]\hat{a}_j(t)$$

$$= \frac{1}{i\hbar}\varepsilon_i\hat{a}_i(t)$$
$$= \frac{\partial}{\partial t}\hat{a}_i(t)$$

则：

$$\hat{a}_i(t) = e^{\frac{1}{i\hbar}\varepsilon_i t}\cdot\hat{a}_i$$

同理：

$$\hat{a}_i^\dagger(t) = e^{-\frac{1}{i\hbar}\varepsilon_i t}\hat{a}_i^\dagger$$

则：

$$\hat{H} = \sum_i \varepsilon_i \hat{a}_i^\dagger\hat{a}_i \qquad \hat{N} = \sum_i \hat{a}_i^\dagger\hat{a}_i \qquad \hat{n}_i = \hat{a}_i^\dagger\hat{a}_i$$
$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$$

- $\hat{a}_i^\dagger$；$\hat{a}_i$ 可起到升降算符的作用.（$|\cdots\rangle$ 表示 $|\cdots n_i\cdots\rangle$）（$\hat{a}^\dagger$ 叫升算符. $\hat{a}$ 叫降算符）

$$[\hat{n}_i, \hat{a}_i^\dagger] = 1 \qquad [\hat{n}_i, \hat{a}_i] = -1$$
$$\Downarrow \qquad\qquad \Downarrow$$

$$\hat{n}_i\hat{a}_i^\dagger|\cdots\rangle = (n_i+1)\hat{a}_i^\dagger|\cdots\rangle \qquad \hat{n}_i\hat{a}_i|\cdots\rangle = (n_i-1)\hat{a}_i|\cdots\rangle$$

$$\left(\langle\cdots|\hat{a}_i\right)\left(\hat{a}_i^\dagger|\cdots\rangle\right) = \langle\cdots|\hat{a}_i^\dagger\hat{a}_i+1|\cdots\rangle \qquad \left(\langle\cdots|\hat{a}_i^\dagger\right)\left(\hat{a}_i|\cdots\rangle\right) = n_i$$
$$= (n_i+1)$$

$$\hat{a}_i^\dagger|\cdots\rangle = \sqrt{n_i+1}\,|\cdots n_i+1\cdots\rangle \qquad \hat{a}_i|\cdots\rangle = \sqrt{n_i}\,|\cdots n_i-1\cdots\rangle$$

- 定义 $|0\rangle$ 为：$a_i|0\rangle = 0$.　　$|0\rangle$：vacuum state.

定义多体态（many body state）

$$|n_1, n_2\cdots\rangle = \frac{1}{\sqrt{n_1!}\sqrt{n_2!}\cdots\sqrt{n_i!}\cdots}(a_1^\dagger)^{n_1}\cdots(a_i^\dagger)^{n_i}\cdots|0\rangle$$

$\downarrow$

构成 Fock space

- n-particle 的另一种态

$$|\vec{x}_1\cdots\vec{x}_n, t\rangle = \frac{1}{\sqrt{n!}}\underbrace{\hat{\psi}^\dagger(x_1)\cdots\hat{\psi}^\dagger(x_n)}_{\text{时间相同.}}|0\rangle$$

$$\hat{\psi}^\dagger(x)|\vec{x}_1\cdots\vec{x}_n, t\rangle = \frac{1}{\sqrt{n!}}\hat{\psi}^\dagger(x)\cdots|0\rangle = \frac{1}{\sqrt{(n+1)!}}\sqrt{(n+1)}\,\hat{\psi}^\dagger(x)\cdots|0\rangle = \sqrt{n+1}\,|\vec{x}\,\vec{x}_1\cdots t\rangle$$

$$\hat{\psi}(x)|0\rangle = 0$$

$$\hat{\psi}(x)|\vec{x}_1\cdots\vec{x}_n, t\rangle = \frac{1}{\sqrt{n!}}\hat{\psi}(x)\hat{\psi}^\dagger(x_1)\cdots\hat{\psi}^\dagger(x_n)|0\rangle = \frac{1}{\sqrt{n!}}\left(\hat{\psi}^\dagger(x_1)\hat{\psi}(x) + \delta^{(3)}(x-x_1)\right)\hat{\psi}^\dagger(x_2)\cdots|0\rangle = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta^{(3)}(x_i-x)|x_1\cdots x_{i-1}x_{i+1}\cdots x_n, t\rangle$$

○ 这种态是 n-particle 的原因.(它确实是 $\hat{N}_V$ 的本征态)

粒子 number op $\hat{N} = \int d^3x' \,\hat{\psi}^\dagger(x',t)\,\hat{\psi}(x',t)$

区域粒子 number $\hat{N}_V = \int_V d^3x' \,\hat{\psi}^\dagger(x',t)\,\hat{\psi}(x',t)$

1° $\hat{N}_V|0\rangle = 0|0\rangle$

$\hat{N}_V|0\rangle = \int_V d^3x' \,\hat{\psi}^\dagger(x',t)\,\hat{\psi}(x',t)|0\rangle = 0$ ($\hat{\psi}(x',t)$ 中只含有 $\hat{a}$)

2° $[\hat{N}_V, \hat{\psi}^\dagger(x,t)] = \int_V d^3x' \,[\hat{\psi}^\dagger(x',t)\hat{\psi}(x',t), \hat{\psi}^\dagger(x,t)]$

$= \int_V d^3x' \left(\hat{\psi}^\dagger(x',t)\hat{\psi}(x',t)\hat{\psi}^\dagger(x,t) - \hat{\psi}^\dagger(x,t)\hat{\psi}^\dagger(x',t)\hat{\psi}(x',t) \right)$

$= \int_V d^3x' \,\hat{\psi}^\dagger(x',t)\,[\hat{\psi}(x',t), \hat{\psi}^\dagger(x,t)]$

$= \begin{cases} \hat{\psi}^\dagger(x,t) & x \in V \\ 0 & x \notin V \end{cases}$

则:

$\hat{N}_V|x_1 \cdots x_n, t\rangle = n_V \cdot |x_1 \cdots x_n, t\rangle$ (n_V 代表 $x_1 \cdots x_n$ 中在 V 中的个数)

○ $|x_1 \cdots x_n, t\rangle$ 内积

$\langle x_1' x_2' \cdots x_n', t | x_1 \cdots x_n, t\rangle = \frac{1}{n!}\langle 0|\psi(x_1') \cdots \psi(x_n')\psi^\dagger(x_1) \cdots \psi^\dagger(x_n)|0\rangle$

$= \frac{1}{n!}\langle 0|\psi(x_n') \cdots \psi(x_1')\psi^\dagger(x_1) \cdots \psi^\dagger(x_n)|0\rangle$

$= \frac{1}{n!}\langle 0| \cdots \psi(x_2')\left(\psi^\dagger(x_1)\psi(x_1') + \delta^{(3)}(x_1 - x_1')\right)\psi^\dagger(x_2) \cdots |0\rangle$

$= \frac{1}{n!}\sum_P P_{\{x_1 \cdots x_n\}}[\delta^{(3)}(x_1 - x_1') \cdots \delta^{(3)}(x_n - x_n')]$

$P_{\{x_1 \cdots x_n\}}$ 代表 $x_1 \cdots x_n$ 之间的 permutation

○ $|x_1 \cdots, t\rangle$ 的完备性

$\mathbb{I} = |0\rangle\langle 0| + \int d^3x_1 \,|x_1, t\rangle\langle x_1, t| + \int d^3x_1 d^3x_2 \,|x_1 x_2, t\rangle\langle x_1 x_2, t| \cdots$

↑ 证明?

○ 定义多体波函数 $\Psi^{(n)}_{[n_1 n_2 \cdots]}(x_1 \cdots x_n, t) = \langle x_1 \cdots x_n; t | n_1 n_2 \cdots\rangle$

$\int d^3x_1 \cdots d^3x_n \,\Psi^{(n)*}_{[n_1' n_2' \cdots]}(x_1 \cdots x_n; t)\,\Psi^{(n)}_{[n_1 n_2 \cdots]}(x_1 \cdots x_n, t)$

$= \int d^3x_1 \cdots d^3x_n \,\langle n_1' n_2' \cdots | x_1 \cdots x_n, t\rangle\langle x_1 \cdots x_n, t | n_1 n_2 \cdots\rangle$

$= \delta_{n_1' n_1}\delta_{n_2' n_2} \cdots$

○ 多体波函数满足 Schrodinger 方程

$$i\hbar\frac{\partial}{\partial t}\Phi^{(n)}_{[n_1 n_2\cdots]}(x_1\cdots x_n;t) = i\hbar\frac{\partial}{\partial t}\langle x_1\cdots x_n,t|n_1,n_2\cdots\rangle$$

$$= i\hbar\frac{\partial}{\partial t}\frac{1}{\sqrt{n!}}\langle 0|\hat{\psi}(x_n,t)\cdots\hat{\psi}(x_1,t)|n_1 n_2\cdots\rangle$$

$$= \sum_i i\hbar\frac{1}{\sqrt{n!}}\langle 0|\hat{\psi}(x_n,t)\cdots\frac{\partial}{\partial t}\hat{\psi}(x_i,t)\cdots\hat{\psi}(x_1,t)|n_1 n_2\cdots\rangle$$

$$= \sum_i\frac{1}{\sqrt{n!}}\langle 0|\hat{\psi}(x_n,t)\cdots D_x\hat{\psi}(x_i,t)\cdots\hat{\psi}(x_1,t)|n_1 n_2\cdots\rangle$$

$$= \sum_i\left(-\frac{\hbar^2}{2m}\nabla_i^2+V(x_i)\right)\Phi^{(n)}_{[n_1 n_2\cdots]}(x_1\cdots x_n,t)$$

○ 多体波函数的例子：

$$\Phi^{(1)}_k(x_1,t)=\langle x_1,t|0\cdots\overset{k\text{th}}{1}\cdots\rangle=\langle 0|\hat{\psi}(x_1)\hat{a}_k^\dagger|0\rangle=\sum_i\langle 0|u_i(x_1)\hat{a}_i(t)\hat{a}_k^\dagger|0\rangle$$

$$=\sum_i\langle 0|u_i(x_1)\hat{a}_i\hat{a}_k^\dagger|0\rangle\cdot e^{\frac{1}{i\hbar}\varepsilon_i t}$$

$$= u_k(x_1)e^{\frac{1}{i\hbar}\varepsilon_k t}\longrightarrow\Phi^{(1)}_k(x)=u_k(x)$$

$$\Phi^{(2)}_{k_1k_2}(x_1,x_2,t)=\langle x_1 x_2,t|0\cdots\overset{k_1}{1}\cdots\overset{k_2}{1}\cdots\rangle=\frac{1}{\sqrt{2}}\langle 0|\hat{\psi}(x_1)\hat{\psi}(x_2)\hat{a}_{k_1}^\dagger\hat{a}_{k_2}^\dagger|0\rangle$$

$(k_1\neq k_2)$

$$=\sum_{ij}\frac{1}{\sqrt{2}}\langle 0|u_i(x_1)\hat{a}_i(t)\,u_j(x_2)\hat{a}_j(t)\,\hat{a}_{k_1}^\dagger\hat{a}_{k_2}^\dagger|0\rangle$$

$$=\sum_{ij}\frac{1}{\sqrt{2}}\langle 0|u_i(x_1)u_j(x_2)\hat{a}_i(t)\hat{a}_j(t)\hat{a}_{k_1}^\dagger\hat{a}_{k_2}^\dagger|0\rangle$$

$$=\frac{1}{\sqrt{2}}\left(u_{k_1}(x_1)u_{k_2}(x_2)+u_{k_1}(x_2)u_{k_2}(x_1)\right)\cdot e^{\frac{1}{i\hbar}(\varepsilon_{k_1}+\varepsilon_{k_2})t}=\Phi^{(2)}_{k_1k_2}(x_1x_2)e^{\square t}$$

$$\Phi^{(2)}_{k_1k_1}(x_1,x_2,t)=\sum_{ij}\frac{1}{\sqrt{2!}}\langle 0|u_i(x_1)u_j(x_2)\hat{a}_i(t)\hat{a}_j(t)\frac{1}{\sqrt{2!}}\hat{a}_{k_1}^\dagger\hat{a}_{k_1}^\dagger|0\rangle$$

$$=\frac{1}{2}\langle 0|u_{k_1}(x_1)u_{k_1}(x_2)\hat{a}_{k_1}(\hat{a}_{k_1}^\dagger\hat{a}_{k_1}+1)\hat{a}_{k_1}^\dagger|0\rangle\exp(\square t)$$

$$=\frac{1}{2}\langle 0|2u_{k_1}(x_1)u_{k_1}(x_2)\hat{a}_{k_1}\hat{a}_{k_1}^\dagger|0\rangle\exp(\square t)$$

$$=u_{k_1}(x_1)u_{k_1}(x_2)\exp\left(\frac{1}{i\hbar}2\varepsilon_{k_1}t\right)=\Phi^{(2)}_{k_1k_1}(x_1x_2)\exp\left(\frac{1}{i\hbar}2\varepsilon_{k_1}t\right)$$

Generalize to n-particle states

$$\Phi^{(n)}_{k_1\cdots k_n}(x_1\cdots x_n)=\frac{1}{\sqrt{n_1!\,n_2!\cdots}}\frac{1}{\sqrt{n!}}\sum_P P[u_{k_1}(x_1)\cdots u_{k_n}(x_n)]$$

$\uparrow$ permute $\{x_1\cdots x_n\}$

是 completely symmetrized product wave function

总结

Heisenberg pic

$$H = \int d^3x\, \hat\varphi^\dagger(x)\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x,t)\right)\hat\varphi(x) = \int d^3x\, \hat\varphi^\dagger(x)\, D_x\, \hat\varphi(x)$$

Equal Time Commutation Relation

$$[\hat\varphi(x,t), \hat\varphi^\dagger(x',t)] = \delta^{(3)}(x-x')$$

$$\frac{1}{i\hbar}[\hat A, \hat H] = \frac{\partial}{\partial t}\hat A$$

example $\frac{1}{i\hbar}[\hat\varphi, \hat H] = \frac{\partial}{\partial t}\hat\varphi = \frac{1}{i\hbar} D_x \hat\varphi$

找正交基

$$\int d^3x\, u_i^*(x)\, u_j(x) = \delta_{ij}$$

$$\sum_i u_i^*(x_1)\, u_j(x_2) = \delta^{(3)}(x_1 - x_2)$$

$$\hat\varphi(x,t) = \sum_i u_i(x)\, \hat a_i(t)$$

$$\hat\varphi^\dagger(x,t) = \sum_j u_j^*(x)\, \hat a_j^\dagger(t)$$

$$[\hat a_i(t), \hat a_j^\dagger(t)] = \delta_{ij}$$

不含时schrodinger方程解做正交基

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right) u_i(x) = \varepsilon_i u_i(x)$$

$$\hat H = \sum_i \varepsilon_i \hat a_i^\dagger(t)\, \hat a_i(t)$$

$$\frac{1}{i\hbar}[\hat a_i(t), \hat H] = \frac{\partial}{\partial t}\hat a_i(t) = \frac{\varepsilon_i}{i\hbar}\hat a_i(t)$$

$$\hat a_i(t) = \hat a_i \exp\left(\frac{1}{i\hbar}\varepsilon_i t\right)$$

$$\hat H = \sum_i \varepsilon_i \hat a_i^\dagger \hat a_i \qquad [\hat a_i, \hat a_j^\dagger] = \delta_{ij}$$

$$\hat N = \int d^3x\, \hat\varphi^\dagger(x,t)\hat\varphi(x,t)$$

$$\hat n_i = \hat a_i^\dagger \hat a_i \qquad \hat N = \sum_i \hat n_i \qquad \hat H = \sum_i \varepsilon_i \hat n_i$$

$$[\hat a_i, \hat n_j] = \delta_{ij}\hat a_i$$

$$[\hat a_i^\dagger, \hat n_j] = -\delta_{ij}\hat a_i^\dagger$$

$\hat N, \hat n_i, \hat H$ 相互对易

构造共同本征态 $|n_1 n_2 \cdots\rangle$, satisfies $\hat n_i|\cdots\rangle = n_i|\cdots\rangle$ — Fock space

$$\hat n_i \hat a_i^\dagger |n_1 n_2\cdots\rangle = (\hat a_i^\dagger \hat n_i - [\hat a_i^\dagger, \hat n_i])|n_1 n_2\cdots\rangle = (\hat a_i^\dagger n_i + \hat a_i^\dagger)|\cdots\rangle = (n_i+1)\hat a_i^\dagger|\cdots\rangle$$

$$\hat n_i \hat a_i |n_1 n_2\cdots\rangle = (\hat a_i \hat n_i - [\hat a_i, \hat n_i])|n_1 n_2\cdots\rangle = (n_i - 1)\hat a_i|\cdots\rangle$$

$$\langle\cdots|\hat a_i \hat a_i^\dagger|\cdots\rangle = \langle\cdots|\hat n_i + 1|\cdots\rangle = (n_i+1)$$

$$\langle\cdots|\hat a_i^\dagger \hat a_i|\cdots\rangle = \langle\cdots|\hat n_i|\cdots\rangle = n_i$$

$$\hat a_i^\dagger|\cdots n_i\cdots\rangle = \sqrt{n_i+1}\,|\cdots n_i+1\cdots\rangle$$

$$\hat a_i|\cdots n_i\cdots\rangle = \sqrt{n_i}\,|\cdots n_i-1\cdots\rangle$$

$$|n_1 n_2\cdots\rangle = \frac{1}{\sqrt{n_1! n_2!\cdots}}(\hat a_1^\dagger)^{n_1}(\hat a_2^\dagger)^{n_2}\cdots|0\rangle$$

Localized state vector

$$|x_1 x_2 \cdots x_n, t\rangle = \frac{1}{\sqrt{n!}}\hat\varphi^\dagger(x_1)\hat\varphi^\dagger(x_2)\cdots\hat\varphi^\dagger(x_n)|0\rangle$$

$$\hat N_V = \int_V d^3x\, \hat\varphi^\dagger(x)\hat\varphi(x)$$

$$\hat\varphi(x)|x_1\cdots x_n,t\rangle = \sum_{i=1}^{n}\frac{1}{n}\delta^{(3)}(x-x_i)|\cdots x_{i-1}x_{i+1}\cdots,t\rangle$$

$$[\hat N_V, \hat\varphi^\dagger(x)] = \begin{cases}\hat\varphi^\dagger(x) & x\in V\\ 0 & x\notin V\end{cases} \Leftarrow \hat N_V|x_1\cdots,t\rangle = n_V|x_1\cdots,t\rangle$$

n_V 是 $\{x_1\cdots x_n\}$ 在 V 中的个数

$$\hat\varphi^\dagger(x)|x_1\cdots x_n,t\rangle = \sqrt{n+1}\,|x, x_1\cdots x_n,t\rangle$$

$$\langle x_1' x_2'\cdots x_n',t|x_1 x_2\cdots x_n,t\rangle = \frac{1}{n!}\sum_{P\{x_1\cdots x_n\}} P[\delta^{(3)}(x_1-x_1')\delta^{(3)}(x_2-x_2')\cdots\delta^{(3)}(x_n-x_n')]$$

$$\mathbb{I} = |0\rangle\langle 0| + \int d^3x_1\, |x_1,t\rangle\langle x_1,t| + \int d^3x_1 d^3x_2\, |x_1 x_2,t\rangle\langle x_1 x_2,t| + \cdots$$

Q

Many body wave function

$$\Phi^{(n)}_{[n_1 n_2 \cdots]}(x_1 \cdots x_n, t) = \langle x_1\, x_2 \cdots x_n; t \,|\, n_1\, n_2 \cdots \rangle$$

$$i\hbar \frac{\partial}{\partial t} \Phi^{(n)}_{[n_1 n_2 \cdots]} = \sum_i D_{x_i} \Phi^{(n)}_{[n_1 \cdots]}(x_1 \cdots x_i \cdots x_n)$$

$$\int d^3x_1 \cdots d^3x_n \; \Phi^{(n)*}_{[n_1' n_2' \cdots]}(x_1 \cdots x_n, t)\, \Phi^{(n)}_{[n_1 n_2 \cdots]}(x_1\, x_2 \cdots, t) = \delta_{n_1' n_1} \delta_{n_2' n_2} \cdots$$

$$\Phi^{(n)}_{k_1 k_2 \cdots k_n}(x_1 \cdots x_n, t) = \exp\left(\frac{1}{i\hbar}(\varepsilon_{k_1} + \varepsilon_{k_2} \cdots + \varepsilon_{k_n})t\right) \sum_{P\{x_1 \cdots x_n\}} \frac{1}{\sqrt{n_1!\, n_2! \cdots}} \frac{1}{\sqrt{n!}} P[u_{k_1}(x_1) \cdots u_{k_n}(x_n)]$$

$$= \langle x_1 \cdots x_n, t \,|\, \frac{1}{\sqrt{n_1!\, n_2! \cdots}} \hat{a}^\dagger_{k_1} \hat{a}^\dagger_{k_2} \cdots \hat{a}^\dagger_{k_n} \,|\, 0 \rangle$$

Quantization of Fermi Particle

- **Anti-Commutation**

$$\{\hat\psi(x,t),\hat\psi^\dagger(x',t)\} = \delta^{(3)}(x-x') \qquad \text{others} = 0$$

- Anti-commutation of $\hat a$; $\hat a^\dagger$; $\{\hat a_i,\hat a_j^\dagger\}=\delta_{ij}, \{\hat a_i,\hat a_j\}=\{\hat a_i^\dagger,\hat a_j^\dagger\}=0$, $\left(\hat\psi(x,t)=\sum_i u_i(x)\hat a_i(t)\quad \hat\psi^\dagger(x,t)=\sum_i u_i^*(x)\hat a_i^\dagger(t)\right)$

$\hat H,\hat n_i,\hat N$:

$$\hat H = \sum_i \varepsilon_i \hat a_i^\dagger \hat a_i \quad ; \hat a_i(t) = \hat a_i \exp\left(\frac{1}{i\hbar}\varepsilon_i t\right)$$

(不受对易关系影响) ↓只由这个展开式引出!

$$[\hat n_i,\hat n_j] = [\hat a_i^\dagger\hat a_i,\hat a_j^\dagger\hat a_j] = \hat a_i^\dagger\hat a_i\hat a_j^\dagger\hat a_j - \hat a_j^\dagger\hat a_j\hat a_i^\dagger\hat a_i = \hat n_i\hat n_j\left(1-(-1)^{2+2}\right)=0$$

($\hat n_i$ 相互对易, 可构建共同态)

$$\hat n_i\hat n_i = \hat a_i^\dagger\hat a_i\hat a_i^\dagger\hat a_i = \hat a_i^\dagger(-\hat a_i^\dagger\hat a_i+1)\hat a_i = \hat a_i^\dagger\hat a_i = \hat n_i \longrightarrow n_i = 1 \text{ or } 0.$$

(Pauli不相容)

$$|n_1 n_2\cdots\rangle = (\hat a_1^\dagger)^{n_1}(\hat a_2^\dagger)^{n_2}\cdots|0\rangle \longrightarrow \text{可写全证 } \langle n_1\cdots|n_1\cdots\rangle = 1$$

升降算符:

$$\hat a_i^\dagger|n_1\cdots 0\cdots\rangle = (-1)^{\sum_{k=1}^{i-1}n_k}(\hat a_1^\dagger)^{n_1}\cdots(\hat a_i^\dagger)\cdots|0\rangle = (-1)^{\sum_{k=1}^{i-1}n_k}|n_1 n_2\cdots 1\cdots\rangle$$

$$\hat a_i|n_1\cdots 1\cdots\rangle = (-1)^{\sum_{k=1}^{i-1}n_k}(\hat a_1^\dagger)^{n_1}\cdots\hat a_i\hat a_i^\dagger\cdots|0\rangle = (-1)^{\sum_{k=1}^{i-1}n_k}|n_1 n_2\cdots 0\cdots\rangle$$

- **Localized state**

$$|x_1\cdots x_n,t\rangle = \frac{1}{\sqrt{n!}}\hat\psi^\dagger(x_1,t)\cdots\hat\psi^\dagger(x_n,t)|0\rangle$$

它是合理的:(因为)

$$[\hat N_V,\hat\psi^\dagger(x)] = \int_V d^3x'\,[\hat\psi^\dagger(x')\hat\psi(x'),\hat\psi^\dagger(x)] = \int d^3x'\left(\hat\psi^\dagger(x')\hat\psi(x')\hat\psi^\dagger(x) + \hat\psi^\dagger(x')\hat\psi^\dagger(x)\hat\psi(x') - \hat\psi^\dagger(x')\hat\psi^\dagger(x)\hat\psi(x') - \hat\psi^\dagger(x)\hat\psi^\dagger(x')\hat\psi(x')\right)$$

$$= \int_V d^3x'\left(\hat\psi^\dagger(x')\{\hat\psi(x'),\hat\psi^\dagger(x)\} - \{\hat\psi^\dagger(x'),\hat\psi^\dagger(x)\}\hat\psi(x')\right)$$

$$= \hat\psi^\dagger(x)\quad (x\in V) \qquad = 0 \quad (x\notin V)$$

$$\hat N_V|x_1\cdots x_n,t\rangle = n_V|x_1\cdots x_n,t\rangle \qquad n_V \text{ 表示 } V \text{ 中 } \{x_1\cdots x_n\} \text{ 的数量}$$

- **Many-particle wave function is totally anti-commutation**

$$\Psi^{(n)}_{[n_1 n_2\cdots]}(x_1 x_2\cdots t) = \operatorname{sgn}P\,\Psi^{(n)}_{[n_1 n_2\cdots]}(x_{i_1},x_{i_2}\cdots,t)$$

$$(i_1,i_2\cdots i_n) = P(1,2\cdots n)$$

- 场算符按Schrodinger方程演化

$$[\hat\psi(x),\hat H] = [\hat\psi(x),\int d^3x'\,\hat\psi^\dagger(x')D_{x'}\hat\psi(x')]$$

$$= \int d^3x'\left(\hat\psi(x)\hat\psi^\dagger(x')D_{x'}\hat\psi(x') + \hat\psi^\dagger(x')\hat\psi(x)D_{x'}\hat\psi(x') - \hat\psi^\dagger(x')\hat\psi(x)D_{x'}\hat\psi(x') - \hat\psi^\dagger(x')D_{x'}\hat\psi(x')\hat\psi(x)\right)$$

$$= \int d^3x'\left(\{\hat\psi(x),\hat\psi^\dagger(x')\}D_{x'}\hat\psi(x') - \hat\psi^\dagger(x')D_{x'}\{\hat\psi(x),\hat\psi(x')\}\right)$$

$$= D_x\hat\psi(x)$$

- 多粒子波函数:

$$\Psi^{(2)}_{k_1k_2}(x_1,x_2,t) = \frac{1}{\sqrt{2!}}\left(u_{k_1}(x_1)u_{k_2}(x_2) - u_{k_2}(x_1)u_{k_1}(x_2)\right)\exp\left(\frac{1}{i\hbar}(\varepsilon_{k_1}+\varepsilon_{k_2})t\right)$$

$$\Psi^{(n)}_{k_1\cdots k_n}(x_1\cdots x_n,t) = \frac{1}{\sqrt{n!}}\begin{vmatrix} u_{k_1}(x_1) & \cdots & u_{k_1}(x_n) \\ u_{k_2}(x_1) & \cdots & u_{k_2}(x_n) \\ \vdots & & \vdots \\ u_{k_n}(x_1) & \cdots & u_{k_n}(x_n)\end{vmatrix}$$

Interacting Particle Systems, Hartree-Fock Approximation

- Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$\hat{H}_0 = \int d^3x \ \hat{\psi}^\dagger(x,t) D_x \hat{\psi}(x,t)$$

$$\hat{H}_1 = \frac{1}{2}\int d^3x' d^3x \ \hat{\psi}^\dagger(x',t)\hat{\psi}^\dagger(x,t) U(x,x') \hat{\psi}(x,t)\hat{\psi}(x',t)$$

$$U(x,x') = U(x',x)$$

- 场算符的时间演化

$$[\hat{\psi}(x), \hat{H}_0] = D_x \hat{\psi}(x)$$

$$[\hat{\psi}(x), \hat{H}_1] = \frac{1}{2}\int d^3x_1 d^3x_2 \ U(x_1,x_2)[\hat{\psi}(x), \hat{\psi}^\dagger(x_2)\hat{\psi}^\dagger(x_1)\psi(x_1)\hat{\psi}(x_2)]$$

$$= \frac{1}{2}\int d^3x_1 d^3x_2 \ U(x_1,x_2)\left(\hat{\psi}(x)\hat{\psi}^\dagger(x_2)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x_2) - \hat{\psi}^\dagger(x_2)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x_2)\hat{\psi}(x)\right)$$

$$= \frac{1}{2}\int d^3x_1 d^3x_2 \ U(x_1,x_2)\left(\left(-\hat{\psi}^\dagger(x_2)\hat{\psi}(x) + \delta^{(3)}(x-x_2)\right)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x_2) - \hat{\psi}^\dagger(x_2)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x_2)\hat{\psi}(x)\right)$$

$$= \frac{1}{2}\int d^3x_1 d^3x_2 \ U(x_1,x_2)\Big(-\hat{\psi}^\dagger(x_2)\left(-\hat{\psi}^\dagger(x_1)\hat{\psi}(x) + \delta^{(3)}(x-x_1)\right)\hat{\psi}(x_1)\hat{\psi}(x_2) + \delta^{(3)}(x-x_2)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x_2)$$
$$- \hat{\psi}^\dagger(x_2)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x_2)\hat{\psi}(x)\Big)$$

$$= \frac{1}{2}\int d^3x_1 d^3x_2 \ U(x_1,x_2)\Big(\hat{\psi}^\dagger(x_2)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x_2)\hat{\psi}(x) - \delta^{(3)}(x-x_1)\hat{\psi}^\dagger(x_2)\hat{\psi}(x_1)\hat{\psi}(x_2)$$
$$+ \delta^{(3)}(x-x_2)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x_2)$$
$$- \hat{\psi}^\dagger(x_2)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x_2)\hat{\psi}(x)\Big)$$

$$= \frac{1}{2}\int d^3x_1 \left(U(x,x_1) + U(x_1,x)\right)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x)$$

$$= \int d^3x_1 \ U(x,x_1)\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}(x)$$

场的 Equation of motion

$$i\hbar\partial_0\hat{\psi}(x) - D_x\hat{\psi}(x) - \int d^3x' \ \hat{\psi}^\dagger(x')\hat{\psi}(x')U(x',x)\hat{\psi}(x) = 0$$

- 场算符用基函数展开

$$\hat{\psi}(x,t) = \sum_i \hat{a}_i(t)\varphi_i(x)$$

Hamilton 算符

$$\hat{H}_0 = \int d^3x \ \hat{\psi}^\dagger(x) D_x \hat{\psi}(x)$$

$$= \sum_{ij}\int d^3x \ \hat{a}_i^\dagger(t)\varphi_i^*(x) D_x \hat{a}_j(t)\varphi_j(x)$$

$$= \sum_{ij}\left(\int d^3x \ \varphi_i^*(x) D_x\varphi_j(x)\right)\hat{a}_i^\dagger(t)\hat{a}_j(t)$$

$$= \sum_{ij} d_{ij}\hat{a}_i^\dagger(t)\hat{a}_j(t)$$

$$d_{ij} = \int d^3x \ \varphi_i^*(x) D_x \varphi_j(x)$$

$$\hat{H}_1 = \frac{1}{2}\int d^3x_1 d^3x_2 \ \hat{\psi}^\dagger(x_2)\hat{\psi}^\dagger(x_1) U(x_1,x_2)\hat{\psi}(x_1)\hat{\psi}(x_2)$$

$$= \sum_{ijkl}\frac{1}{2}\int d^3x_1 d^3x_2 \ \hat{a}_i^\dagger(t)\varphi_i^*(x_2)\hat{a}_j^\dagger(t)\varphi_j^*(x_1) U(x_1,x_2)\hat{a}_k(t)\varphi_k(x_1)\hat{a}_l(t)\varphi_l(x_2)$$

$$= \sum_{ijkl} U_{ijkl}\cdot\hat{a}_i^\dagger(t)\hat{a}_j^\dagger(t)\hat{a}_k(t)\hat{a}_l(t)$$

$$U_{ijkl} = \frac{1}{2}\int d^3x_1 d^3x_2 \ \varphi_i^*(x_2)\varphi_j^*(x_1) U(x_1,x_2)\varphi_k(x_1)\varphi_l(x_2)$$

- Hartree-Fock 近似.

用自由场的 Fermi Particle 量子化中的结论.

$$|\Psi\rangle = |n_1 n_2 \cdots\rangle = (\hat{a}_1^{+})^{n_1} (\hat{a}_2^{+})^{n_2} \cdots |0\rangle$$

取:

$$\hat{a}_i(t) = \hat{a}_i \exp\left(\frac{1}{i\hbar}\varepsilon_i t\right) \quad (\text{是一种近似})$$

认为:

$$\hat{\psi}(x,t) = \sum_i \hat{a}_i(t)\varphi_i(x)$$

由场算符的时间演化方程:

$$i\hbar\partial_0\hat{\psi}(x) - D_x\hat{\psi}(x) - \int d^3x' \hat{\psi}^{+}(x')\hat{\psi}(x')U(x',x)\hat{\psi}(x) = 0$$

$$\langle\Psi|\hat{a}_l^{+}(t)\left(i\hbar\partial_0\hat{\psi}(x) - D_x\hat{\psi}(x) - \int d^3x' \hat{\psi}^{+}(x')\hat{\psi}(x')U(x',x)\hat{\psi}(x)\right)|\Psi\rangle = 0$$

$$\begin{aligned}\langle\Psi|\hat{a}_l^{+}(t)i\hbar\partial_0\hat{\psi}(x)|\Psi\rangle &= \sum_i \langle\Psi|\hat{a}_l^{+}(t)i\hbar\frac{1}{i\hbar}\varepsilon_i\hat{a}_i(t)\varphi_i(x)|\Psi\rangle \\ &= \sum_i \langle\Psi|\hat{a}_l^{+}(t)\varepsilon_i\hat{a}_i(t)\varphi_i|\Psi\rangle \\ &= \langle\Psi|\hat{a}_l^{+}(t)\hat{a}_l(t)\varepsilon_l\varphi_l(x)|\Psi\rangle \quad \text{若 } i\neq l\ |\Psi\rangle \text{ 和 } \hat{a}_l^{+}\hat{a}_i|\Psi\rangle \text{ 正交} \\ &= n_l\varepsilon_l\varphi_l(x)\end{aligned}$$

$$\begin{aligned}\langle\Psi|\hat{a}_l^{+}(t)D_x\hat{\psi}(x)|\Psi\rangle &= \sum_i \langle\Psi|\hat{a}_l^{+}(t)\hat{a}_i(t)D_x\varphi_i(x)|\Psi\rangle \\ &= \langle\Psi|\hat{a}_l^{+}(t)\hat{a}_l(t)D_x\varphi_l(x)|\Psi\rangle \\ &= n_l D_x\varphi_l(x)\end{aligned}$$

$$\langle\Psi|\hat{a}_l^{+}(t)\int d^3x' \hat{\psi}^{+}(x')\hat{\psi}(x')U(x',x)\hat{\psi}(x)|\Psi\rangle = \sum_{ijk}\int d^3x' \langle\Psi|\hat{a}_l^{+}(t)\hat{a}_i^{+}(t)\hat{a}_j(t)\hat{a}_k(t)|\Psi\rangle\varphi_i^{*}(x')\varphi_j(x')U(x',x)\varphi_k(x)$$

$$\langle\Psi|\hat{a}_l^{+}(t)\hat{a}_i^{+}(t)\hat{a}_j(t)\hat{a}_k(t)|\Psi\rangle = -\delta_{lj}\delta_{ik}\, n_l n_i + \delta_{lk}\delta_{ij}\, n_l n_i = n_i n_l(\delta_{lk}\delta_{ij} - \delta_{lj}\delta_{ik})$$

代入得:

$$n_l\varepsilon_l\varphi_l(x) - n_l D_x\varphi_l(x) - \sum_{ijk}\int d^3x'\, n_i n_l(\delta_{lk}\delta_{ij} - \delta_{lj}\delta_{ik})\varphi_i^{*}(x')\varphi_j(x')U(x',x)\varphi_k(x) = 0$$

$$\begin{aligned}n_l\varepsilon_l\varphi_l(x) - n_l D_x\varphi_l(x) &- \sum_i\int d^3x'\, n_i n_l\varphi_i^{*}(x')\varphi_i(x')U(x',x)\varphi_l(x) \\ &+ \sum_i\int d^3x'\, n_i n_l\varphi_i^{*}(x')\varphi_l(x')U(x',x)\varphi_i(x) = 0 \\ \varepsilon_l\varphi_l(x) - D_x\varphi_l(x) &- \sum_i\int d^3x'\, n_i\varphi_i^{*}(x')\varphi_i(x')U(x',x)\varphi_l(x) \\ &+ \sum_i\int d^3x'\, n_i\varphi_i^{*}(x')\varphi_i(x)U(x',x)\varphi_l(x') = 0\end{aligned}$$

密度矩阵定义为:

$$\rho(x',x) = \sum_i n_i\varphi_i^{*}(x')\varphi_i(x) = \rho^{*}(x,x') \quad ; \quad \rho(x) = \rho(x,x)$$

Hartree-Fock Equation.

$$D_x\varphi_l(x) + \int d^3x'\,\rho(x',x')U(x',x)\varphi_l(x) - \int d^3x'\,\rho(x',x)U(x',x)\varphi_l(x') = \varepsilon_l\varphi_l(x)$$

- Hartree-Fock 近似下的 φ 的正交性:

Hartree-Fock Equation

$$D_x\varphi_l(x) + \int d^3x' \sum_i n_i\left(\varphi_i^{*}(x')\varphi_i(x')U(x',x)\varphi_l(x) - \varphi_i^{*}(x')\varphi_l(x')U(x',x)\varphi_i(x)\right) = \varepsilon_l\varphi_l(x)$$

$\int d^3x\, \varphi_k^*(x) \cdots$

$$d_{kl} + \sum_i n_i \left(U_{kiil} - U_{kili} \right) = \varepsilon_l \int d^3x\, \varphi_k^*(x)\varphi_l(x)$$

$\int d^3x (\cdots)^* \varphi_l(x)$

$$d_{lk}^* + \sum_i n_i \left(U_{kiil} - U_{kili} \right) = \varepsilon_k \int d^3x\, \varphi_k^*(x)\varphi_l(x)$$

由于：$d_{kl} = d_{lk}^*$

$$\int d^3x\, \varphi_k^*(x)\varphi_l(x) = \delta_{kl}$$

$$\langle \Phi | \hat{H} | \Phi \rangle = \sum_i n_i d_{ii} + \frac{1}{2} \sum_{ij} n_i n_j \left(U_{ijji} - U_{ijij} \right)$$ $\rightarrow$ 这个试子我自己设算!

$$= \sum_i n_i \varepsilon_i - \frac{1}{2} \sum_{ij} n_i n_j \left(U_{ijji} - U_{ijij} \right)$$

自然单位制

- 由于 $[\hbar][T]^{-1} = [E] = [m][L]^2[T]^{-2}$

$$\begin{cases} [\hbar] = [m][L]^2[T]^{-1} \\ [c] = [L][T]^{-1} \end{cases}$$

任何单位可写为：

$$[m]^{\alpha}[L]^{\beta}[T]^{\gamma} = [m]^{x}[\hbar]^{y}[c]^{z}.$$

取 $[\hbar]=[c]=1$

则 $[m] = [E] = [L]^{-1}$

可用 $[m]^{x}$ 来表示所有物理量的单位.

- $g = (1, -1, -1, -1)$

Non-charged Klein-Gordon Field quantization

- Klein-Gordon Field 的 Hamiltonian 导出

$$\mathcal{L}(\phi,\dot\phi,\nabla\phi)=\frac{1}{2}\hbar^2\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x^\mu}-\frac{1}{2}m^2c^2\phi^2\quad ;L=L[\phi,\dot\phi] \qquad -(1)$$

Euler-Lagrange 方程

$$\frac{\partial\mathcal{L}}{\partial\phi}=\frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right) \qquad -(2)$$

$$\overset{(1)}{\underset{(2)}{}}\Big|\ (\Box+m^2)\phi(x)=0 \qquad \Box=\partial^\mu\partial_\mu$$

正则共轭场:(自然单位制 $\hbar=c=1$)

$$\pi(x)=\frac{\partial\mathcal{L}}{\partial\dot\phi}=\dot\phi$$

Hamiltonian 密度:

$$\mathcal{H}=\pi\dot\phi-\mathcal{L}=\frac{1}{2}(\pi^2+(\nabla\phi)^2+m^2\phi^2)$$

Hamiltonian

$$H=\int d^3x\cdot\frac{1}{2}(\pi^2(x)+(\nabla\phi(x))^2+m^2\phi^2(x))=H[\pi,\phi]$$

- 量子化

$$\hat H=\int d^3x\,\frac{1}{2}(\hat\pi(\vec x,t)^2+(\nabla\hat\phi(\vec x,t))^2+m^2\hat\phi(\vec x,t)^2)$$

Equal Time Commutation Relation $[\hat\phi(\vec x,t),\hat\pi(\vec x',t)]=i\delta^{(3)}(\vec x-\vec x')$

$$[\hat\phi(\vec x,t),\hat\phi(\vec x',t)]=[\hat\pi(\vec x,t),\hat\pi(\vec x',t)]=0$$

Hamilton 运动方程:

$$\frac{\partial}{\partial t}\hat\phi(\vec x,t)=-i[\hat\phi(\vec x,t),\hat H]=\hat\pi(\vec x,t)$$

$$\frac{\partial}{\partial t}\hat\pi(\vec x,t)=-i[\hat\pi(\vec x,t),\hat H]=(\nabla^2-m^2)\hat\phi(\vec x,t)$$

$$\frac{\partial^2}{\partial t^2}\hat\phi(\vec x,t)=(\nabla^2-m^2)\hat\phi(\vec x,t)$$

场按基底展开 $\omega_p=\sqrt{\vec p^2+m^2}$; $-\omega_p^2 u_p(\vec x)=(\nabla^2-m^2)u_p(\vec x)$ (稳时解) $u_p(\vec x)=N_pe^{i\vec p\cdot\vec x}$ $N_p=\frac{1}{\sqrt{2(\vec p^2+m^2)^{1/2}(2\pi)^3}}$

$$\hat\phi(\vec x,t)=\int d^3p\cdot\frac{1}{\sqrt{2(\vec p^2+m^2)^{1/2}(2\pi)^3}}\cdot e^{i\vec p\cdot\vec x}\,\hat a_p(t)=\int d^3p\cdot u_p(\vec x)\cdot\hat a_p(t)=\int d^3p\cdot N_p\cdot e^{i\vec p\cdot\vec x}\,\hat a_p(t) \qquad -(1)$$

完备条件:

$$\int d^3x\cdot\frac{1}{\sqrt{2(\vec p'^2+m^2)^{1/2}(2\pi)^3}}e^{-i\vec p'\cdot\vec x}\hat\phi(\vec x,t)=\int d^3x\frac{1}{\sqrt{2(\vec p'^2+m^2)^{1/2}(2\pi)^3}}\cdot e^{-i\vec p'\cdot\vec x}\cdot\int d^3p\frac{1}{\sqrt{2(\vec p^2+m^2)^{1/2}(2\pi)^3}}\cdot e^{i\vec p\cdot\vec x}\cdot\hat a_p(t)$$

$$=\int d^3x\,d^3p\ e^{i(\vec p-\vec p')\cdot\vec x}\frac{1}{2\ (2\pi)^3}\cdot\frac{1}{(\vec p'^2+m^2)^{1/4}(\vec p^2+m^2)^{1/4}}\hat a_p(t)$$

$$=\int d^3p\ \frac{1}{2}\ \frac{1}{(\vec p'^2+m^2)^{1/4}(\vec p^2+m^2)^{1/4}}\delta^{(3)}(\vec p-\vec p')\hat a_p(t)$$

$$=\frac{1}{2}\frac{1}{(\vec p'^2+m^2)^{1/2}}\cdot\hat a_{p'}(t)$$

$$\hat a_{p'}(t)=\sqrt{2(\vec p'^2+m^2)^{1/2}}\frac{1}{(2\pi)^{3/2}}\int d^3x\cdot e^{-i\vec p'\cdot\vec x}\hat\phi(\vec x,t) \qquad -(2)$$

可用(1)的形式来表示 $\hat\phi(\vec x,t)=\int d^3p\frac{1}{\sqrt{2(\vec p^2+m^2)^{1/2}(2\pi)^3}}e^{i\vec p\cdot\vec x}\cdot\left(\sqrt{2(\vec p^2+m^2)^{1/2}}\frac{1}{(2\pi)^{3/2}}\int d^3x'\cdot e^{-i\vec p\cdot\vec x'}\hat\phi(\vec x',t)\right)$

$$=\int d^3p\cdot d^3x'\frac{1}{(2\pi)^3}\cdot e^{i\vec p\cdot(\vec x-\vec x')}\hat\phi(\vec x',t)$$

$$=\int d^3x'\ \delta^{(3)}(x-x')\hat\phi(\vec x',t)$$

$$=\hat\phi(\vec x,t) \qquad -(3)$$

说明了(1)的展开形式和合理! 完备性: $\int d^3p\ e^{i\vec p\cdot\vec x}e^{-i\vec p\cdot\vec x'}=(2\pi)^3\delta^{(3)}(x-x')$

- $\hat{a}_p(t)$ 的运动方程

$$\ddot{\hat{a}}_p(t) = -(\vec{p}^2+m^2)\hat{a}_p(t)$$

解

$$\hat{a}_p(t) = \hat{a}_p^{(1)} e^{-iW_p t} + \hat{a}_p^{(2)} e^{iW_p t} \qquad W_p = \sqrt{\vec{p}^2+m^2}$$

由于是实数场 $\phi^* = \phi$，ϕ 是 Hermitian $\phi^\dagger = \phi$

$$\hat{\phi}(\vec{x},t) = \int d^3p \frac{1}{\sqrt{2\cdot W_p\cdot(2\pi)^3}}\cdot\left(\hat{a}_p^{(1)} e^{i(\vec{p}\cdot\vec{x}-W_p t)} + \hat{a}_p^{(2)} e^{i(\vec{p}\cdot\vec{x}+W_p t)}\right)$$

$$= \int d^3p \frac{1}{\sqrt{2\cdot W_p\cdot(2\pi)^3}}\cdot\left(\hat{a}_p^{(1)} e^{i(\vec{p}\cdot\vec{x}-W_p t)} + \hat{a}_{-p}^{(2)} e^{i(-\vec{p}\cdot\vec{x}+W_p t)}\right)$$

$$(\hat{a}_p^{(1)})^\dagger = \hat{a}_{-p}^{(2)}$$

$$\hat{\phi}(x,t) = \int d^3p \frac{1}{\sqrt{2W_p(2\pi)^3}}\left(\hat{a}_p\, e^{i(\vec{p}\cdot\vec{x}-W_p t)} + \hat{a}_p^\dagger\, e^{-i(\vec{p}\cdot\vec{x}-W_p t)}\right)$$

$$\hat{\pi}(x,t) = \frac{\partial}{\partial t}\hat{\phi}(x,t) = \int d^3p \frac{1}{\sqrt{2W_p(2\pi)^3}}\cdot(-iW_p t)\left(\hat{a}_p\, e^{i(\vec{p}\cdot\vec{x}-W_p t)} - \hat{a}_p^\dagger\, e^{-i(\vec{p}\cdot\vec{x}-W_p t)}\right)$$

- $\hat{a}_p$ 的对易关系：

$$[\hat{a}_p, \hat{a}_{p'}^\dagger] = \delta^{(3)}(\vec{p}-\vec{p}') \qquad [\hat{a}_p, \hat{a}_{p'}] = [\hat{a}_p^\dagger, \hat{a}_{p'}^\dagger] = 0$$

- 定义 $u_p(x,t) = N_p e^{-ip\cdot x} = \frac{1}{\sqrt{2W_p(2\pi)^3}} e^{-i(W_p t - p\cdot x)}$

$$(\partial_0^2 - \nabla^2 + m^2)\,u_p(x,t) = 0$$

- 此时：

$$\xrightarrow{\quad} \text{正能量 指 } -iWt$$

$$\hat{\phi}(x,t) = \int d^3p\left(\hat{a}_p u_p(x,t) + \hat{a}_p^\dagger u_p^*(x,t)\right) = \hat{\phi}^{(+)}(x,t) + \hat{\phi}^{(-)}(x,t)$$

$$\hat{\pi}(x,t) = \frac{\partial}{\partial t}\hat{\phi}(x,t) = \int d^3p\cdot(-iW_p t)\left(\hat{a}_p u_p(x,t) - \hat{a}_p^\dagger u_p^*(x,t)\right) = \hat{\pi}^{(+)}(x,t) - \hat{\pi}^{(-)}(x,t)$$

- 定义 scalar product of two klein-Gordon Wave functions ϕ and χ.

$$(\phi,\chi) = i\int d^3x\, \phi^*(x,t)\overleftrightarrow{\partial_0}\chi(x,t)$$

$$\equiv i\int d^3x\cdot\left(\phi^*(x,t)\frac{\partial\chi(x,t)}{\partial t} - \frac{\partial\phi^*(x,t)}{\partial t}\chi(x,t)\right)$$

- 表达 $\hat{a}_p$ 的简单形式：

$$\hat{a}_p = i\int d^3x\, u_p^*(x,t)\overleftrightarrow{\partial_0}\hat{\phi}(x,t)$$

$$\hat{a}_p^\dagger = -i\int d^3x\, u_p(x,t)\overleftrightarrow{\partial_0}\hat{\phi}(x,t)$$

- Hamiltonian

$$\hat{H} = \frac{1}{2}\int d^3p\, W_p\left(\hat{a}_p^\dagger\hat{a}_p + \hat{a}_p\hat{a}_p^\dagger\right) = \frac{1}{2}\int d^3p\, W_p\left(2\hat{a}_p^\dagger\hat{a}_p + \delta^{(3)}(0)\right)$$

- Normal ordering 引入：(生成算符似乎总在左)

$$:\hat{\phi}\hat{\chi}: = \hat{\phi}^{(-)}\hat{\chi}^{(-)} + \hat{\phi}^{(-)}\hat{\chi}^{(+)} + \hat{\chi}^{(-)}\hat{\phi}^{(+)} + \hat{\phi}^{(+)}\hat{\chi}^{(+)}$$

为什么叫 $a_p^\dagger$ 生成算符，
设有态 $|E\rangle$ 是 $\hat{H}$ 本征态、
则：
$$\hat{H}|E\rangle = E|E\rangle$$
$$\hat{H}\hat{a}_{p_1}^\dagger|E\rangle = \hat{a}_{p_1}^\dagger\hat{H}|E\rangle + W_{p_1}\cdot\hat{a}_{p_1}^\dagger|E\rangle$$
$$= \underbrace{(E+W_{p_1})}_{\Downarrow}\cdot\hat{a}_{p_1}^\dagger|E\rangle$$
生成了一个 P_1 动量的粒子.

$\hat{a}_p$ 叫湮灭算符同理！
真空态：$|0\rangle$：$\hat{a}_p|0\rangle = 0$！

○ 动量算符

经典场论，时空平移 → 能动量张量

$$P_\mu = \int d^3x \, \Theta_{0\mu} = \int d^3x \left(\pi \frac{\partial\phi}{\partial x^\mu} - g_{0\mu} \mathcal{L} \right)$$

$$\vec{P} = -\int d^3x \; \pi \, \nabla\phi \qquad \text{(相当于}(P_x, P_y, P_z)\text{)}$$

量子化：

$$\hat{\vec{P}} = -\frac{1}{2}\int d^3x \left(\hat{\pi}(x,t)\nabla\hat{\phi}(x,t) + \nabla\hat{\phi}(x,t)\hat{\pi}(x,t) \right)$$

$$= \frac{1}{2}\int d^3p \; \vec{p} \left(\hat{a}_p^\dagger \hat{a}_p + \hat{a}_p \hat{a}_p^\dagger \right)$$

○ Noether 定理量子化出的角动量算符

$$\hat{L} = -\frac{1}{2}\int d^3x : \left(\hat{\pi} \; x \times \nabla\hat{\phi} + (x \times \nabla\hat{\phi})\hat{\pi} \right):$$

$$= i \int d^3p \;\; \hat{a}_p^\dagger (p \times \nabla_p) \hat{a}_p$$

Charged KG 场.

- Lagrangian 到 Hamiltonian.

$$\mathcal{L} = \frac{\partial\phi^*}{\partial x_\mu}\frac{\partial\phi}{\partial x^\mu} - m^2\phi^*\phi \qquad L = L[\phi, \phi^*, \dot\phi, \dot\phi^*]$$

正则共轭场

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi^*$$

$$\pi^* = \frac{\partial\mathcal{L}}{\partial\dot\phi^*} = \dot\phi$$

Hamiltonian：

$$H = \int d^3x\,(\pi\partial_0\phi + \pi^*\partial_0\phi^* - \mathcal{L})$$

$$= \int d^3x\,(\pi^*\pi + \nabla\phi^*\nabla\phi + m^2\phi\phi^*)$$

$$\hat H = \int d^3x\,(\pi^\dagger\pi + \nabla\phi^\dagger\nabla\phi + m^2\phi\phi^\dagger)$$

量子化：

$$[\hat\phi(x,t), \hat\pi(x',t)] = i\delta^{(3)}(x-x') = [\hat\phi^\dagger(x,t), \hat\pi^\dagger(x',t)]$$

展开：

$$\hat\phi(x,t) = \int d^3p\cdot(\hat a_p u_p(\vec x,t) + \hat b_p^\dagger u_p^*(\vec x,t))$$

$$\hat\phi^\dagger(x,t) = \int d^3p\,(\hat a_p^\dagger u_p^*(\vec x,t) + \hat b_p u_p(\vec x,t))$$

生成湮灭算符对易：

$$[\hat a_p, \hat a_{p'}^\dagger] = [\hat b_p, \hat b_{p'}^\dagger] = \delta^{(3)}(p-p') \quad \text{others} = 0$$

Hamiltonian：

$$\hat H = :\int d^3p\; \omega_p\,(\hat a_p\hat a_p^\dagger + \hat b_p^\dagger\hat b_p):$$

$$= \int d^3p\; \omega_p\,(\hat a_p^\dagger\hat a_p + \hat b_p^\dagger\hat b_p)$$

动量：

$$\hat P = \int d^3p\; \vec p\,(\hat a_p^\dagger\hat a_p + \hat b_p^\dagger\hat b_p) = \int d^3p\; \vec p\,(\hat n_p^{(a)} + \hat n_p^{(b)})$$

角动量

$$\hat L = -i\int d^3p\cdot[\hat a_p^\dagger(p\times\nabla_p)\hat a_p + \hat b_p^\dagger(p\times\nabla_p)\hat b_p]$$

- 诺特定理，经典场的变化：

$$\phi' = \phi e^{i\alpha} \qquad \phi^{*\prime} = \phi^* e^{-i\alpha}$$

$$Q = \int d^3x\; j^0(x) = -i\int d^3x\cdot\left(\frac{\partial\mathcal{L}}{\partial\pi^*}\phi - \frac{\partial\mathcal{L}}{\partial\pi}\phi^*\right)$$

$$= -i\int d^3x\,(\pi\phi - \pi^*\phi^*) \qquad \pi = \dot\phi^* \quad \pi^* = \dot\phi$$

$$= i\int d^3x\; \phi^*\overleftrightarrow{\partial_0}\phi = (\phi,\phi)$$

量子：

$$\hat Q = -i\int d^3x\; :(\hat\pi\hat\phi - \hat\pi^\dagger\hat\phi^\dagger):$$

$$= \int d^3p\,(\hat a_p^\dagger\hat a_p - \hat b_p^\dagger\hat b_p)$$

对称变化.

- 坐标变换，w是表示变换的参数.

$$x \to x' = L(w, x)$$

场变换

$\phi(x) \to \phi'(x') = \Lambda(w)\phi(x)$ 　　$\phi(x)$是多值时，$\Lambda(w)$是矩阵.

态的变化：

$|\alpha\rangle \to |\alpha'\rangle = \hat{U}(w)|\alpha\rangle$ 　　保证内积不变 $\hat{U}^{\dagger}(w) = \hat{U}^{-1}(w)$

假设场变换满足

$$\langle \beta' | \hat{\phi}(x') | \alpha' \rangle = \Lambda(w) \langle \beta | \hat{\phi}(x) | \alpha \rangle \qquad \sim (0)$$

场算符变化

$$\begin{cases} \hat{U}^{-1}(w)\hat{\phi}(x')\hat{U}(w) = \Lambda(w)\hat{\phi}(x) \\ \hat{U}^{-1}(w)\hat{\phi}(x)\hat{U}(w) = \Lambda(w)\hat{\phi}(L^{-1}(w,x)) \end{cases}$$

$\underbrace{U(w_1)U(w_2) = U(w')}_{\Downarrow\text{坐标变换在Hilbert空间中表示}}$ 　where $w' = w_1 w_2$ 　$-(0.1)$

取：$\hat{U}(\delta w) = 1 - i\hat{G}(\delta w)$，$(\hat{G}^{\dagger}(\delta w) = \hat{G}(\delta w))$

则

$$i[\hat{\phi}(x), \hat{G}] = -\partial_\mu \hat{\phi}(x)\,\delta x^\mu + (\Lambda(\delta w) - 1)\hat{\phi}(x) \qquad -(1)$$

取：

$$\begin{cases} x'^\mu = x^\mu + \varepsilon^\mu + \delta w^\mu{}_\nu x^\nu & \text{(坐标变换)} \\ \hat{G} = -\varepsilon_\mu \hat{P}^\mu + \frac{1}{2}\delta w_{\mu\nu}\underset{\uparrow\text{对应经典场Noether定理中Lorentz转动的守恒荷.}}{\hat{M}^{\mu\nu}} & \text{(Ansatz)} \end{cases}$$

1° 纯平移. 当 $\Lambda(\delta w) = 1$ (pure translation) (1)式化作如下形式.

$$i[\hat{\phi}(x), \hat{P}^\mu] = -\partial^\mu \hat{\phi}(x) \qquad -(2)$$

当：$x'^\mu = x^\mu + a^\mu$ 时.

$$\hat{U}(a) = \exp(i a \cdot \hat{P})$$

(0.1)式变为：

$$\hat{\phi}(x+a) = e^{ia\hat{P}}\hat{\phi}(x)e^{-ia\hat{P}} \qquad -(3)$$

用(2)，也可看出(3)的右边是左边的Tylor展开！(证明见(4.85))

(0)对任何算符都应满足！

$$\partial_\mu \hat{F}(x) = i[\hat{P}_\mu, \hat{F}(x)] \quad ; \quad \hat{F}(x+a) = e^{ia\cdot\hat{P}}\hat{F}(x)e^{-ia\cdot\hat{P}}$$

2° Lorentz转动：

$$\Lambda(\delta w) = 1 + \frac{1}{2}\delta w_{\mu\nu} I^{\mu\nu}$$

$I^{\mu\nu}$ 是Lorentz群的生成元. $(I^{\mu\nu})_{r.s}$

(1)式变为：

$$i[\hat{\phi}(x), \hat{M}^{\mu\nu}] = (x^\nu\partial^\mu - x^\mu\partial^\nu)\hat{\phi}(x) - I^{\mu\nu}\hat{\phi}(x)$$

3° 内部自由度 internal degrees of freedom

$$\Lambda_{rs}(\varepsilon) = \delta_{rs} + i\varepsilon\lambda_{rs}$$

算符变化生成元：

$$\hat{G} = -\varepsilon\hat{Q}$$ ← 对应 Noether Theorem 中的 Q.

(0,1) 成为：

$$[\hat{\phi}_r(x), \hat{Q}] = \lambda_{rs}\hat{\phi}_s(x)$$

$$[\hat{H}, \hat{Q}] = 0$$

不变对易条件 Invariant Commutation Relation.

° Charged Klein-Gordon 场 场算符 $\hat\varphi$, $\hat\varphi^\dagger$ 之间的对易关系.

$$i\Delta(x-y) := [\hat\varphi(x), \hat\varphi^\dagger(y)] \quad —(1)$$

(4.58): $$\hat\varphi(x) = \int d^3p\,(\hat a_p u_p(x) + \hat b_p^\dagger u_p^*(x)) = \hat\varphi^{(+)}_{(x)} + \hat\varphi^{(-)}_{(x)} \quad —(2)$$

代入(1):

$$i\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) \qquad p^0 = \omega_p = \sqrt{\vec p^2+m^2}$$

$$= i\Delta^{(+)}(x-y) + i\Delta^{(-)}(x-y) \quad —(3)$$

$$\Delta(x-y) = -\int\frac{d^3p}{(2\pi)^3}\frac{\sin p\cdot(x-y)}{\omega_p} \quad \text{\} 视为定义式!} \quad —(4)$$

1° $\Delta(x-y)$ 是 Lorentz 不变的. 取 $z = x-y$

(3)

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{-i(\omega_p z_0 - \vec p\cdot\vec z)} - e^{i(\omega_p z_0 - \vec p\cdot\vec z)}\right)$$

$$= \int\frac{d^4p}{(2\pi)^3}\frac{1}{2\omega_p}\left(\delta(p_0-\omega_p) - \delta(p_0+\omega_p)\right)\cdot e^{-i(p_0 z_0 - \vec p\cdot\vec z)}$$

$$= \int\frac{d^4p}{(2\pi)^3}\frac{\mathrm{sgn}(p_0)}{2\omega_p}\left(\delta(p_0-\omega_p)+\delta(p_0+\omega_p)\right)e^{-ip\cdot z}$$

$$= \int\frac{d^4p}{(2\pi)^3}\,\mathrm{sgn}(p_0)\,\delta(p^2-m^2)\,e^{-ip\cdot z}$$

每一项都 Lorentz invariant.

$$\frac{1}{2\omega_p}\cdot\left(\delta(p_0-\omega_p)+\delta(p_0+\omega_p)\right) = \delta\left((p_0-\omega_p)(p_0+\omega_p)\right) = \delta(p_0^2-\omega_p^2)$$

$$= \delta(p_0^2 - \vec p^2 - m^2)$$

$$= \delta(p^2 - m^2)$$

2° $\Delta(x-y)$ 是奇 function.

由(4)

3° 边界条件. boundary condition at vanishing time difference:

a. $\Delta(0,\vec x) = 0$ 由(3) (将体积分转换为球积分)

b. $\frac{\partial}{\partial x^0}\Delta(x^0,\vec x)\big|_{x^0=0} = -\delta^{(3)}(\vec x)$

由(3)

$$\frac{\partial}{\partial x^0}\Delta(x^0,\vec x)\Big|_{x^0=0} = \frac{1}{i}\cdot\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\cdot\left(-i\omega_p\, e^{-ip\cdot x} - (i\omega_p)e^{ip\cdot x}\right)\Big|_{x_0=0}$$

$$= -\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left(e^{i\vec p\cdot\vec x} + e^{-i\vec p\cdot\vec x}\right) = -\delta^{(3)}(\vec x)$$

c. $\nabla\Delta(x^0,x)|_{x^0=0} = \int\frac{d^3p}{(2\pi)^3}\frac{\vec p\cos(\vec p\cdot\vec x)}{\omega_p} = 0$ (体积分化为球积分可证!)

d. $x^0=0$ 时 { i 所有空间求导 / ii 所有时间2阶导 } 都会消失.

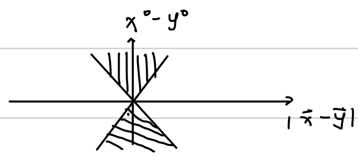
4° $\Delta(x)$ 满足各向同性 KG 方程. homogeneous K-G-Equation

$$(\Box+m^2)\Delta(x) = 0$$

因: $\Delta(x) = \frac{1}{i}[\hat\varphi(x), \hat\varphi^\dagger(0)]$, $\hat\varphi(x)$ 满足 KGE

5. $\Delta(x-y)=0$ for $(x-y)^2<0$ (类空间隔)

由 1°, 3°(a).



◦ 对于观测量 observable Greiner (4.118) / microcausality.

$$\hat{O}(x) = \hat{\phi}^\dagger(x)\, O(x)\, \hat{\phi}(x)$$

$$[\hat{O}(x), \hat{O}(y)] = O(x)O(y)\left(\hat{\phi}^\dagger(x)\hat{\phi}(y) + \hat{\phi}^\dagger(y)\hat{\phi}(x)\right) i\Delta(x-y). \begin{cases} =0 & (x-y)^2<0 \\ \neq 0 & (x-y)^2>0 \end{cases}$$

标量场 Feynman Propagator

○ Charged K-G 场：/ $\Delta_F(x-y)$ 的具体表达

$$i\Delta_F(x-y) = \langle 0|T(\hat\phi(x)\,\hat\phi^\dagger(y))|0\rangle$$

$$T(\hat A(x)\,\hat B(y)) = \hat A(x)\hat B(y)\Theta(x_0-y_0) \pm \hat B(y)\hat A(x)\,\Theta(y_0-x_0)$$

+：bosonic　　−：fermionic.

$x^0>y^0$ 时：$\langle 0|T(\hat\phi(x)\hat\phi^\dagger(y))|0\rangle = \langle 0|\hat\phi(x)\hat\phi^\dagger(y)|0\rangle = \langle 0|\hat\phi^{(+)}(x)\,\hat\phi^{\dagger(-)}(y)|0\rangle$　　$(x_0>y_0)$

$$\hat\phi^{(+)}(x) = \int d^3p\,\hat a_p\,u_p(x) \qquad \hat\phi^{\dagger(+)}(x) = \int d^3p\,\hat a_p^\dagger\,u_p^*(x)$$

$$\hat\phi^{(-)}(x) = \int d^3p\,\hat b_p^\dagger\,u_p^*(x) \qquad \hat\phi^{\dagger(-)}(x) = \int d^3p\,b_p\,u_p(x)$$

$$i\Delta_F(x-y) = \int d^3p\cdot u_p(x)\,u_p^*(y) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip\cdot(x-y)} = i\Delta^{(+)}(x-y)\quad(x_0>y_0)$$

$$= \langle 0|\hat\phi^\dagger(y)\hat\phi(x)|0\rangle = \langle 0|\hat\phi^\dagger(y)\hat\phi^{(-)}(x)|0\rangle = \int d^3p\cdot u_p(y)u_p^*(x)$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\cdot e^{ip\cdot(x-y)}$$

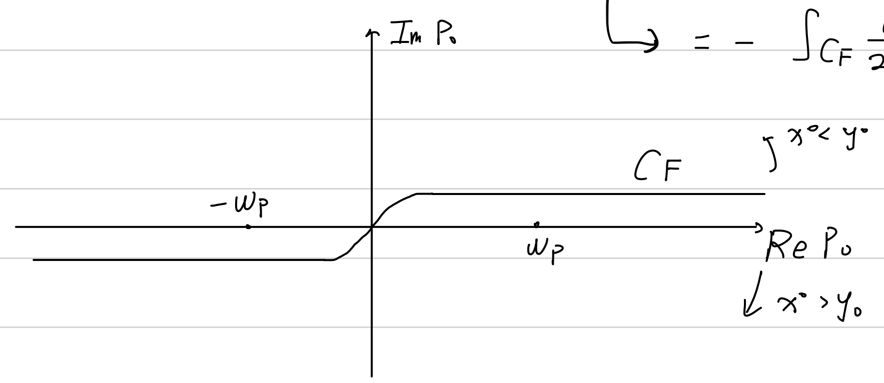
$$= -i\Delta^{(-)}(x-y)\qquad (x_0<y_0)$$

$$i\Delta_F(x-y) = \Theta(x_0-y_0)\,i\Delta^{(+)}(x-y) - \Theta(y_0-x_0)\,i\Delta^{(-)}(x-y)$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(\Theta(x_0-y_0)e^{-ip\cdot(x-y)} + \Theta(y_0-x_0)e^{ip\cdot(x-y)}\right) \longleftarrow p=(\omega_p,\vec p)$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\underbrace{\left(\Theta(x_0-y_0)e^{-i\omega_p(x^0-y^0)}\cdot e^{i\vec p\cdot(\vec x-\vec y)} + \Theta(y_0-x_0)e^{i\omega_p(x^0-y^0)}e^{i\vec p\cdot(\vec x-\vec y)}\right)}$$

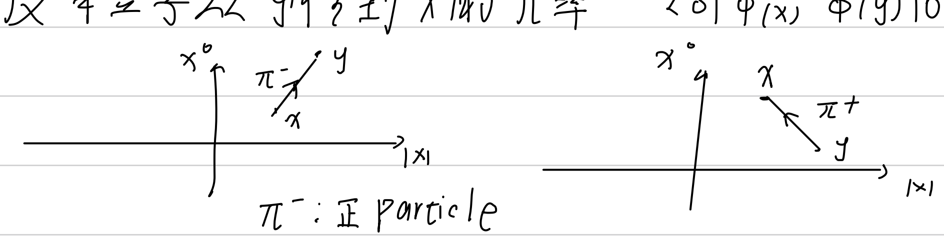
$$\hookrightarrow = -\int_{C_F}\frac{dp^0}{2\pi i}\frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p)(p^0+\omega_p)}\cdot e^{i\vec p\cdot\vec x}$$



$$\Delta_F(x-y) = \int_{C_F}\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2-m^2} = \int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{1}{p^2-m^2+i\epsilon}$$

$\langle 0|T(\phi(y)\,\phi^\dagger(x))|0\rangle$

○ 物理解释：$\hat\phi^\dagger(x)$ 在 Charged KG 场中代表生成了一个粒子，在 x. 则 $\langle 0|\hat\phi(y)\hat\phi^\dagger(x)|0\rangle$ 代表 x 生成的粒子和 y 生成粒子内积. 若 $x^0<y^0$，代表正粒子从 x 传到 y 的几率！ $x^0<y^0$，代表反粒子从 y 传到 x 的几率　$\langle 0|\hat\phi^\dagger(x)\,\hat\phi(y)|0\rangle$



π^-：正 particle　　π^+：反粒子.

Spin-1/2 Field.

Dirac Equation

- Dirac Equation　γ是一个4维矩阵.

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

(5.2)式下面一段说 ψ 有4个变量,满足相对论旋量的变化规律.

Lagrangian (自带charge)

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = i\psi^\dagger\dot{\psi} + i\psi^\dagger\vec{\alpha}\cdot\nabla\psi - m\psi^\dagger\beta\psi$$

$$\bar{\psi} = \psi^\dagger\gamma^0 \quad \beta = \gamma^0 \quad \beta^2 = 1 \quad \vec{\alpha} = \gamma_0 \cdot \vec{\gamma}$$

正则共轭场:

$$\pi_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger \qquad \pi_{\psi^\dagger} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}^\dagger} = 0$$

Hamiltonian

$$\mathcal{H} = \psi^\dagger(-i\vec{\alpha}\cdot\nabla + \beta m)\psi$$

$$H = \int d^3x\, \psi^\dagger(x,t)(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi(x,t)$$

为保证 $\mathcal{L}^* = \mathcal{L}$

$$\mathcal{L}' \equiv \frac{1}{2}(\mathcal{L} + \mathcal{L}^\dagger) = \frac{1}{2}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}\bar{\psi}(-i\gamma^\mu\overleftarrow{\partial}_\mu - m)\psi$$

$$= \frac{1}{2}\bar{\psi}i\gamma^\mu\overleftrightarrow{\partial}_\mu\psi - m\bar{\psi}\psi$$

$$\pi_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger \qquad \pi_{\psi^\dagger} = 0$$

Commutation between Hamiltonian & helicity op

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$
$$(i\gamma^{0}\partial_{0} + i\gamma^{j}\partial_{j} - m)\psi = 0$$
$$i\partial_{0}\psi + (i\gamma^{0}\gamma^{j}\partial_{j} - m\gamma^{0})\psi = 0$$
$$i\partial_{0}\psi = \boxed{-(i\gamma^{0}\gamma^{j}\partial_{j} - m\gamma^{0})}\psi$$

$$\hat{H} = -(i\gamma^{0}\gamma^{j}\partial_{j} - m\gamma^{0})\psi = (m\gamma^{0} - i\gamma^{0}\gamma^{j}\partial_{j})\psi.$$
$$\psi = \exp(ip\cdot x)\, u(p)$$
$$\hat{H} = (m\gamma^{0} - i\gamma^{0}\gamma^{j}(ip_j)) = m\gamma^{0} + \gamma^{0}\gamma^{j}p_j$$

$$\gamma^{0} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \qquad \gamma^{0}\gamma^{j} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^{j} \\ -\sigma^{j} & 0 \end{bmatrix} = \begin{bmatrix} -\sigma^{j} & 0 \\ 0 & \sigma^{j} \end{bmatrix}$$

$$H = m\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^{j} \\ -\sigma^{j} & 0 \end{bmatrix} p_j$$
$$= m\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + \begin{bmatrix} -\sigma^{j} & 0 \\ 0 & \sigma^{j} \end{bmatrix} p_j$$

$$= \begin{bmatrix} -\sigma^{j}p_j & m \\ m & \sigma^{j}p_j \end{bmatrix} = \begin{bmatrix} \vec{\sigma}\cdot\vec{p} & m \\ m & -\vec{\sigma}\cdot\vec{p} \end{bmatrix}$$

Commute

$$[H, \Sigma\cdot P] = \left[\begin{bmatrix} \vec{\sigma}\cdot\vec{p} & m \\ m & -\vec{\sigma}\cdot\vec{p} \end{bmatrix}, \begin{bmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{bmatrix}\right]$$

$$= \begin{bmatrix} (\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) & m\vec{\sigma}\cdot\vec{p} \\ m\vec{\sigma}\cdot\vec{p} & -(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) \end{bmatrix} - \begin{bmatrix} (\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) & m(\vec{\sigma}\cdot\vec{p}) \\ m\vec{\sigma}\cdot\vec{p} & -(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) \end{bmatrix}$$
$$= 0$$

Ehrenfest's Theorem (接 Latex笔记 QFT 中的 Dirac 方程部分)

- sign operator (符号算符)

$$\Lambda = \frac{H_D}{\sqrt{H_D^2}} = \frac{\vec{\alpha}\cdot\hat{\vec{P}} + \beta m_0}{\sqrt{H_D^2}} \qquad \Lambda = \Lambda^\dagger = \Lambda^{-1}$$

在动量表示中：

$$\Lambda = \frac{\vec{\alpha}\cdot\vec{P} + \beta m_0}{E_P}$$

Sign op 作用于本征态 $\Psi_{\vec{P},\lambda,\lambda_s}$ 上：

$$\Lambda \Psi_{\vec{P},\lambda,\lambda_s} = \lambda \Psi_{\vec{P},\lambda,\lambda_s}$$

定义 λ-state：

$$\Psi_\lambda = \sum_{\lambda_s} \int d^3p \; A_{\lambda_s}(\vec{P})\cdot \Psi_{\vec{P},\lambda,\lambda_s}$$

- 投影算符：

$$\Lambda_\pm = \frac{1}{2}(1 \pm \Lambda)$$

- 定义 Even/odd operator

Even op ： $\lambda = +1/-1 \longrightarrow \lambda = +1/-1$

Odd op ： $\lambda = \pm 1 \longrightarrow \lambda = \mp 1$

$$A = \overset{\text{Even}}{[A]} + \overset{\text{odd}}{\{A\}}$$

$$A\Psi_+ = [A]\Psi_+ + \{A\}\Psi_+$$

$$A\Psi_- = [A]\Psi_- + \{A\}\Psi_-$$

$$\Lambda A \Lambda \Psi_+ = \Lambda([A]\Psi_+ + \{A\}\Psi_+) = [A]\Psi_+ - \{A\}\Psi_+$$

$$\Lambda A \Lambda \Psi_- = \Lambda(-[A]\Psi_- - \{A\}\Psi_-) = [A]\Psi_- - \{A\}\Psi_-$$

$$[A] = \frac{1}{2}(A + \Lambda A \Lambda)$$

$$\{A\} = \frac{1}{2}(A - \Lambda A \Lambda)$$

- Heisenberg picture / velocity operator：

$$\frac{d\hat{\vec{x}}}{dt} = \frac{1}{i}\cdot[\hat{\vec{x}}, H_D]$$

$$= \frac{1}{i}\cdot[\hat{\vec{x}}, \vec{\alpha}\cdot\hat{\vec{P}} + m_0\beta]$$

$$= \frac{1}{i}[\hat{\vec{x}}, \vec{\alpha}\cdot\hat{\vec{P}}]$$

$$\frac{d\hat{x}_i}{dt} = \frac{1}{i}\cdot[x_i, \alpha_i \hat{P}_i] = \alpha_i$$

$$\frac{d\hat{\vec{x}}}{dt} = \vec{\alpha}$$

$\Leftarrow$ $\{\alpha_i, \alpha_j\} = 0 \quad (i \neq j)$ $\qquad \alpha_i\alpha_i = \mathbb{I} \longrightarrow \alpha_i$ 的本征值是 ± 1

$\{\alpha_i, \beta\} = 0 \qquad \beta\beta = \mathbb{I}$

$\frac{d\hat{x}_i}{dt}$ 的本征值是 $1 \Rightarrow c\cdot 1 = c \Rightarrow$ 速度本征值是光速 → 不合理.

— $[\vec{\alpha}]$

$$[\alpha_i] = \frac{1}{2}(\alpha_i + \Lambda\alpha_i\Lambda) = \frac{1}{2}\left(\alpha_i + \frac{\vec{\alpha}\cdot\hat{\vec{P}} + \beta m_0}{\sqrt{H_D^2}}\cdot\alpha_i\cdot\frac{\vec{\alpha}\cdot\hat{\vec{P}} + \beta m_0}{\sqrt{H_D^2}}\right)$$

$$(\vec{\alpha}\cdot\hat{\vec{P}} + \beta m_0)\alpha_i \underset{\{\alpha_i,\alpha_j\}=0 \;\; \{\alpha_i,\beta\}=0}{=} -\alpha_i(\vec{\alpha}\cdot\hat{\vec{P}} + \beta m_0) + 2\alpha_i\alpha_i\hat{P}_i \underset{\alpha_i\alpha_i = \mathbb{I}}{=} -\alpha_i H_D + 2\hat{P}_i \qquad \{\alpha_i, H_D\} = 2\hat{P}_i$$

$$= \frac{1}{2}\left(\alpha_i + \frac{(-\alpha_i H_D + 2\hat{P}_i)H_D}{H_D^2}\right) = \frac{1}{2}\left(\alpha_i - \alpha_i + 2\frac{\hat{P}_i}{\sqrt{H_D^2}}\cdot\frac{H_D}{\sqrt{H_D^2}}\right) = \frac{\hat{P}_i}{\sqrt{H_D^2}}\cdot\Lambda$$

$$[\vec{\alpha}] = \frac{\hat{\vec{p}}}{\sqrt{H_D^2}} \cdot \Lambda$$

$$\left[\frac{d\hat{\vec{x}}}{dt}\right] = [\vec{\alpha}] = \frac{\hat{\vec{p}}}{\sqrt{H_D^2}} \cdot \Lambda \Rightarrow \left[\frac{d\hat{\vec{x}}}{dt}\right]\Psi_\lambda = \frac{\hat{\vec{p}}}{E_p}\cdot\lambda\cdot\Psi_\lambda \begin{cases} = +\dfrac{c\hat{\vec{p}}}{E_p}\Psi_+ & \text{positive-free-solution} \\ = -\dfrac{c\hat{\vec{p}}}{E_p}\Psi_- & \text{negative-free-solution} \end{cases}$$

positive 的结果和经典很像. negative 多了一个负号!

- Wave Packet of Dirac Waves. ⟶ 波包中心遵从经典轨迹!

$$\frac{d\hat{\vec{x}}}{dt} = \frac{1}{i}[\hat{\vec{x}}, H_D] = \frac{1}{i}[\hat{\vec{x}}, \vec{\alpha}\cdot\hat{\vec{p}} + \beta m_0] = \vec{\alpha}$$

$$\frac{d\alpha_i}{dt} = \frac{1}{i}[\alpha_i, H_D] = \frac{1}{i}(\alpha_i H_D - H_D\alpha_i) = \frac{1}{i}(2\alpha_i H_D - H_D\alpha_i - \alpha_i H_D)$$

$$= \frac{1}{i}\left(2\alpha_i H_D - \alpha_i(\vec{\alpha}\cdot\hat{\vec{p}} + \beta m_0) - (\vec{\alpha}\cdot\hat{\vec{p}} + \beta m_0)\alpha_i\right)$$

$$= \frac{1}{i}\left(2\alpha_i H_D - \alpha_i(\sim) + \alpha_i(\vec{\alpha}\cdot\hat{\vec{p}} + \beta m_0) - 2\hat{p}_i\right)$$

$$= \frac{2\alpha_i}{i} H_D - \frac{2}{i}\hat{p}_i$$

$$\frac{d\vec{\alpha}}{dt} = 2i\hat{\vec{p}} - 2i\vec{\alpha} H_D$$

$$= \left(-\frac{\hat{\vec{p}}}{H_D} + \vec{\alpha}\right)(-2iH_D)$$

$$\alpha(t) = \frac{\hat{\vec{p}}}{H_D} + \left(\vec{\alpha}(0) - \frac{\hat{\vec{p}}}{H_D}\right)\cdot\exp(-2iH_D t)$$

$$\hat{x}(t) = \hat{\vec{x}}(0) + \frac{\hat{\vec{p}}}{H_D}t + \frac{i}{2H_D}\left(\vec{\alpha}(0) - \frac{\hat{\vec{p}}}{H_D}\right)\cdot\left(\exp(-2iH_D t) - 1\right)$$

最后一项是振动项.(可证明,振动项对 $\Psi_\pm$ 作用期望值为0:

$$\Lambda_\pm\left(\vec{\alpha}(0) - \frac{\hat{\vec{p}}}{H_D}\right)\Lambda_\pm = 0$$

以 Λ_+ 为例: $\Lambda_+ = \frac{1}{2}(1+\Lambda)$. $\quad \vec{\alpha}(0) = \vec{\alpha}$

$$(1+\Lambda)\left(\vec{\alpha}(0) - \frac{\hat{\vec{p}}}{H_D}\right)(1+\Lambda) = \left(1 + \frac{H_D}{\sqrt{H_D^2}}\right)\left(\vec{\alpha} + \vec{\alpha}\frac{H_D}{\sqrt{H_D^2}} - \frac{\hat{\vec{p}}}{H_D} - \frac{\hat{\vec{p}}}{H_D}\cdot\Lambda\right)$$

$$= (1+\Lambda)\left((1-\Lambda)\vec{\alpha} + \frac{2\hat{\vec{p}}}{\sqrt{H_D^2}} - (1+\Lambda)\frac{\hat{\vec{p}}}{H_D}\right)$$

$$= (1-\Lambda\Lambda)\vec{\alpha} + 2(1+\Lambda)\frac{\hat{\vec{p}}}{\sqrt{H_D^2}} - (1+\Lambda^2+2\Lambda)\frac{\hat{\vec{p}}}{H_D}$$

$$= 2(1+\Lambda)\left(\frac{\hat{\vec{p}}}{\sqrt{H_D^2}} - 2\frac{\hat{\vec{p}}}{H_D}\right)$$

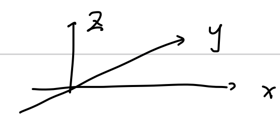
由于:

$$\langle -|\Lambda_+ \sim \Lambda_+|-\rangle = 0 \Rightarrow H_D = \sqrt{H_D^2} \Rightarrow (1+\Lambda)\sim(1+\Lambda) = 0$$

Solution of Dirac Equation from Lorentz Transformation!

Plane Wave in arbitrary direction.

- Rest Frame solution(静粒子解)



静系 rest frame 中静止自由粒子 Dirac Equation:

$$\begin{cases} (i\gamma^\mu \partial_\mu - m_0)\psi = 0 \\ (\vec{\alpha}\cdot\hat{\vec{p}} + \beta m)\psi = E_p\psi = i\frac{\partial}{\partial t}\psi \end{cases}$$

rest condition: β 具体表示:

$$\begin{cases} \hat{\vec{p}}\,\psi^r = 0 \qquad \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \\ \beta m\,\psi^r = i\frac{\partial}{\partial t}\psi^r \end{cases}$$

$$m\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}\psi^r = i\frac{\partial}{\partial t}\psi^r$$

$$\Downarrow$$

$$\psi^r = w^r(0)\cdot\exp(-i\varepsilon_r\cdot m_0 t)$$

$$w^1(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad w^2(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad w^3(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad w^4(0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\varepsilon_r = \begin{cases} +1 & r = 1 \text{ or } 2 \text{ 正能解} \\ -1 & r = 3 \text{ or } 4 \text{ 负能解} \end{cases}$$

- plane Wave for moving particle. 由静止粒子解到运动粒子解.

— S 和 Λ 间关系

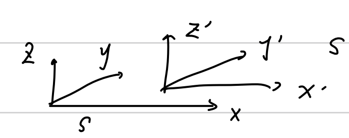
Minimal Lorentz Transformation leads to S op:

$$S = \mathbb{I} - \frac{i}{4}\cdot\sigma_{\mu\nu}\cdot\Delta w^{\mu\nu}$$

Minimal Lorentz Transformation represented by Generator.

$$\Lambda(\Delta w)^{\mu\nu} = \mathbb{I} + \Delta w^{\mu\nu} = \mathbb{I} + w_1 I_1 \cdots + w_6 I_6$$

不考虑转动: → Lorentz Trans "$\frac{w}{N}$"



$$\begin{cases} \Lambda(\Delta w) = \mathbb{I} + \Delta w = \mathbb{I} + w_1 I_1 + w_2 I_2 + w_3 I_3 \\ \qquad = \mathbb{I} + \frac{w}{N}(\cos\alpha I_1 + \cos\beta I_2 + \cos\gamma\cdot I_3) \\ S(\Delta w) = \mathbb{I} - \frac{i}{4}\sigma_{\mu\nu}\left(\frac{w}{N}(\cos\alpha I_1^{\mu\nu} + \cos\beta I_2^{\mu\nu} + \cos\gamma I_3^{\mu\nu})\right) \end{cases}$$

→ 代表 S' 相对 S velocity $\frac{w}{N}(\cos\alpha, \cos\beta, \cos\gamma)$ (N 很大)

将上面微变动做 N 次

$$\begin{cases} \Lambda(w) = \exp(w(\cos\alpha I_1 + \cos\beta I_2 + \cos\gamma I_3)) \\ S(w) = \exp\left(-\frac{i}{4}\sigma_{\mu\nu}\, w\cdot(\cos\alpha I_1^{\mu\nu} + \cos\beta I_2^{\mu\nu} + \cos\gamma I_3^{\mu\nu})\right) \end{cases}$$

Lorentz Trans 中 w 和 β 的关系:

$\tanh(w) = \beta$ (若仅有 V_x, 易看出.) ← ? Latex Lorentz Trans 知识!

Lorentz Trans 生成元 operator

$$I_1^{\mu\nu} = \begin{pmatrix} 0&1&0&0\\ -1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \quad I_2^{\mu\nu} = \begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ -1&0&0&0\\ 0&0&0&0 \end{pmatrix} \quad I_3^{\mu\nu} = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ -1&0&0&0 \end{pmatrix}$$

— σ 表达   和理的 $\sigma_{\mu\nu}$ 表达式: (Latex QFT 知识)

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \qquad \gamma^0 = \beta \quad \gamma^1 = \beta\alpha_1 \quad \gamma^2 = \beta\alpha_2 \quad \gamma^3 = \beta\alpha_3$$

$$\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0&1\\1&0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0&-i\\i&0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1&0\\0&-1 \end{pmatrix}$$

— $S(w)$的表达.

$$S(w) = \exp\left[-\frac{i}{4} w\, \sigma_{\mu\nu} \cdot \begin{pmatrix} 0 & \cos\alpha & \cos\beta & \cos\gamma \\ -\cos\alpha & 0 & 0 & 0 \\ -\cos\beta & 0 & 0 & 0 \\ -\cos\gamma & 0 & 0 & 0 \end{pmatrix}\right]$$

$$= \exp\left(-\frac{i}{4} w\cdot 2\cdot (\sigma_{01}\cos\alpha + \sigma_{02}\cos\beta + \sigma_{03}\cos\gamma)\right)$$

由 $\sigma_{\mu\nu}$ 的定义：

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

$$\sigma_{0i} = \frac{i}{2}[\gamma_0, \gamma_i] = -\frac{i}{2}[\gamma^0, \gamma^i] = -\frac{i}{2}[\beta, \beta\alpha_i] = -\frac{i}{2}(\beta\beta\alpha_i - \beta\alpha_i\beta)$$

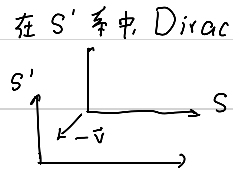
$$\Leftarrow \begin{cases} \{\alpha_i, \beta\} = 0 \\ \beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1, \quad \{\alpha_i, \alpha_j\} = 0 \end{cases}$$

$$= -\frac{i}{2}(\alpha_i + \alpha_i) = -i\alpha_i$$

$$S(w) = \exp\left(-\frac{i}{4} w\cdot 2\cdot(-i)(\alpha_x\cdot\cos\alpha + \alpha_y\cos\beta + \alpha_z\cos\gamma)\right)$$

$$= \exp\left(+\frac{1}{2} w\cdot \vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right).$$

其中，$\vec{v}$是在S'系中Dirac粒子的运动velucity! $\left((\cos\alpha, \cos\beta, \cos\gamma) = -\frac{\vec{v}}{|\vec{v}|} \quad \tanh w = \beta > 0\right)$



— $S(w)$的具体表达

$$S(w) = \exp\left(+\frac{1}{2} w\, \vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right) = 1 + \frac{1}{2} w\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|} + \frac{1}{2!}\cdot\left(\frac{1}{2} w\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right)^2 \cdots$$

$$= 1 + \frac{1}{2} w\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|} + \frac{1}{2}\frac{w^2}{4|\vec{v}|^2}(\vec{\alpha}\cdot\vec{v})^2 + \cdots$$

$$\Leftarrow \begin{cases} (\vec{\alpha}\cdot\vec{v})^2 = (\alpha_i v_i)(\alpha_j v_j) = \alpha_i\alpha_j v_i v_j = \frac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i) v_i v_j \\ \quad = \frac{1}{2}\{\alpha_i, \alpha_j\} v_i v_j \\ \quad = \frac{1}{2}\cdot 2\cdot\delta_{ij}\cdot v_i v_j \\ \quad = v_i v_i \\ \quad = |\vec{v}|^2 \end{cases}$$

$$\left(\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right)^2 = 1$$

$$= 1 + \frac{1}{1!}\cdot\left(\frac{1}{2} w\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right) + \frac{1}{3!}\left(\frac{1}{2} w\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right)^3 + \frac{1}{5!}\cdot\left(\frac{1}{2} w\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right)^5 \cdots$$

$$+ \frac{1}{2!}\left(\frac{1}{2} w\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right)^2 + \frac{1}{4!}\cdot\left(\frac{1}{2} w\cdot\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right)^4 + \frac{1}{6!}\left(\frac{1}{2} w\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right)^6 \cdots$$

$$= 1 + \left(\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right)\left(\frac{1}{2}w + \frac{1}{3!}\left(\frac{1}{2}w\right)^3 + \frac{1}{5!}\left(\frac{1}{2}w\right)^5 \cdots\right)$$

$$+ \left(\frac{1}{2!}\cdot\left(\frac{1}{2}w\right)^2 + \frac{1}{4!}\left(\frac{1}{2}w\right)^4 + \frac{1}{6!}\left(\frac{1}{2}w\right)^6 \cdots\right)$$

$$= \left(\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|}\right)\cdot\sinh\left(\frac{1}{2}w\right) + \cosh\left(\frac{1}{2}w\right)$$

$$\tanh(w) = \beta = \frac{\sinh(w)}{\cosh(w)} \;;\quad 2\cosh\left(\frac{w}{2}\right)\sinh\left(\frac{w}{2}\right) = \sinh(w) \;;\quad \cosh(w) = 2\cosh^2\left(\frac{w}{2}\right) - 1 = 2\sinh^2\left(\frac{w}{2}\right) + 1$$

$$\cosh^2\left(\frac{w}{2}\right) = 1 + \sinh^2\left(\frac{w}{2}\right)$$

$$\Downarrow$$

$$\beta = \frac{2\cosh\left(\frac{w}{2}\right)\sinh\left(\frac{w}{2}\right)}{2\cosh^2\left(\frac{w}{2}\right) - 1}$$

$$= \frac{2\cosh\left(\frac{w}{2}\right)\sqrt{\cosh^2\left(\frac{w}{2}\right) - 1}}{2\cosh^2\left(\frac{w}{2}\right) - 1}$$

$$\beta^2(2\cosh^2(\tfrac{w}{2})-1)^2 = 4\cosh^2(\tfrac{w}{2})(\cosh^2(\tfrac{w}{2})-1)$$

$$\beta^2(2x-1)^2 = 4x\cdot(x-1) \qquad \cosh^2(\tfrac{w}{2})=x$$

$$\beta^2(4x^2+1-4x) = 4x^2-4x$$

$$4(1-\beta^2)x^2+4(\beta^2-1)x-\beta^2=0$$

$$x = \frac{-4(\beta^2-1)\pm\sqrt{16(\beta^2-1)^2+16\beta^2(1-\beta^2)}}{8(1-\beta^2)}$$

$$= \frac{-4(\beta^2-1)\pm\sqrt{16\beta^4+16-32\beta^2+16\beta^2-16\beta^4}}{8(1-\beta^2)}$$

$$= \frac{-4(\beta^2-1)\pm\sqrt{16(1-\beta^2)}}{8(1-\beta^2)}$$

$$= \frac{4\pm4\sqrt{1-\beta^2}\frac{1}{1-\beta^2}}{8}$$

$$= \frac{1}{2}\left(1\pm\frac{1}{\sqrt{1-\beta^2}}\right) = \frac{1}{2}\left(1+\frac{1}{\sqrt{1-\beta^2}}\right)$$

$$\cosh^2(\tfrac{w}{2}) = \tfrac{1}{2}(1+\tfrac{E}{m_0}) = \frac{E+m_0}{2m_0}$$

$$\cosh(\tfrac{w}{2}) = \sqrt{\frac{E+m_0}{2m_0}}$$

$$\beta=\tanh(w) = \frac{2\cosh(\frac{w}{2})\sinh(\frac{w}{2})}{\cosh^2(\frac{w}{2})+\sinh^2(\frac{w}{2})} = \frac{2\tanh(\frac{w}{2})}{1+\tanh^2(\frac{w}{2})}$$

$$\Downarrow x=\tanh(w/2)$$

$$\beta\cdot x^2-2x+\beta=0$$

$$x = \frac{4\pm\sqrt{4-4\beta^2}}{2\beta}$$

$$= \frac{1\pm\sqrt{1-\beta^2}}{\beta} \qquad (\because \tanh\tfrac{w}{2}<1)$$

$$= \frac{1-\sqrt{1-\beta^2}}{\beta} = \frac{1-\frac{m_0}{E}}{|\vec{p}|\cdot\frac{1}{E}} = \frac{E-m_0}{|\vec{p}|}$$

$$\tanh(\tfrac{w}{2}) = \frac{E-m_0}{|\vec{p}|} = \frac{(E-m_0)|\vec{p}|}{|\vec{p}|^2} = \frac{|\vec{p}|}{E+m_0}$$

$$\cosh(\tfrac{w}{2}) = \sqrt{\frac{E+m}{2m}} \qquad \sinh(\tfrac{w}{2}) = \cosh(\tfrac{w}{2})\tanh(\tfrac{w}{2}) = \sqrt{\frac{E+m}{2m}}\cdot\frac{|\vec{p}|}{E+m_0}$$

$$= (\vec{\alpha}\cdot\frac{\vec{v}}{|\vec{v}|})\cdot\sinh(\tfrac{w}{2})+\cosh(\tfrac{w}{2}) \qquad \alpha_i=\begin{pmatrix}0&\sigma_i\\ \sigma_i&0\end{pmatrix} \quad \alpha_x=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix} \quad \alpha_y=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix} \quad \alpha_z=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}$$

$$= \sqrt{\frac{E+m}{2m}}\cdot\mathbb{I}+\sqrt{\frac{E+m}{2m}}\cdot\frac{|\vec{p}|}{E+m}\begin{pmatrix}0&0&\frac{p_z}{|p|}&\frac{p_x-ip_y}{|p|}\\0&0&\frac{p_x+ip_y}{|p|}&-\frac{p_z}{|p|}\\\frac{p_z}{|p|}&\frac{p_x-ip_y}{|p|}&0&0\\\frac{p_x-ip_y}{|p|}&-\frac{p_z}{|p|}&0&0\end{pmatrix}$$

$$= \sqrt{\frac{E+m}{2m}}\begin{pmatrix}1&0&\frac{p_z}{E+m}&\frac{p_-}{E+m}\\0&1&\frac{p_+}{E+m}&-\frac{p_z}{E+m}\\\frac{p_z}{E+m}&\frac{p_-}{E+m}&1&0\\\frac{p_+}{E+m}&-\frac{p_z}{E+m}&0&-1\end{pmatrix} \qquad \left\{\begin{aligned}p_+&=p_x+ip_y\\p_-&=p_x-ip_y\end{aligned}\right.$$

对于S'系中的spinor（表示$\vec{p}$粒子的Spinor）：

$$\psi^{r\prime}(x') = S\cdot\psi^r(x)$$

$$= S\,w^r(0)\cdot\exp(-i\varepsilon_r\cdot m_0 t)$$

$$= S\,w^r(0)\cdot\exp(-i\varepsilon_r\cdot p_\mu x^\mu) = S\,w^r(0)\cdot\exp(-i\varepsilon_r p'_\mu x'^\mu)$$

这里 P_μ 和 P_μ' 是同一个.

$$= \sqrt{\frac{E+m}{2m}} \cdot \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_-}{E+m} \\ 0 & 1 & \frac{p_+}{E+m} & -\frac{p_z}{E+m} \\ \frac{p_z}{E+m} & \frac{p_-}{E+m} & 1 & 0 \\ \frac{p_+}{E+m} & -\frac{p_z}{E+m} & 0 & 1 \end{pmatrix} \cdot W^r(0) \cdot \exp(-i\varepsilon_r P_\mu x'^\mu)$$

$$= W^r(\vec{p}) \exp(-i\varepsilon_r P_\mu \cdot x'^\mu)$$

$$[W^1(\vec{p}), W^2(\vec{p}), W^3(\vec{p}), W^4(\vec{p})] = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_-}{E+m} \\ 0 & 1 & \frac{p_+}{E+m} & -\frac{p_z}{E+m} \\ \frac{p_z}{E+m} & \frac{p_-}{E+m} & 1 & 0 \\ \frac{p_+}{E+m} & -\frac{p_z}{E+m} & 0 & 1 \end{pmatrix}$$

- $W^r(\vec{p})$ 满足： $(\not{p} - \varepsilon_r m) W^r(\vec{p}) = 0$

proof:

在 S' 系中, Dirac Equation 能够得到满足!

$$i(\gamma^\mu \partial_\mu - m)\psi = 0$$

而：

$$\psi^r(x) = W^r(\vec{p}) \cdot \exp(-i\varepsilon_r P_\mu x^\mu)$$

1° $\varepsilon_r = +1$ $(r = 1 \text{ or } 2)$:

$$(\gamma^\mu P_\mu - m) W^r(\vec{p}) = 0$$

2° $\varepsilon_r = -1$ $(r = 3 \text{ or } 4)$:

$$(-\gamma^\mu P_\mu - m) W^r(\vec{p}) = 0 \Rightarrow (\gamma^\mu P_\mu + m) W^r(\vec{p}) = 0$$

$$(\gamma^\mu P_\mu - \varepsilon_r m) W^r(\vec{p}) = 0 \qquad (\not{p} - \varepsilon_r m) W^r(\vec{p}) = 0$$

End of Proof.

- $W^r(\vec{p})$ 满足： $\overline{W}^r(\vec{p})(\not{p} - \varepsilon_r m_0 c) = 0$

Proof: 由上一个 $W^r(\vec{p})$ 的性质：

$$(\not{p} - \varepsilon_r m) W^r(\vec{p}) = 0$$

Complex Conjugate :

$$W^{r\dagger}(\vec{p})(\not{p} - \varepsilon_r m)^\dagger = 0$$

$$W^{r\dagger}(\vec{p})(\gamma^{\mu\dagger} P_\mu - \varepsilon_r m) = 0$$

$$\Downarrow \longleftarrow \quad \gamma^{0\dagger} = \gamma^0 \quad \gamma^{k\dagger} = -\gamma^k$$

$$W^{r\dagger}(\vec{p})(\gamma^0 P_0 - \gamma^1 P_1 - \gamma^2 P_2 - \gamma^3 P_3 - \varepsilon_r m) = 0$$

$$\Downarrow \leftarrow \quad \gamma^0 \cdot \gamma^0 = \mathbb{I} \quad \{\gamma^0, \gamma^k\} = 0$$

$$W^{r\dagger}(\vec{p})(\gamma^0 P_0 - \gamma^1 P_1 - \gamma^2 P_2 - \gamma^3 P_3 - \varepsilon_r m) \cdot \gamma^0$$

$$= W^{r\dagger}(\vec{p})\gamma^0(\gamma^0 P_0 + \gamma^1 P_1 + \gamma^2 P_2 + \gamma^3 P_3 - \varepsilon_r m)$$

$$= \overline{W}(\vec{p})(\not{p} - m_0) = 0$$

End of Proof

- Normalisation relation (Direct calculation!)

$$\overline{w}^{r}(\vec{p})\, w^{r'}(\vec{p}) = \delta_{rr'}\, \varepsilon_r$$

$$w^{r\dagger}(\varepsilon_r\vec{p})\, w^{r'}(\varepsilon_{r'}\vec{p}) = \frac{E}{m}\delta_{rr'}$$

这样的正交性保证了粒子spinor之间的正交性：

$$\psi^{1,2}_{\vec{p}}(x) = w^{1,2}(\vec{p})\exp(-i p_0 x^0 + i\vec{p}\cdot\vec{x}) \rightarrow \text{能量 } p_0,\ \text{动量 } \vec{p}$$

$$\psi^{3,4}_{-\vec{p}}(x) = w^{3,4}(-\vec{p})\exp(i p_0 x^0 + i\vec{p}\cdot\vec{x}) \rightarrow \text{能量} -p_0,\ \text{动量 } \vec{p}$$

$$\langle \psi^{r'}_{p'} | \psi^{r}_{p}\rangle = \int d^3x\ w^{r'\dagger}(\vec{p}')\, w^{r}(\vec{p})\cdot \exp\big(-i(\varepsilon_r p_\mu x^\mu - \varepsilon_{r'} p'_\mu x^\mu)\big)$$

$$= w^{r'\dagger}(\vec{p}')\, w^{r}(\vec{p})\cdot (2\pi)^3\cdot \delta^{(3)}(\varepsilon_r\vec{p} - \varepsilon_{r'}\vec{p}')\cdot \exp\big(-i(\varepsilon_r p_0 - \varepsilon_{r'} p_0')x^0\big)$$

$$= \frac{E}{m}(2\pi)^3\cdot\delta^{(3)}(\vec{p}' - \vec{p})\,\delta_{r,r'}$$

- closure relation：(24)

$$\sum_{r=1}^{4} \varepsilon_r w^{r}_{\alpha}(\vec{p})\, \overline{w}^{r}_{\beta}(\vec{p}) = \delta_{\alpha\beta}$$

$$\sum_{r=1}^{4} w^{r}_{\alpha}(\varepsilon_r\vec{p})\, w^{r\dagger}_{\beta}(\varepsilon_r\vec{p}) = \frac{E}{m}\delta_{\alpha\beta}$$

PCT Symmetry.

Parity — $\mathcal{P}$ operator.

- Dirac 自由粒子满足 Dirac 方程：

$$(i\hbar\not{\partial} - m_0 c)\psi = 0$$

由 Dirac 方程的 Lorentz 协变性：

$$\psi'(x') = S(\Lambda)\psi(x)$$ (详细见 Latex 笔记)

$$\left(i\hbar S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu} - m_0 c\right)\psi'(x') = 0$$

为了保证在 All 参考系中方程形式相同

$$S(\Lambda)\cdot\gamma^\mu\cdot S^{-1}(\Lambda)\cdot\Lambda^\nu{}_\mu = \gamma^\nu$$

$$\Big\downarrow \Lambda^\nu{}_\mu\cdot\Lambda_\nu{}^\rho = \delta^\rho_\mu$$

$$S(\Lambda)\cdot\gamma^\rho\cdot S^{-1}(\Lambda) = \Lambda_\nu{}^\rho\cdot\gamma^\nu$$

Spatial Reflection 也是 Lorentz transformation!

$$\Lambda^\mu{}_\nu = \begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix} \qquad \Lambda_\mu{}^\nu = \begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$$

Spinor transformation 满足：

$$\Lambda_\nu{}^\rho\cdot\gamma^\nu = \mathcal{P}\gamma^\rho\mathcal{P}^{-1}$$

$$\Lambda_\nu{}^\rho\Lambda_\rho{}^\sigma\cdot\gamma^\nu = \mathcal{P}\gamma^\rho\cdot\Lambda_\rho{}^\sigma\cdot\mathcal{P}^{-1}$$

$$\Big\downarrow \begin{cases}\Lambda_\nu{}^\rho\Lambda_\rho{}^\sigma = \delta^\sigma_\nu\\ \Lambda_\rho{}^\sigma = \begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}\end{cases}$$

$$\delta^\sigma_\nu\gamma^\nu = \mathcal{P}\sum_{\rho=0}^{3}\gamma^\rho g^{\rho\sigma}\mathcal{P}^{-1}$$

$$\mathcal{P}^{-1}\gamma^\sigma\mathcal{P} = \gamma^\sigma g^{\sigma\sigma}$$

Solution:

$$\mathcal{P} = e^{i\varphi}\gamma^0 \qquad \mathcal{P}^{-1} = e^{-i\varphi}\gamma^0$$

验证满足对 $\mathcal{P}$ 的要求：

$$\mathcal{P}^{-1}\gamma^\sigma\mathcal{P} = e^{-i\varphi}\gamma^0\gamma^\sigma\cdot e^{i\varphi}\gamma^0 = \gamma^0\gamma^\sigma\gamma^0$$

$$= (-\gamma^\sigma\gamma^0 + \gamma^\sigma\gamma^0 + \gamma^0\gamma^\sigma)\gamma^0$$

$$= (-\gamma^\sigma\gamma^0 + \{\gamma^\sigma,\gamma^0\})\gamma^0$$

$$\Big\downarrow \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$$

$$= (-\gamma^\sigma\gamma^0 + 2g^{\sigma 0})\gamma^0$$

$$= -\gamma^\sigma + 2g^{\sigma 0}\gamma^0$$

$$= \begin{cases}-\gamma^\sigma & \sigma\neq 0\\ \gamma^\sigma & \sigma = 0\end{cases}$$

$$= \gamma^\sigma\cdot g^{\sigma\sigma}$$

In all transformation of spase Inversion:

$$\psi'(x') = \psi'(\vec{x}',t') = \mathcal{P}\psi(\vec{x},t) = e^{i\varphi}\gamma^0\psi(\vec{x},t)$$

变换后的 $\psi'(x')$ 满足：

$$(i\hbar\gamma^\mu\frac{\partial}{\partial x'^\mu} - m_0 c)\psi'(x') = 0$$

Charge conjugation

- 正常有相互作用的场方程

$$(i\hbar \not\partial - \frac{e}{c}\not A - m_0 c)\psi = 0$$

反电荷对应方程

$$(i\hbar \not\partial + \frac{e}{c}\not A - m_0 c)\psi_c = 0$$

Noticed：

$$\begin{cases} \left(i\hbar \frac{\partial}{\partial x^\mu}\right)^* = -i\hbar \frac{\partial}{\partial x^\mu} \\ A_\mu^* = A_\mu \end{cases}$$

原方程：

$$\left((i\hbar\partial_\mu - \frac{e}{c}A_\mu)\gamma^\mu - m_0 c\right)\psi = 0$$

$$\Big\downarrow \text{conjugate}$$

$$\left((-i\hbar\partial_\mu - \frac{e}{c}A_\mu)\gamma^{\mu *} - m_0 c\right)\psi^* = 0$$

$$\Big\downarrow \text{Times } U \text{ from left}$$

$$U\left((i\hbar\partial_\mu + \frac{e}{c}A_\mu)\gamma^{\mu *} + m_0 c\right)\psi^* = 0$$

$$\left((i\hbar\partial_\mu + \frac{e}{c}A_\mu)U\gamma^{\mu *}U^{-1} + m_0 C\right)U\psi^* = 0$$

$$\Big\downarrow \quad \begin{cases} \psi_c(x) = U\cdot\psi^*(x) \\ \text{假设 } U\cdot\gamma^{\mu *}U^{-1} = -\gamma^\mu \end{cases}$$

$$\left((i\hbar\partial_\mu + \frac{e}{c}A_\mu)\gamma^\mu - m_0 C\right) U\cdot\psi^* = 0$$

$$\Big\downarrow$$

$$(i\hbar\not\partial + \frac{e}{c}\not A - m_0 C)U\psi^* = 0$$

得到了正电子对应的 Dirac field Equation

- 求解 U 算符。

$$U = C\cdot\gamma^0$$

$$U\psi^* = \psi_c = C\cdot\psi$$

U满足算符关系：

$$\boxed{\gamma^2} \quad \gamma^2\cdot\gamma^2 \quad \boxed{\gamma^2\gamma^2 = -1}$$

$$\{\gamma^2, \gamma^2\} = -2$$

$$\boxed{(i\gamma^2)(i\gamma^2) = -\gamma^2\gamma^2 = 1}$$

$$(\gamma^2)^{-1} = -\gamma^2$$

$$(i\gamma^2)^{-1} =$$

$$i\gamma^2\,\gamma^\mu\, i\gamma^2 = -\gamma^2\gamma^\mu\gamma^2 = +\gamma^2\gamma^2\gamma^\mu = \boxed{-\gamma^\mu}$$

$$\gamma$$

$$U\cdot\gamma^{\mu *}\cdot U^{-1} = -\gamma^\mu$$

$$(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = C\cdot\gamma^0\cdot\gamma^{\mu *}\cdot\gamma^0\cdot C^{-1} = -\gamma^\mu$$

$$\Big\downarrow \quad \gamma^0\cdot\gamma^{\mu *}\cdot\gamma^0 = \gamma^{\mu T}$$

$$C\cdot\gamma^{\mu T}\cdot C^{-1} = -\gamma^\mu$$

$$(C^{-1})^T\gamma^\mu C^T = -\gamma^{\mu T}$$

$$\Big\downarrow \quad \begin{cases} \gamma^{1T} = -\gamma^1 & \gamma^{2T} = \gamma^2 \\ \gamma^{3T} = -\gamma^3 & \gamma^{0T} = \gamma^0 \end{cases}$$

C^T commute with γ^1 & γ^3 / Anti commute γ^2 & γ^0

Solution：

$$C = i\gamma^2\gamma^0$$

$$U\psi^* = C\gamma^0\psi^* = i\gamma^2\gamma^0\cdot\gamma^0\cdot\psi^* = i\gamma^2 K\cdot\psi$$

$\hat{C} = i\gamma^2 K$ (K is operator of complex conjugate)

∘ Usually

$$T^{-1}\Psi(x)T = -\zeta^{*}\boxed{\underline{C}}\gamma_5\Psi(Tx)$$

$$C = \begin{bmatrix} & +1 & & \\ -1 & & & \\ & & & -1 \\ & & +1 & \end{bmatrix} = \begin{bmatrix} i\sigma_2 & \\ & -i\sigma_2 \end{bmatrix} = i\begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix} = i\begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}\cdot\begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} = i\gamma^2\gamma^0$$

$$= i\begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix}$$

Time Reversal; T operator.

- From Dirac Equation to T operator.

$$t' = -t \qquad \vec{x}' = \vec{x}$$

Dirac Equation

$$i\hbar \frac{\partial \psi(\vec{x},t)}{\partial t} = \left[c\vec{\alpha}\cdot(-i\hbar\vec{\nabla}) + \beta m_0 c^2 \right] \cdot \psi(\vec{x},t)$$

Suppose T operator

$$T\psi(\vec{x},t) = \psi'(\vec{x}',t')$$

Time T operator from left

$$T i\hbar T^{-1} \cdot T \cdot \frac{\partial}{\partial t}\psi(\vec{x},t) = \left[c \cdot T\vec{\alpha}T^{-1} T(-i\hbar\vec{\nabla})T^{-1} + T\beta T^{-1} \cdot m_0 c^2 \right] \cdot T \cdot \psi(\vec{x},t)$$

认为 T op 满足：

$$\begin{cases} T \cdot i T^{-1} = -i & (T\text{ 中有 complex conjugate}) \\ T \cdot \vec{\alpha} \cdot T^{-1} = -\vec{\alpha} \\ T\beta T^{-1} = \beta \end{cases}$$

则 $\left(\frac{\partial}{\partial t'} = -\frac{\partial}{\partial t},\ \nabla' = \nabla \right)$

$$i\hbar \frac{\partial}{\partial t'} \cdot \psi'(\vec{x}',t') = \left(c\vec{\alpha}\cdot(-i\hbar\vec{\nabla}') + \beta m_0 c^2 \right) \psi'(\vec{x}',t')$$

$\psi'(x',t')$ satisfy Dirac Eq.

T 的具体表示

$$T = T_0 \cdot K$$

$$T_0 = -i\alpha_1\alpha_3 = i\gamma^1\gamma^3$$

$$T_0 = i\alpha_3\alpha_1$$

$$\boxed{\begin{aligned} &(i\gamma^1\gamma^3)(+i\gamma^1\gamma^3) = T_0 T_0 \\ &= -\gamma^1\gamma^3\gamma^1\gamma^3 = +\gamma^1\gamma^1\gamma^3\gamma^3 = +1 \end{aligned}}$$ ← reverse of T_0

- Usually

$$-C\gamma_5 = -i \cdot (i\gamma^2\gamma^0) i\gamma^0\gamma^1\gamma^2\gamma^3 = +i\gamma^2\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = +i\gamma^2\gamma^1\gamma^2\gamma^3$$

$$= -i\gamma^2\gamma^2\gamma^1\gamma^3$$

$$= \boxed{+i\gamma^1\gamma^3}$$

- T_0 explicitly

$$T_0 = i\gamma^1\gamma^3 = i\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = i\begin{bmatrix} -\sigma_1\sigma_3 & 0 \\ 0 & -\sigma_1\sigma_3 \end{bmatrix} = i\begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$\sigma_1\sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = (-i)\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = (-i)\sigma_2$$

Quantization

- Equal-Time anti-commutation Rules.

$$\{\hat{\psi}_\alpha(x,t),\hat{\psi}^\dagger_\beta(x',t)\}=\delta_{\alpha\beta}\,\delta^3(x-x')$$

$$\{\hat{\psi}_\alpha(x,t),\hat{\psi}_\beta(x',t)\}=\{\hat{\psi}^\dagger_\alpha(x,t),\hat{\psi}^\dagger_\beta(x',t)\}=0$$

- 平面波展开:

$$\psi^{(r)}_p(x,t)=(2\pi)^{-3/2}\cdot\sqrt{\frac{m}{\omega_p}}\cdot w_r(p)\cdot e^{-i\varepsilon_r(\omega_p t-\vec{p}\cdot\vec{x})}\qquad \begin{cases} r=1,2 & \varepsilon_r=+1\\ r=3,4 & \varepsilon_r=-1\end{cases}\qquad \omega_p=\sqrt{\vec{p}^2+m^2}$$

$$(i\gamma^\mu\partial_\mu-m)\psi^{(r)}_p(x,t)=0$$

$$(\gamma^\mu p_\mu-\varepsilon_r m)\,w_r(p)=0$$

正交,归一化 $w_r(p)$ (Dirac Spinors)

$$w_{r'}^\dagger(\varepsilon_{r'}p)\,w_r(\varepsilon_r p)=\frac{\omega_p}{m}\delta_{rr'}$$

$$\bar{w}_{r'}(p)\,w_r(p)=\varepsilon_r\,\delta_{rr'}$$

$$\sum_{r=1}^{4}w_{r\alpha}(\varepsilon_r p)\,w^\dagger_{r\beta}(\varepsilon_r p)=\frac{\omega_p}{m}\delta_{\alpha\beta}$$

$$\sum_{r=1}^{4}\varepsilon_r\,w_{r\alpha}(p)\,\bar{w}_{r\beta}(p)=\delta_{\alpha\beta}$$

展开:

$$\hat{\psi}(x,t)=\sum_{r=1}^{4}\int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{\omega_p}}\cdot\hat{a}(p,r)\,w_r(p)\,e^{-i\varepsilon_r p\cdot x}$$

$$\hat{H}=\int d^3p\left(\sum_{r=1}^{2}\omega_p\,\hat{a}^\dagger(p,r)\hat{a}(p,r)-\sum_{r=3}^{4}\omega_p\,\hat{a}^\dagger(p,r)\hat{a}(p,r)\right)$$

取:

$$E_0=-\sum_p\sum_{r=3}^{4}\omega_p$$

$$\begin{cases}\hat{H}'=\hat{H}-E_0\\ \quad=\sum_p\left(\sum_{r=1}^{2}\omega_p\hat{n}_{pr}+\sum_{r=3}^{4}\omega_p\hat{\bar{n}}_{pr}\right)\qquad\text{(Dirac 之海)}\\ \hat{\bar{n}}_{pr}=1-\hat{a}^\dagger(p,r)\hat{a}(p,r)=\hat{a}(p,r)\hat{a}^\dagger(p,r)\end{cases}$$

Vacuum:

$$\hat{n}_{pr}|0\rangle=0\qquad r=1,2$$

重定义:

$$\hat{\bar{n}}_{pr}|0\rangle=0\qquad r=3,4$$

$$\begin{cases}u(p,+s)=w_1(p)\\ u(p,-s)=w_2(p)\\ v(p,-s)=w_3(p)\\ v(p,+s)=w_4(p)\end{cases}\qquad\begin{cases}\hat{b}(p,+s)=\hat{a}(p,1)\\ \hat{b}(p,-s)=\hat{a}(p,2)\\ \hat{d}^\dagger(p,-s)=\hat{a}(p,3)\\ \hat{d}^\dagger(p,+s)=\hat{a}(p,4)\end{cases}$$

$$\{\hat{b}(p,s),\hat{b}^\dagger(p',s')\}=\delta^{(3)}(p-p')\delta_{ss'}\qquad\{\hat{d}(p,s),\hat{d}^\dagger(p',s')\}=\delta^{(3)}(p-p')\delta_{ss'}$$

$$\hat{\psi}(x,t)=\sum_s\int\frac{d^3p}{(2\pi)^{3/2}}\cdot\sqrt{\frac{m}{\omega_p}}\cdot\left(\hat{b}(p,s)\,u(p,s)\,e^{-ip\cdot x}+\hat{d}^\dagger(p,s)\cdot v(p,s)\,e^{ip\cdot x}\right)$$

Hamiltonian

$$\hat{H}'=\sum_s\int d^3p\;\omega_p\left(\hat{b}^\dagger(p,s)\hat{b}(p,s)+\hat{d}^\dagger(p,s)\hat{d}(p,s)\right)$$

生成反粒子

Charge

$$\hat{Q} = e\int d^3x\, \hat{\psi}^\dagger(x)\hat{\psi}(x)$$

$$= e\sum_s \int d^3p \cdot \left(\hat{b}^\dagger(p,s)\hat{b}(p,s) + \hat{d}(p,s)\hat{d}^\dagger(p,s)\right)$$

Charge of Dirac sea

$$\hat{Q}_0 = e\sum_s \int d^3p\ \delta^{(3)}(0)$$

$$\hat{Q}' = \hat{Q} - Q_0 = e\sum_s \int d^3p \cdot \left(\hat{b}^\dagger(p,s)\hat{b}(p,s) - \hat{d}^\dagger(p,s)\hat{d}(p,s)\right)$$

Spin

$$\hat{\vec{S}} \cdot \frac{\vec{p}}{|p|} = \frac{1}{2}\int d^3p \left(\hat{b}^\dagger(p,s)\hat{b}(p,s) + \hat{d}^\dagger(p,s)\hat{d}(p,s) - \hat{b}^\dagger(p,-s)\hat{b}(p,-s) - \hat{d}^\dagger(p,-s)\hat{d}(p,-s)\right)$$

Feynman Propagator for Dirac Field.

- Feynman propagator 定义为

$$i S_{F\alpha\beta}(x-y) = \langle 0 | T(\hat{\psi}_\alpha(x) \hat{\bar{\psi}}_\beta(y)) | 0 \rangle$$

$$= \Theta(x_0-y_0)\langle 0 | \hat{\psi}_\alpha(x) \hat{\bar{\psi}}_\beta(y) | 0 \rangle - \Theta(y_0-x_0)\langle 0 | \hat{\bar{\psi}}_\beta(y) \hat{\psi}_\alpha(x) | 0 \rangle$$

结果：

$$S_{F\alpha\beta}(x-y) = (i\not\partial + m)_{\alpha\beta} \Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{\not p + m}{p^2 - m^2 + i\epsilon}$$

$$\not\partial = \partial^\mu \gamma_\mu$$

Spin-1 field — Maxwell and Proca Equations.

Maxwell equation

- Maxwell equation:

$$\begin{cases} \nabla\cdot E = \rho \\ \nabla\cdot B = 0 \\ \nabla\times B - \frac{\partial E}{\partial t} = \vec{j} \\ \nabla\times E + \frac{\partial B}{\partial t} = 0 \end{cases}$$

- Electro-magnetic Field strength tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

- F 的性质:

$$F_{\mu\nu}F^{\mu\nu} = -2(E^2 - B^2)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \longleftrightarrow \begin{cases} B = \nabla\times A \\ E = -\frac{\partial A}{\partial t} - \nabla A_0 \end{cases} \quad A^\mu = (A_0, \vec{A})$$

$\downarrow$

F 是反对称的 $F^{\mu\nu} = -F^{\nu\mu}$

- Maxwell Equation 用 F 表示:

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad j^\nu = (\rho, \vec{j})$$

$$\partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0$$

- Maxwell 方程用 A 表示:

$$\begin{cases} \partial_\mu F^{\mu\nu} = j^\nu \\ \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0 \end{cases}$$

$\Downarrow$

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial_\mu\partial^\mu)A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu$$

$$\Downarrow$$

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu \qquad \text{Maxwell-Eq}$$

$$\partial^\lambda(\partial^\mu A^\nu - \partial^\nu A^\mu) + \partial^\nu(\partial^\lambda A^\mu - \partial^\mu A^\lambda) + \partial^\mu(\partial^\nu A^\lambda - \partial^\lambda A^\nu) = 0$$

(恒成立)

- A 的 Gauge invariance:

$$A'^\mu = A^\mu + \partial^\mu\Lambda$$

$$F'^{\mu\nu} = \partial^\mu(A^\nu + \partial^\nu\Lambda) - \partial^\nu(A^\mu + \partial^\mu\Lambda) = F^{\mu\nu}$$

Lagrange density and conserved quantities.

- $$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

$$\left\{ F_{\mu\nu} F^{\mu\nu} = -2(E^2 - B^2)\right.$$

$$= -\frac{1}{4}(-2)(E^2 - B^2) - j_\mu A^\mu$$

$$= \frac{1}{2}(E^2 - B^2) - \rho A_0 + \vec{j}\cdot\vec{A}$$

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j_\mu A^\mu$$

$$= -\frac{1}{2}\left((\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\nu A_\mu)(\partial^\mu A^\nu)\right) - j_\mu A^\mu$$

Action:

$$W = \int d^4x\, \mathcal{L} = \int d^4x \left[\left(-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\nu A_\mu)(\partial^\mu A^\nu)\right) - j_\mu A^\mu\right]$$

Euler-lagrange Equation

$$\frac{\partial \mathcal{L}}{\partial(\phi)} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0$$

$$\frac{\partial\left((\partial_\rho A_\sigma)(\partial^\rho A^\sigma)\right)}{\partial(\partial_\mu A^\nu)} = \frac{\partial\left((\partial_\mu A^\nu)\cdot(\partial^\mu A_\nu) + \cdots\right)}{\partial(\partial_\mu A^\nu)} = \frac{\partial\left(g^{\mu\mu} g_{\nu\nu}(\partial_\mu A_\nu)^2\right)}{\partial(\partial_\mu A^\nu)}$$

$$= 2\,\partial^\mu A_\nu$$

$$\frac{\partial\left((\partial_\rho A_\sigma)(\partial^\sigma A^\rho)\right)}{\partial(\partial_\mu A^\nu)} = \frac{\partial\left((\partial_\mu A^\nu)(\partial_\nu A^\mu) + (\partial_\nu A^\mu)(\partial_\mu A^\nu) + \cdots\right)}{\partial(\partial_\mu A^\nu)} = 2\partial_\nu A^\mu$$

Euler-lagrange Equation.

$$\frac{\partial \mathcal{L}}{\partial(A^\nu)} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) = 0$$

$$(-j_\nu) - \partial_\mu\left(-\partial^\mu A_\nu + \partial_\nu A^\mu\right) = 0$$

$$(\partial_\mu \partial^\mu) A_\nu - \partial_\nu(\partial_\mu A^\mu) = j_\nu$$

$$\Box A^\nu + \partial^\nu(\partial_\mu A^\mu) = j^\nu \quad ✓$$

- Gauge-invariance and current conservation. 之间的关系!

$$W = \int d^4x\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu\right)$$

$$W' = \int d^4x\left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu(A^\mu + \partial^\mu \Lambda)\right]$$

$$= \int d^4x\left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu\right] - \int d^4x\, j_\mu \partial^\mu \Lambda$$

$$= W - \int d^4x\left[\partial^\mu(j_\mu \Lambda) - \Lambda\, \partial^\mu j_\mu\right]$$

$$= W + \int d^4x\cdot \Lambda\underbrace{(\partial^\mu j_\mu)}_{\partial^\mu j_\mu = 0} - \int d^4x\cdot \underbrace{\partial^\mu(j_\mu \Lambda)}_{\text{surface term!}}$$

$$= W$$

。能动量张量：

$$\Theta^{\mu\nu} = -\mathcal{L}\, g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial^\nu \phi(x))$$

$$\Theta^{\mu\nu} = -\left(-\frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} - j_\rho A^\rho\right) g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)}(\partial^\nu A^\lambda)$$

$$= \left(\frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} + j_\rho A^\rho\right) g^{\mu\nu} + (-\partial^\mu A_\lambda + \partial_\lambda A^\mu)(\partial^\nu A^\lambda)$$

$$= \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu} - F^{\mu\sigma}(\partial^\nu A_\sigma) + g^{\mu\nu} j_\sigma A^\sigma$$

— Four divergence of energy-momentum tensor.

$$\partial_\mu \Theta^{\mu\nu} = \partial_\mu\left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu} - F^{\mu\sigma}(\partial^\nu A_\sigma) + g^{\mu\nu} j_\sigma A^\sigma\right)$$

$$= \frac{1}{4} \partial^\nu (F_{\alpha\beta} F^{\alpha\beta}) - \partial_\mu\left[F^{\mu\sigma}(\partial^\nu A_\sigma)\right] + \partial^\nu (j_\sigma A^\sigma)$$

Maxwell - Equation.

$$\partial_\mu F^{\mu\sigma} = j^\sigma$$

$$= \frac{1}{4} \partial^\nu (F_{\alpha\beta}) F^{\alpha\beta} + \frac{1}{4} F_{\alpha\beta} \partial^\nu (F^{\alpha\beta}) - (\partial_\mu F^{\mu\sigma})(\partial^\nu A_\sigma)$$
$$- F^{\mu\sigma} \partial_\mu (\partial^\nu A_\sigma) + (\partial^\nu j_\sigma) A^\sigma + j_\sigma (\partial^\nu A^\sigma)$$

$$= \frac{1}{2} \partial^\nu (F_{\alpha\beta}) F^{\alpha\beta} - j^\sigma (\partial^\nu A_\sigma) - F^{\mu\sigma} \partial_\mu (\partial^\nu A_\sigma) + (\partial^\nu j_\sigma) A^\sigma$$
$$+ j_\sigma (\partial^\nu A^\sigma)$$

$$= \frac{1}{2} \partial^\nu (\partial_\alpha A_\beta - \partial_\beta A_\alpha) F^{\alpha\beta} - F^{\alpha\beta} \partial_\alpha (\partial^\nu A_\beta) + (\partial^\nu j_\sigma) A^\sigma$$

$$= -\frac{1}{2} \partial^\nu (\partial_\alpha A_\beta + \partial_\beta A_\alpha) F^{\alpha\beta} + (\partial^\nu j_\sigma) A^\sigma$$

反对称!

$$\partial_\mu \Theta^{\mu\nu} = (\partial^\nu j_\sigma) A^\sigma$$

— Energy - Momentum tensor is not Gauge Invariant!

$$\Theta^{\mu\nu} = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu} - F^{\mu\sigma}(\partial^\nu A_\sigma) + g^{\mu\nu} j_\sigma A^\sigma$$

Gauge transformation: $A^\mu \to A^\mu + \partial^\mu \Lambda$:

$$\Theta'^{\mu\nu} = \Theta^{\mu\nu} - F^{\mu\sigma} \partial^\nu \partial_\sigma \Lambda + g^{\mu\nu} j_\sigma \partial^\sigma \Lambda$$

$$= \Theta^{\mu\nu} - \partial_\sigma\left(F^{\mu\sigma} \partial^\nu \Lambda - g^{\mu\nu} j^\sigma \Lambda\right)$$
$$+ (\partial_\sigma F^{\mu\sigma})(\partial^\nu \Lambda) - g^{\mu\nu} (\partial^\sigma j_\sigma) \Lambda$$

Maxwell Equation: $\partial_\sigma F^{\mu\sigma} = -\partial_\sigma F^{\sigma\mu} = -j^\mu$

current conservation: $\partial_\sigma j^\sigma = 0$

$$= \Theta^{\mu\nu} - \partial_\sigma\left(F^{\mu\sigma} \partial^\nu \Lambda - g^{\mu\nu} j^\sigma \Lambda\right) - j^\mu (\partial^\nu \Lambda)$$

— 能动量张量可加入 Divergence of arbitrary rank-3 tensor. — Anti-symmetric in first two indices $\chi^{\sigma\mu\nu} = -\chi^{\mu\sigma\nu}$

reason:

$$\tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma \chi^{\sigma\mu\nu}$$

$$\tilde{P}^\nu = \int d^3x \,(\Theta^{0\nu} + \partial_\sigma \cdot \chi^{\sigma 0\nu})$$

$$= P^\nu + \int d^3x \cdot \partial_\sigma \cdot \chi^{\sigma 0 \nu}$$

$$= P^\nu + \underbrace{\int d^3x\, \partial_0 \chi^{00\nu}}_{=0\ (\chi^{00\nu} = -\chi^{00\nu})} + \underbrace{\int d^3x \cdot \partial_i \chi^{i0\nu}}_{\text{surface term.}}$$

$$= P^\nu$$

Modified energy momentum tensor:

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma \cdot \chi^{\sigma\mu\nu}$$

$$= \Theta^{\mu\nu} + \partial_\sigma \cdot (F^{\mu\sigma} A^\nu)$$

$$= \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \cdot g^{\mu\nu} - F^{\mu\sigma} (\partial^\nu A_\sigma) + g^{\mu\nu} \cdot j_\sigma \cdot A^\sigma + \underline{\partial_\sigma \cdot (F^{\mu\sigma} A^\nu)}$$

$$= \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu} - F^{\mu\sigma} (\partial^\nu A_\sigma) + F^{\mu\sigma} \cdot \partial_\sigma A^\nu + (\partial_\sigma F^{\mu\sigma}) A^\nu + g^{\mu\nu} j_\sigma \cdot A^\sigma$$

Maxwell equation:

$$(\partial_\sigma F^{\sigma\mu}) = j^\mu$$

$$= \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu} + F^{\mu\sigma} \cdot F_\sigma{}^\nu - j^\mu A^\nu + g^{\mu\nu} \cdot j_\sigma \cdot A^\sigma$$

Modify 的好处: 1° $j^\mu = 0$ 时, T 是 Gauge Invariant 的!
2° $j^\mu = 0$ 时, T 是 Symmetric Tensor!

— Modified energy density:

$$w = T^{00} = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \cdot g^{00} + F^{0\sigma} \cdot F_\sigma{}^0 - j^0 A^0 + g^{00} j_\sigma A^\sigma$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \quad F, \text{definition.}$$

$$F_{\alpha\beta} F^{\alpha\beta} = -2(E^2 - B^2) \quad \text{property of } F.$$

$$= -\frac{1}{2}(E^2 - B^2) + (-E^1) \cdot (E^1) \cdot (-1) + (-E^2)(E^2)(-1) + (-E^3) E^3 \cdot (-1) - \vec{j} \cdot \vec{A}$$

$$= \frac{1}{2}(B^2+E^2) - \vec{j}\cdot\vec{A}.$$

—— Modified momentum density:

$$p^k = T^{0k} = \vec{E}\times\vec{B} - j^0\vec{A}$$

◦ Lorentz 对称性 → 角动量和自旋守恒

$$j_\mu = \frac{1}{2}\delta w^{\nu\lambda}\cdot M_{\mu\nu\lambda}$$

$$M_{\mu\nu\lambda} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_r)}(I_{\nu\lambda})_{rs}\phi_s - (\Theta_{\mu\nu}x_\lambda - \Theta_{\mu\lambda}x_\nu)$$

$$\phi'_r(x') = \phi_r(x) + \frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})_{rs}\phi_s(x)$$

$$A'^\alpha(x') = A^\alpha(x) + \frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})^\alpha{}_\beta A^\beta(x)$$

$$M_{\mu\nu\lambda} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\alpha)}(I_{\nu\lambda})^\alpha{}_\beta A^\beta - (\Theta_{\mu\nu}x_\lambda - \Theta_{\mu\lambda}x_\nu)$$

1° $$\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\alpha)} = \frac{\partial\left[-\frac{1}{2}\left((\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\nu A_\mu)(\partial^\mu A^\nu)\right) - j_\mu A^\mu\right]}{\partial(\partial^\mu A^\alpha)}$$

$$= -\partial_\mu A_\alpha + \partial_\alpha A_\mu$$

$$= F_{\alpha\mu}$$

2° $$\Theta^{\mu\nu} = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\cdot g^{\mu\nu} - F^{\mu\sigma}(\partial^\nu A_\sigma) + g^{\mu\nu}\cdot j_\sigma\cdot A^\sigma.$$

3° $$A'^\alpha(x') = A^\alpha(x) + \frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})^\alpha{}_\beta A^\beta(x)$$

$$= A^\alpha(x) + \delta w^\alpha{}_\beta\cdot A^\beta(x)$$

$$\frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})^\alpha{}_\beta = \delta w^\alpha{}_\beta$$

$$\frac{1}{2}\delta w_{\mu\nu}(I^{\mu\nu})^\alpha{}_\beta = \delta w_{\mu\nu}\cdot g^{\mu\alpha}\cdot g^\nu{}_\beta$$

$$0 = \delta w_{\mu\nu}\left(g^{\mu\alpha}\cdot g^\nu{}_\beta - \frac{1}{2}(I^{\mu\nu})^\alpha{}_\beta\right)$$

$$(I^{\mu\nu})^\alpha{}_\beta \underset{\Downarrow}{=} 2g^{\mu\alpha}\cdot g^\nu{}_\beta$$

反对称部分:(反对称部分在与 $\delta w_{\mu\nu}$ 缩并时才不消失!)

$$\frac{1}{2}\left[(I^{\mu\nu})^\alpha{}_\beta - (I^{\nu\mu})^\alpha{}_\beta\right] = g^{\mu\alpha}g^\nu{}_\beta - g^{\nu\alpha}g^\mu{}_\beta$$

$$(I^{\mu\nu})^\alpha{}_\beta = g^{\mu\alpha}g^\nu{}_\beta - g^{\nu\alpha}g^\mu{}_\beta$$

$$M_{\mu\nu\lambda} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\alpha)} (I_{\nu\lambda})^\alpha{}_\beta A^\beta - (\Theta_{\mu\nu} x_\lambda - \Theta_{\mu\lambda} x_\nu)$$

$$= - F_{\mu\alpha} (g_\nu{}^\alpha g_{\lambda\beta} - g_\lambda{}^\alpha g_{\nu\beta}) \cdot A^\beta - (\Theta_{\mu\nu} \cdot x_\lambda - \Theta_{\mu\lambda} x_\nu)$$

$$= \Theta_{\mu\lambda} x_\nu - \Theta_{\mu\nu} x_\lambda + F_{\mu\lambda} A_\nu - F_{\mu\nu} A_\lambda$$

$$j_\mu = \frac{1}{2} \delta w^{\nu\lambda} \cdot M_{\mu\nu\lambda}$$

守恒荷： $j_0 = \frac{1}{2} \delta w^{\nu\lambda} \cdot M_{0\nu\lambda} = \frac{1}{2} \delta w^{\nu\lambda} (\Theta_{0\lambda} x_\nu - \Theta_{0\nu} x_\lambda + F_{0\lambda} A_\nu - F_{0\nu} A_\lambda)$

spin : $S^{n\ell} = \int d^3x \cdot (F^{0\ell} \cdot A^n - F^{0n} \cdot A^\ell)$

three-vector spin:

$$\vec{S} = \int d^3x \ \vec{E} \times \vec{A}$$

reason:

Lorentz transformation

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu = (\delta^\mu_\nu + \delta w^\mu{}_\nu) \cdot x^\nu$$

坐标系绕 z 轴转动：

$$\Lambda = \mathbb{I} + \Delta\varphi \cdot I_6$$

$$I_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \Delta W_{6\,\nu\lambda} \overset{\text{将第一个指标变为下指标!}}{=} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

坐标系绕 x 轴转动：

$$\Lambda = \mathbb{I} + \Delta\varphi \cdot I_4$$

$$I_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \longrightarrow \Delta W_{4\,\nu\lambda} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

坐标系绕 y 轴转动：

$$\Lambda = \mathbb{I} + \Delta\varphi\, I_5$$

$$I_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \Delta W_{5\,\nu\lambda} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

仅看转动 Lorentz 变化：

$$j_0 = \frac{1}{2} \Big(\overset{\Delta\varphi_4}{(\delta W_4)_{\nu\lambda}} + \overset{\Delta\varphi_5}{(\delta W_5)_{\nu\lambda}} + \overset{\Delta\varphi_6}{(\delta W_6)_{\nu\lambda}} \Big) (\Theta_0{}^\lambda x^\nu - \Theta_0{}^\nu x^\lambda + F_0{}^\lambda A^\nu - F_0{}^\nu A^\lambda)$$

由于只有空间坐标转动：

$$j_0 = \frac{1}{2} \big(\Delta\varphi_4 (\delta W_4)_{n\ell} + \Delta\varphi_5 (\delta W_5)_{n\ell} + \Delta\varphi_6 (\delta W_6)_{n\ell} \big) \boxed{(\Theta_0{}^\ell x^n - \Theta_0{}^n x^\ell + F_0{}^\ell A^n - F_0{}^n A^\ell)}$$

$\Downarrow$ 关于 n, ℓ 反对称!

$$= - (\Delta\varphi_4 \cdot \delta_n^2 \delta_\ell^3 + \Delta\varphi_5 \cdot \delta_n^3 \delta_\ell^1 + \Delta\varphi_6 \cdot \delta_n^1 \delta_\ell^2) \cdot (\Theta^{0\ell} x^n - \Theta^{0n} x^\ell + \underbrace{F^{0\ell} A^n - F^{0n} A^\ell}_{\Downarrow})$$

$$S^{n\ell} = F^{0\ell} A^n - F^{0n} A^\ell .$$

轴子电动力学.

- Modified Lagrangian and Equation of Motion.

$$\mathcal{L}_\theta = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \theta\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$
$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$
$$+ \theta\,\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) + 4\theta\,\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu)(\partial_\rho A_\sigma)$$

Equation of Motion

$$\partial_\mu \left\{ \frac{\partial(\mathcal{L})}{\partial(\partial_\mu A_\nu)} \right\} = \frac{\partial(\mathcal{L})}{\partial(A_\nu)} = 0$$

Explicitly

$$\partial_\mu\left(-\partial^\mu A^\nu + \partial^\nu A^\mu + 4\theta\,\varepsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma) + 4\theta\,\varepsilon^{\rho\sigma\mu\nu}(\partial_\rho A_\sigma)\right) = 0$$
$$-(\partial_\mu\partial^\mu)A^\nu + (\partial^\nu\partial_\mu)A^\mu + 8\theta\,\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu\partial_\rho A_\sigma) = 0$$

由于 $\partial_\mu\partial_\rho = \partial_\rho\partial_\mu$，但 Levi-civita 关于 μ,ρ Anti-symmetry，则.

EoM:
$$-\partial^2 A^\nu + \partial^\nu\partial_\mu A^\mu = 0$$

和电动力学相同.

- 当 $\theta = \theta(x)$，不是常数时.

Equation of motion:

$$\partial_\mu\left(-\partial^\mu A^\nu + \partial^\nu A^\mu + 4\theta\,\varepsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma) + 4\theta\,\varepsilon^{\rho\sigma\mu\nu}(\partial_\rho A_\sigma)\right) = 0$$

$$-(\partial_\mu\partial^\mu)A^\nu + (\partial^\nu\partial_\mu)A^\mu + 8\theta\,\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu\partial_\rho A_\sigma) + 8(\partial_\mu\theta)\,\varepsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma) = 0$$

Equation of Motion 成为了.

$$-\partial^2 A^\nu + \partial^\nu\partial_\mu A^\mu + 8\,\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu\theta)(\partial_\rho A_\sigma) = 0$$

Wilson Loop In SU(2) Gauge Theory.

∘ Wilson Loop $U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x)$ $\quad U_\mu(x) = \exp(ia A_\mu(x))$

(a) $A = A^\dagger$ $(\sigma = \sigma^\dagger)$ $\quad A_\mu = A_\mu^a T^a \quad T^a = \sigma_a/2$

1° 展开 $U_{\mu\nu}(x)$ 到 a^2 阶,

$$U_{\mu\nu}(x) = \left(1 + iaA_\mu(x) - \tfrac{1}{2}a^2 A_\mu^2(x)\right)\left(1 + iaA_\nu(x+\mu) - \tfrac{1}{2}a^2 A_\nu^2(x+\mu)\right)$$
$$\left(1 - iaA_\mu(x+\nu) - \tfrac{1}{2}a^2 A_\mu^2(x+\nu)\right)\left(1 - iaA_\nu(x) - \tfrac{1}{2}a^2 A_\nu^2(x)\right)$$

$$= 1 + iaA_\mu(x) + iaA_\nu(x+\mu) - iaA_\mu(x+\nu) - iaA_\nu(x)$$
$$- a^2 A_\mu(x) A_\nu(x+\mu) + a^2 A_\mu(x) A_\mu(x+\nu) + a^2 A_\mu(x) A_\nu(x)$$
$$+ a^2 A_\nu(x+\mu) A_\mu(x+\nu) + a^2 A_\nu(x+\mu) A_\nu(x)$$
$$- a^2 A_\mu(x+\nu) A_\nu(x)$$
$$- \tfrac{1}{2}a^2 A_\mu^2(x) - \tfrac{1}{2}a^2 A_\nu^2(x+\mu) - \tfrac{1}{2}a^2 A_\mu^2(x+\nu) - \tfrac{1}{2}a^2 A_\nu^2(x)$$

$$= 1 + ia\left(A_\mu(x) - A_\mu(x+\nu)\right) + ia\left(A_\nu(x+\mu) - A_\nu(x)\right)$$
$$+ a^2\left(-A_\mu(x) A_\nu(x+\mu) - A_\mu(x+\nu) A_\nu(x) + A_\mu(x) A_\nu(x)\right)$$
$$+ a^2 A_\nu(x+\mu) A_\mu(x+\nu) - a^2 A_\mu(x+\nu) A_\nu(x)$$
$$+ a^2 A_\mu(x) A_\mu(x+\nu) + a^2 A_\nu(x+\mu) A_\nu(x)$$
$$- \tfrac{1}{2}a^2 A_\mu^2(x) - \tfrac{1}{2}a^2 A_\nu^2(x+\mu) - \tfrac{1}{2}a^2 A_\mu^2(x+\nu) - \tfrac{1}{2}a^2 A_\nu^2(x)$$

展开到2阶.

$$= 1 - ia^2 \partial_\nu A_\mu - ia^2 \partial_\mu A_\nu + a^2\left(-A_\mu A_\nu + A_\nu A_\mu + A_\mu A_\mu + A_\nu A_\nu\right)$$
$$- a^2 A_\mu^2 - a^2 A_\nu^2$$

$$= 1 - ia^2\left(\partial_\nu A_\mu + \partial_\mu A_\nu\right) + a^2\left\{(A_\mu + A_\nu)(A_\mu - A_\nu) - A_\mu^2 - A_\nu^2\right\}$$

$$= 1 - ia^2\left(\partial_\nu A_\mu + \partial_\mu A_\nu\right) + a^2\left(-A_\mu A_\nu + A_\nu A_\mu\right)$$

(b) 场强张量定义为.

$$F_{\mu\nu} = \partial_\nu A_\mu + \partial_\mu A_\nu - \frac{1}{i}\left(A_\nu A_\mu - A_\mu A_\nu\right)$$

按生成元展开, $[\sigma^i, \sigma^j] = i\varepsilon_{ijk}\sigma^k$

$$F_{\mu\nu} = \left(\partial_\nu A_\mu^a + \partial_\mu A_\nu^a\right)\frac{\sigma^a}{2} + i\left(A_\nu^a A_\mu^b\right)\frac{1}{4}[\sigma^a, \sigma^b]$$

$$= \left(\partial_\nu A_\mu^a + \partial_\mu A_\nu^a\right)\frac{\sigma^a}{2} + i\left(A_\nu^a A_\mu^b\right)\frac{1}{4}\, i\, \varepsilon^{abc}\sigma^c$$

$$F_{\mu\nu}^a = \left(\partial_\nu A_\mu^a + \partial_\mu A_\nu^a\right) - \frac{1}{2} A_\nu^b A_\mu^c \varepsilon^{bca}$$

plane-wave-expansion — Maxwell field.

∘ fixed momentum field mode. $k=(k^0,\vec{k})$ $k^0=|\vec{k}|$ (photons) $k^0\neq|\vec{k}|$ (virtual photons)

$$A^{\mu}(k,x,\lambda)=N_k\,\varepsilon^{\mu}(k,\lambda)\cdot\exp(-i\,k_\mu x^\mu)$$

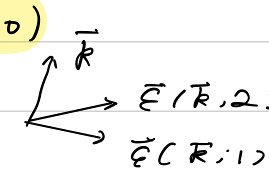
Maxwell Equation:

$$\Box A^{\nu}-\partial^{\nu}(\partial_\mu A^{\mu})=j^{\nu}$$

∘ Field Mode (给定 k – four vector, 不一定是 $k^2=0$)

— $\lambda=1$ $\varepsilon(k,1)=(0,\vec{\varepsilon}(\vec{k},1))$

— $\lambda=2$ $\varepsilon(k,2)=(0,\vec{\varepsilon}(\vec{k},2))$



— $\lambda=3$ In a special frame, define:

$n=(1,0,0,0)$ ⟶ In other fram (Lorentz Transformation)

$$\varepsilon(k,3)=\frac{k-n(k\cdot n)}{((k\cdot n)^2-k^2)^{1/2}}\xrightarrow{\text{In special frame}}\left(0,\frac{\vec{k}}{|\vec{k}|}\right)$$

$$\varepsilon(k,3)\cdot\varepsilon(k,3)=\frac{(k-n(k\cdot n))(k-n(k\cdot n))}{(k\cdot n)^2-k^2}$$

$$=\frac{k\cdot k+n\cdot n(k\cdot n)^2-2n\cdot k(k\cdot n)}{(k\cdot n)^2-k^2}$$

$$=\frac{-(k\cdot n)^2+k^2}{(k\cdot n)^2-k^2}=-1$$

$$k\cdot\varepsilon(k,3)=\frac{k\cdot(k-n(k\cdot n))}{((k\cdot n)^2-k^2)^{1/2}}=\frac{k^2-(k\cdot n)^2}{((k\cdot n)^2-k^2)^{1/2}}=-((k\cdot n)^2-k^2)^{1/2}$$

— $\lambda=0$

$$\varepsilon(k,4)=n$$

— Completeness relation

$$\sum_{\lambda=0}^{3}g_{\lambda\lambda}\,\varepsilon_\mu(k,\lambda)\varepsilon_\nu(k,\lambda)=g_{\mu\nu}$$

— Isolate transverse mode, 横波

$$\sum_{\lambda=1}^{2}\varepsilon_\mu(k,\lambda)\varepsilon_\nu(k,\lambda)=-g_{\mu\nu}+n_\mu n_\nu-\frac{[k_\mu-n_\mu(k\cdot n)][k_\nu-n_\nu(k\cdot n)]}{(k\cdot n)^2-k^2}$$

$$=-g_{\mu\nu}-\frac{-((k\cdot n)^2-k^2)n_\mu n_\nu+k_\mu k_\nu+n_\mu n_\nu(k\cdot n)^2-k_\mu n_\nu(k\cdot n)-k_\nu n_\mu(k\cdot n)}{(k\cdot n)^2-k^2}$$

$$=-g_{\mu\nu}-\frac{k_\mu k_\nu+n_\mu n_\nu k^2-(k_\mu n_\nu+k_\nu n_\mu)(k\cdot n)}{(k\cdot n)^2-k^2}$$

$$(k^2=0,\ \text{virtual photons})=-g_{\mu\nu}-\frac{k_\mu k_\nu}{(k\cdot n)^2}+\frac{k_\mu n_\nu+k_\nu n_\mu}{k\cdot n}$$

∘ Spin Sum of polarization vector.

$$[\varepsilon_-^\mu(k,r)]^* = -\frac{1}{\sqrt{2}}\frac{[r|\gamma^\mu|k\rangle}{[r|k]} \qquad [\varepsilon_+^\mu(k,r)]^* = \frac{1}{\sqrt{2}}\frac{[k|\gamma^\mu|r\rangle}{\langle r|k\rangle}$$

$\langle p|\gamma^\mu|q] = [q|\gamma^\mu|p\rangle$ property 5°

$\langle p|q\rangle^* = [qp]$ property 2°

$$[\varepsilon_-^\mu(k,r)]^* = -\frac{1}{\sqrt{2}}\frac{\langle r|\gamma^\mu|k]^*}{\langle k\;r\rangle^*} \qquad [\varepsilon_+^\mu(k,r)]^* = \frac{1}{\sqrt{2}}\frac{\langle k|\gamma^\mu|r]^*}{[r|k]^*}$$

$$\varepsilon_-^\mu(k,r) = -\frac{1}{\sqrt{2}}\frac{\langle r|\gamma^\mu|k]}{[k\;r]} \qquad \varepsilon_+^\mu(k,r) = \frac{1}{\sqrt{2}}\frac{\langle k|\gamma^\mu|r]}{\langle r|k\rangle}$$

$$\sum_{\lambda=\pm}\varepsilon_\mu^\lambda(k,n)(\varepsilon_\nu^\lambda(k,n))^* = \varepsilon_\mu^+(k,n)(\varepsilon_\nu^+(k,n))^* + \varepsilon_\mu^-(k,n)(\varepsilon_\nu^-(k,n))^*$$

$$= \frac{1}{2}\frac{\langle k|\gamma^\mu|n]}{[n\;k]}\frac{[k|\gamma^\nu|n\rangle}{\langle n|k\rangle} + \frac{1}{2}\frac{\langle n|\gamma^\mu|k][n|\gamma^\nu|k\rangle}{\langle k\;n\rangle[n\;k]}$$

$\langle k|\gamma^\mu|n] = [n|\gamma^\mu|k\rangle$ (property 5°)

$$= \frac{1}{2}\frac{1}{[n\;k]\langle n|k\rangle}\{[n|\gamma^\mu|k\rangle[k|\gamma^\nu|n\rangle + [n|\gamma^\nu|k\rangle[k|\gamma^\mu|n\rangle\}$$

$\underbrace{\not p = |p\rangle[p| + |p]\langle p|}_{\text{property } 0°}$; $\langle k|\gamma^\nu|n\rangle = 0$ 类似于 property 5°, unknow.

$$= \frac{1}{2}\frac{1}{[n\;k]\langle n|k\rangle}\{[n|\gamma^\mu\not k\gamma^\nu|n\rangle + [n|\gamma^\nu\not k\gamma^\mu|n\rangle\}$$

$$= \frac{1}{2}\frac{1}{[n\;k]\langle n\;k\rangle}\{[n|\gamma^\mu\gamma^\rho\gamma^\nu|n\rangle + [n|\gamma^\nu\gamma^\rho\gamma^\mu|n\rangle\}k_\rho$$

$\gamma^\mu\gamma^\rho + \gamma^\rho\gamma^\mu = 2g^{\mu\rho}$

$$= \frac{1}{2}\frac{1}{[n\;k]\langle n\;k\rangle}\{k_\rho[n|2g^{\mu\rho}\gamma^\nu|n\rangle + k_\rho[n|2g^{\nu\rho}\gamma^\mu|n\rangle - [n|\not k\gamma^\mu\gamma^\nu|n\rangle - [n|\not k\gamma^\nu\gamma^\mu|n\rangle\}$$

$$= \frac{1}{[nk]\langle nk\rangle}\Big(k^\mu[n|\gamma^\nu|n\rangle + k^\nu[n|\gamma^\mu|n\rangle - \frac{1}{2}[n|\not k\gamma^\mu\gamma^\nu|n\rangle - [n|\not k\gamma^\nu\gamma^\mu|n\rangle\Big\}$$

$\langle p|\not q|p] = 2p\cdot q$

$\Downarrow$ property 3°

$\langle n|\gamma^\nu|n] = 2n^\nu$

$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

$$= \frac{1}{[nk]\langle nk\rangle}\left(2k^\mu n^\nu + 2k^\nu n^\mu - g^{\mu\nu}[n|\not k|n\rangle\right)$$

property 3°: $\langle p|q\rangle = \sqrt{|2p\cdot q|}\,e^{i\phi}$, $[q|p] = \sqrt{|2p\cdot q|}\,e^{-i\phi}$

property 3° $[n|\not k|n\rangle = 2k\cdot n$