$$= \frac{1}{2n\cdot k}\left(2k^{\mu}n^{\nu} + 2k^{\nu}n^{\mu} - g^{\mu\nu}(2k\cdot n)\right)$$

$$= -g^{\mu\nu} + \frac{k^{\mu}n^{\nu} + k^{\nu}n^{\mu}}{n\cdot k} \qquad \text{(上下标写反了…)}$$

Spin = 2. Massless 圆偏振极化张量.

。spin — 1 圆偏振极化张量.

$$(\hat{p}\cdot J)^{\mu}{}_{\nu}\,\varepsilon^{\nu}(p,\lambda)=\lambda\,\varepsilon^{\mu}(p,\lambda)\qquad(\lambda=\pm 1)$$

$$\varepsilon^{\mu}(p,+)=\frac{1}{\sqrt{2}\,|\vec{p}|\times|\vec{p}_T|}\begin{pmatrix}0\\ -p^1p^3+ip^2|\vec{p}|\\ -p^2p^3-ip^1|\vec{p}|\\ |\vec{p}_T|^2\end{pmatrix}\qquad \varepsilon^{\mu}(p,-)=\frac{1}{\sqrt{2}\,|\vec{p}|\times|\vec{p}_T|}\begin{pmatrix}0\\ p^1p^3+ip^2|\vec{p}|\\ p^2p^3-ip^1|\vec{p}|\\ -|\vec{p}_T|^2\end{pmatrix}\qquad |\vec{p}_T|=\sqrt{(p^1)^2+(p^2)^2}$$

spin — 2 圆偏振极化张量

$$\hat{p}\cdot(J^{\mu}{}_{\alpha}\,\delta^{\nu}{}_{\beta}+\delta^{\mu}{}_{\alpha}\,J^{\nu}{}_{\beta})\,\varepsilon^{\mu\nu}(p,\lambda)=2\lambda\,\varepsilon^{\mu\nu}(p,\lambda)\qquad(\lambda=\pm 1)$$

一种构造方式:

$$\varepsilon^{\mu\nu}(p,\lambda)=\varepsilon^{\mu}(p,\lambda)\,\varepsilon^{\nu}(p,\lambda)$$

$$=\frac{1}{2|\vec{p}|^2\times|\vec{p}_T|^2}\begin{pmatrix}0\\ -\lambda p^1p^3+ip^2|\vec{p}|\\ -\lambda p^2p^3-ip^1|\vec{p}|\\ \lambda|\vec{p}_T|^2\end{pmatrix}_{\mu}\otimes\begin{pmatrix}0\\ -\lambda p^1p^3+ip^2|\vec{p}|\\ -\lambda p^2p^3-ip^1|\vec{p}|\\ \lambda|\vec{p}_T|^2\end{pmatrix}_{\nu}$$

线偏振极化:

$$h^{\mu\nu}_{+}(\vec{k})=\frac{1}{2}\left(\varepsilon^{\mu\nu}(\vec{k},+)+\varepsilon^{\mu\nu}(\vec{k},-)\right)$$

$$=\frac{1}{2}\left(\frac{1}{2|\vec{p}|^2\times|\vec{p}_T|^2}\right)\left\{\begin{pmatrix}0\\ -p^1p^3+ip^2|\vec{p}|\\ -p^2p^3-ip^1|\vec{p}|\\ |\vec{p}_T|^2\end{pmatrix}_{\mu}\otimes\begin{pmatrix}0\\ -p^1p^3+ip^2|\vec{p}|\\ -p^2p^3-ip^1|\vec{p}|\\ |\vec{p}_T|^2\end{pmatrix}_{\nu}\right.$$

$$\left.+\begin{pmatrix}0\\ p^1p^3+ip^2|\vec{p}|\\ p^2p^3-ip^1|\vec{p}|\\ -|\vec{p}_T|^2\end{pmatrix}_{\mu}\otimes\begin{pmatrix}0\\ p^1p^3+ip^2|\vec{p}|\\ p^2p^3-ip^1|\vec{p}|\\ -|\vec{p}_T|^2\end{pmatrix}_{\nu}\right\}$$

$$h^{\mu\nu}_{-}(\vec{k})=\frac{-i}{2}\left(\varepsilon^{\mu\nu}(\vec{k},+)-\varepsilon^{\mu\nu}(\vec{k},-)\right)$$

$$=\frac{-i}{2}\left(\frac{1}{2|\vec{p}|^2\times|\vec{p}_T|^2}\right)\left\{\begin{pmatrix}0\\ -p^1p^3+ip^2|\vec{p}|\\ -p^2p^3-ip^1|\vec{p}|\\ |\vec{p}_T|^2\end{pmatrix}_{\mu}\otimes\begin{pmatrix}0\\ -p^1p^3+ip^2|\vec{p}|\\ -p^2p^3-ip^1|\vec{p}|\\ |\vec{p}_T|^2\end{pmatrix}_{\nu}\right.$$

$$\left.-\begin{pmatrix}0\\ p^1p^3+ip^2|\vec{p}|\\ p^2p^3-ip^1|\vec{p}|\\ -|\vec{p}_T|^2\end{pmatrix}_{\mu}\otimes\begin{pmatrix}0\\ p^1p^3+ip^2|\vec{p}|\\ p^2p^3-ip^1|\vec{p}|\\ -|\vec{p}_T|^2\end{pmatrix}_{\nu}\right\}$$

Quantization of photon field　Maxwell场的量子化.

Hamiltonian

- Feynman Gauge

Maxwell 场的 lagrangian density 是：

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu$$

现在改写为（j=0 时）

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\xi(\partial_\sigma A^\sigma)^2 \quad (\xi=1,\ \text{Feynman Gauge})$$

—— 提取有意义项：

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}(\partial_\sigma A^\sigma)(\partial_\lambda A^\lambda)$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\nu A_\mu)(\partial^\mu A^\nu) - \frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu)$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}\partial^\mu((\partial_\nu A_\mu)A^\nu - (\partial_\nu A^\nu)A_\mu) \underbrace{- \frac{1}{2}(\partial^\mu\partial_\nu A_\mu)A^\nu + \frac{1}{2}(\partial^\mu\partial_\nu A^\nu)A_\mu}_{0}$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}\partial^\mu((\partial_\nu A_\mu)A^\nu - (\partial_\nu A^\nu)A_\mu)$$

$$\int d^4x\,\mathcal{L} = \int d^4x\left(-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)\right)$$

只有第一项有意义！

最终！

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)$$

- Hamiltonian Formalism：

$$L = \int d^3x\left(-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)\right) = L[A^\mu, \partial_0 A^\mu]$$

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^\mu)} = \frac{\partial\left(-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)\right)}{\partial(\partial_0 A^\mu)} = -\partial^0 A_\mu = -\partial_0 A^\mu = -\dot{A}^\mu$$

$$H = \int d^3x(\pi_\mu \dot{A}^\mu - \mathcal{L}) = \int d^3x\left(-\partial^0 A_\mu \cdot \partial_0 A^\mu + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)\right)$$

$$= \int d^3x\left(-\frac{1}{2}\pi^\mu\pi_\mu + \frac{1}{2}(\partial_k A_\nu)(\partial^k A^\nu)\right) \qquad k=1,2,3.$$

$$= \int d^3x\left(-\frac{1}{2}\pi^0\pi_0 + \frac{1}{2}\vec{\pi}\cdot\vec{\pi} - \frac{1}{2}(\partial_k A_\nu)(\partial_k A^\nu)\right)$$

$$= \int d^3x\left(-\frac{1}{2}\pi^0\pi^0 + \frac{1}{2}\vec{\pi}\cdot\vec{\pi} - \frac{1}{2}(\nabla A^0)\cdot(\nabla A^0) + \sum_{i=1}^{3}\frac{1}{2}(\nabla A^i)(\nabla A^i)\right)$$

$$= \int d^3x\left(\sum_{i=1}^{3}\frac{1}{2}[(\pi^i)^2 + (\nabla A^i)^2] - \frac{1}{2}[\pi^0\pi^0 + (\nabla A^0)\cdot(\nabla A^0)]\right)$$

$$\Big\}\ \pi^\mu = -\dot{A}^\mu$$

$$= \int d^3x\left(\sum_{i=1}^{3}\frac{1}{2}[(\dot{A}^i)^2 + (\nabla A^i)\cdot(\nabla A^i)] - \frac{1}{2}[(\dot{A}^0)^2 + (\nabla A^0)^2]\right)$$

$$= H[\pi^\mu, A^\mu]$$

Fourier Decomposition.

- Field operator 可被展开为：

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{2W_k(2\pi)^3}} \sum_{\lambda=0}^{3} \left(a_{k,\lambda}\, \epsilon^\mu(k,\lambda) e^{-ik\cdot x} + a^\dagger_{k,\lambda}\, e^{ik\cdot x} \right)$$

— Conjugate field：

$$\begin{aligned} \pi^\mu &= -\dot{A}^\mu \\ &= i \int \frac{d^3k}{\sqrt{2W_k(2\pi)^3}} \cdot W_k \sum_{\lambda=0}^{3} \left(a_{k,\lambda}\, \varepsilon^\mu(k,\lambda)\, e^{-ik\cdot x} - a^\dagger_{k,\lambda}\, \varepsilon^\mu(k,\lambda)\, e^{ik\cdot x} \right) \end{aligned}$$

ETCR

- ETCR 定义为：

$$[A^\mu(x,t), \pi^\nu(x',t)] = i g^{\mu\nu}\delta^{(3)}(x-x') \qquad \pi^\mu = -\partial^0 A^\mu$$

$$[A^\mu(x,t), \dot{A}^\nu(x',t)] = -i g^{\mu\nu}\delta^{(3)}(x-x') \quad ; \quad \text{others} = 0$$

—— Lorentz Gauge 不可能满足！（Lorentz Gauge 是在说：$\partial_\mu A^\mu = 0$）

$$[\partial_\mu A^\mu(x,t), A^\nu(x',t)] = [\partial_0 A^0 + \nabla\cdot\vec{A}, A^\nu]$$

$$= [\partial_0 A^0(x,t), A^\nu(x',t)] + \nabla\cdot[\vec{A}(x,t), A^\nu(x',t)]$$

$$= -[\pi^0(x,t), A^\nu(x',t)]$$

$$= -g^{\nu 0}\delta^{(3)}(x-x') \neq 0$$

$$\partial_\mu A^\mu(x,t) \neq 0 \ !$$

- ETCR for creation and annihilation operator.

Fourier decomposition of Field operator：

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}} \sum_{\lambda=0}^{3} \left(a_{k,\lambda}\,\varepsilon^\mu(k,\lambda)\,e^{-ik\cdot x} + a^+_{k,\lambda}\,\varepsilon^\mu(k,\lambda)\,e^{ik\cdot x}\right)$$

由 proca 场中的推导：

$$\begin{cases} \varepsilon^\mu(k,\lambda)\,\varepsilon_\mu(k,\lambda') = g_{\lambda\lambda'} \\ (A(x), A'(x)) = i\int d^3x\, A^{\mu *}(x)\overleftrightarrow{\partial_0} A'_\mu(x) \\ A\overleftrightarrow{\partial_0} B = A(\partial_0 B) - (\partial_0 A)B \end{cases}$$

$$\begin{cases} (A(k,\lambda), A(x)) = g_{\lambda\lambda}\, a_{k,\lambda} \\ (A^*(k,\lambda), A(x)) = -g_{\lambda\lambda}\, a^+_{k,\lambda} \end{cases}$$

写为：

$$a_{k,\lambda} = i g_{\lambda\lambda}\int d^3x\, A^{\mu *}(k,\lambda)\overleftrightarrow{\partial_0} A_\mu(x)$$

$$a^+_{k,\lambda} = -i g_{\lambda\lambda}\int d^3x\, A^{\mu}(k,\lambda)\overleftrightarrow{\partial_0} A_\mu(x)$$

近一步：

$$a_{k,\lambda} = i g_{\lambda\lambda}\int d^3x \left(A^{\mu *}(k,\lambda)\partial_0 A_\mu(x) - \partial_0 A^{\mu *}(k,\lambda)\, A_\mu(x)\right)$$

$$A^\mu(k,\lambda) = \frac{1}{\sqrt{2\omega_k(2\pi)^3}}\,\varepsilon^\mu(k,\lambda)\,e^{-ik\cdot x}$$

$$\Downarrow$$

$$\partial_0 A^{\mu *}(k,\lambda) = \frac{i\omega_k}{\sqrt{2\omega_k(2\pi)^3}}\,\varepsilon^\mu(k,\lambda)\,e^{ik\cdot x}$$

$$= i g_{\lambda\lambda}\int d^3x \left(A^{\mu *}(k,\lambda)\,\dot{A}_\mu(x) - i\omega_k A^{\mu *}(k,\lambda) A_\mu(x)\right)$$

$$= i g_{\lambda\lambda}\int d^3x\, A^{\mu *}(k,\lambda)\left(\dot{A}_\mu(x) - i\omega_k A_\mu(x)\right) \qquad -(1)$$

$$a^+_{k,\lambda} = -i g_{\lambda\lambda}\int d^3x\, A^{\mu}(k,\lambda)\overleftrightarrow{\partial_0} A_\mu(x)$$

$$= -i g_{\lambda\lambda}\int d^3x\, A^{\mu}(k,\lambda)\left(\dot{A}_\mu(x) + i\omega_k A_\mu(x)\right) \qquad -(2)$$

Commutation Between Creation & annhilation operator:

$$[a_{k,\lambda}, a^{+}_{k',\lambda'}] = [-i g_{\lambda\lambda} \int d^3x\, A^{\mu *}(k,\lambda)(\dot{A}_\mu(x) - i w_k A_\mu(x)),$$
$$-i g_{\lambda'\lambda'} \int d^3x' A^{\mu}(k',\lambda')(\dot{A}_\mu(x') + i w_{k'} A_\mu(x'))]$$
$$= g_{\lambda\lambda}\, g_{\lambda'\lambda'} \int d^3x\, d^3x' \cdot A^{\mu *}(k,\lambda,x)\, A^{\nu}(k',\lambda',x')$$
$$[\dot{A}_\mu(x) - i w_k A_\mu(x),\ \dot{A}_\nu(x') + i w_{k'} A_\nu(x')]$$

$$= g_{\lambda\lambda}\, g_{\lambda'\lambda'} \cdot \int d^3x\, d^3x'\, A^{\mu *}(k,\lambda,x)\, A^{\nu}(k',\lambda',x')$$
$$(i w_{k'} [\dot{A}_\mu(x), A_\nu(x')] + i w_k [\dot{A}_\nu(x'), A_\mu(x)])$$

commutation relation:
$[A^\mu(x,t), \dot{A}^\nu(x',t)] = -i g^{\mu\nu} \delta^{(3)}(x-x')$; others $= 0$

$$= g_{\lambda\lambda}\, g_{\lambda'\lambda'} \int d^3x\, d^3x'\, A^{\mu *}(k,\lambda,x)\, A^{\nu}(k',\lambda',x')$$
$$(\ i w_{k'} (i g_{\nu\mu}) \delta^{(3)}(x-x') + i w_k \cdot (i g_{\mu\nu}) \delta^{(3)}(x-x'))$$

$$\left\{ \begin{aligned} A^\mu(k,\lambda,x) &= \frac{1}{\sqrt{2 w_k (2\pi)^3}} \varepsilon^\mu(k,\lambda)\, e^{-ik\cdot x} \\ A^\nu(k',\lambda',x') &= \frac{1}{\sqrt{2 w_{k'} (2\pi)^3}} \varepsilon^\nu(k',\lambda')\, e^{-ik'\cdot x'} \end{aligned} \right.$$

$$= g_{\lambda\lambda}\, g_{\lambda'\lambda'} \int d^3x\, d^3x' \cdot \frac{1}{\sqrt{2 w_k (2\pi)^3}} \varepsilon^\mu(k,\lambda)\, e^{ik\cdot x} \frac{1}{\sqrt{2 w_{k'} (2\pi)^3}} \varepsilon^\nu(k',\lambda') e^{-ik'\cdot x'}$$
$$(-w_{k'} g_{\nu\mu} - w_k g_{\mu\nu})\, \delta^{(3)}(x-x')$$

$$= g_{\lambda\lambda}\, g_{\lambda'\lambda'} \cdot \int d^3x \frac{1}{\sqrt{2w_k}\ \sqrt{2w_{k'}}} \cdot \frac{1}{(2\pi)^3} \cdot \varepsilon^\mu(k,\lambda)\, \varepsilon^\nu(k',\lambda')\, e^{i(k-k')\cdot x}$$
$$(-w_{k'} - w_k)\, g_{\mu\nu}$$

$$= -g_{\lambda\lambda}\, g_{\lambda'\lambda'}\, g_{\mu\nu}\, \varepsilon^\mu(k,\lambda)\, \varepsilon^\nu(k,\lambda')\, \delta^{(3)}(k-k')$$
$$= -g_{\lambda\lambda}\, g_{\lambda'\lambda'}\, \varepsilon^\mu(k,\lambda)\, \varepsilon_\mu(k,\lambda)\, \delta^{(3)}(k-k')$$

$\varepsilon^\mu(k,\lambda)\, \varepsilon_\mu(k,\lambda') = g_{\lambda,\lambda'}$

$$= -g_{\lambda\lambda'}\, \delta^{(3)}(k-k')$$

physical quantity 物理量.

- Hamiltonian 按前文的 hamilton formalism :

$$H = \int d^3x \left(\sum_{i=1}^{3} \frac{1}{2} [(\dot{A}^i)^2 + (\nabla A^i)\cdot(\nabla A^i)] - \frac{1}{2} [(\dot{A}^0)^2 + (\nabla A^0)^2] \right)$$

Normal ordering (creation to left)

$$H = \int d^3k \, w_k \left(\sum_{\lambda=1}^{3} a^\dagger_{k\lambda} a_{k\lambda} - a^\dagger_{k0} a_{k0} \right)$$

- Momentum :

$$P = \int d^3k \, \vec{k} \left(\sum_{\lambda=1}^{3} a^\dagger_{k\lambda} a_{k\lambda} - a^\dagger_{k0} a_{k0} \right)$$

problem 1. ⇓

- Hamiltonian & Field operator mode Expansion (Not Normalise As Greiner's book)

$$H = \int d^3x \left(\Sigma_i \frac{1}{2} ((\dot{A}^i)^2 + (\nabla A^i)^2) - \frac{1}{2} ((\dot{A}^0)^2 + (\nabla A^0)^2) \right)$$

$$A^\mu(x) = \int \widetilde{d^3k} \sum_{\lambda=0}^{3} e^\mu(k,\lambda) \left(a_{\vec{k},\lambda} e^{-ik\cdot x} + a^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right).$$

$$A^i(x) = \int \widetilde{d^3k} \sum_{\lambda=0}^{3} e^i(k,\lambda) \left(a_{\vec{k},\lambda} e^{-ik\cdot x} + a^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right)$$

$$\dot{A}^i(x) = \int \widetilde{d^3k} \sum_{\lambda=0}^{3} e^i(k,\lambda) \left(- a_{\vec{k},\lambda} e^{-ik\cdot x} + a^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right) (i k^0)$$

$$\vec{\nabla} A^i(x) = \int \widetilde{d^3k} \, \Sigma_{\lambda=0}^{3} e^i(k,\lambda) \left(a_{\vec{k},\lambda} e^{-ik\cdot x} - a^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right) (i \vec{k})$$

$$= \int \widetilde{d^3k} \, \Sigma_{\lambda=0}^{3} e^i(k,\lambda) \left(+ a_{\vec{k},\lambda} e^{-ik\cdot x} - a^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right) (i \vec{k})$$

$$2H = \int d^3x \left(\Sigma_i ((\dot{A}^i)^2 + (\nabla A^i)^2) - ((\dot{A}^0)^2 + (\nabla A^0)^2) \right)$$

$$= \int d^3x \Big\{ \Sigma_i \int \widetilde{d^3k} \sum_{\lambda=0}^{3} e^i(k,\lambda) \left(- a_{\vec{k},\lambda} e^{-ik\cdot x} + a^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right) (i k^0)$$

$$\times \int \widetilde{d^3k'} \sum_{\lambda=0}^{3} e^i(k',\lambda) \left(- a_{\vec{k}',\lambda} e^{-ik'\cdot x} + a^\dagger_{\vec{k}',\lambda} e^{ik'\cdot x} \right) (i k'^0)$$

$$+ \int \widetilde{d^3k} \, \Sigma_{\lambda=0}^{3} e^i(k,\lambda) \left(+ a_{\vec{k},\lambda} e^{-ik\cdot x} - a^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right) (i \vec{k}) \cdot (i \vec{k}')$$

$$\int \widetilde{d^3k'} \, \Sigma_{\lambda=0}^{3} e^i(k',\lambda) \left(+ a_{\vec{k}',\lambda} e^{-ik'\cdot x} - a^\dagger_{\vec{k}',\lambda} e^{ik'\cdot x} \right)$$

$$- \int \widetilde{d^3k} \sum_{\lambda=0}^{3} e^0(k,\lambda) \left(- a_{\vec{k},\lambda} e^{-ik\cdot x} + a^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right) (i k^0)$$

$$\times \int \widetilde{d^3k'} \sum_{\lambda=0}^{3} e^0(k',\lambda) \left(- a_{\vec{k}',\lambda} e^{-ik'\cdot x} + a^\dagger_{\vec{k}',\lambda} e^{ik'\cdot x} \right) (i k'^0)$$

$$- \int \widetilde{d^3k}\ \sum_{\lambda=0}^{3} e^0(k,\lambda)\left(+a_{\vec k,\lambda}\, e^{-ik\cdot x} - a^\dagger_{\vec k,\lambda}\, e^{ik\cdot x}\right)(i\vec k)\cdot(i\vec k')$$

$$\int \widetilde{d^3k'}\ \sum_{\lambda=0}^{3} e^0(k',\lambda)\left(+a_{\vec k',\lambda}\, e^{-ik'\cdot x} - a^\dagger_{\vec k',\lambda}\, e^{ik'\cdot x}\right)\Big\}$$

Normal Ordering:

$$2H = \int d^3x \Big\{ \sum_i \int \widetilde{d^3k} \times \int \widetilde{d^3k'} \sum_{\lambda=0}^{3}\sum_{\lambda'=0}^{3} e^i(k',\lambda')\, e^i(k,\lambda)$$

$$\begin{pmatrix} +a_{\vec k,\lambda} a_{\vec k',\lambda'} e^{-i(k+k')\cdot x} + a^\dagger_{\vec k,\lambda} a^\dagger_{\vec k',\lambda'} e^{i(k+k')\cdot x} \\ - a^\dagger_{\vec k,\lambda} a_{\vec k',\lambda'} e^{i(k-k')\cdot x} - a^\dagger_{\vec k',\lambda'} a_{\vec k,\lambda} e^{i(k'-k)\cdot x} \end{pmatrix} \times (-1)\times(k^0 k'^0)$$

$$+ \int \widetilde{d^3k} \times \int \widetilde{d^3k'} \sum_{\lambda=0}^{3}\sum_{\lambda'=0}^{3} e^i(k',\lambda')\, e^i(k,\lambda)$$

$$\begin{pmatrix} +a_{\vec k,\lambda} a_{\vec k',\lambda'} e^{-i(k+k')\cdot x} + a^\dagger_{\vec k,\lambda} a^\dagger_{\vec k',\lambda'} e^{i(k+k')\cdot x} \\ - a^\dagger_{\vec k,\lambda} a_{\vec k',\lambda'} e^{i(k-k')\cdot x} - a^\dagger_{\vec k',\lambda'} a_{\vec k,\lambda} e^{i(k'-k)\cdot x} \end{pmatrix} \times (-1)\times(\vec k\cdot \vec k')$$

$$- \int \widetilde{d^3k} \times \int \widetilde{d^3k'} \sum_{\lambda=0}^{3}\sum_{\lambda'=0}^{3} e^0(k',\lambda')\, e^0(k,\lambda)$$

$$\begin{pmatrix} +a_{\vec k,\lambda} a_{\vec k',\lambda'} e^{-i(k+k')\cdot x} + a^\dagger_{\vec k,\lambda} a^\dagger_{\vec k',\lambda'} e^{i(k+k')\cdot x} \\ - a^\dagger_{\vec k,\lambda} a_{\vec k',\lambda'} e^{i(k-k')\cdot x} - a^\dagger_{\vec k',\lambda'} a_{\vec k,\lambda} e^{i(k'-k)\cdot x} \end{pmatrix} \times (-1)\times(k^0 k'^0)$$

$$- \int \widetilde{d^3k} \times \int \widetilde{d^3k'} \sum_{\lambda=0}^{3}\sum_{\lambda'=0}^{3} e^0(k',\lambda')\, e^0(k,\lambda)$$

$$\begin{pmatrix} +a_{\vec k,\lambda} a_{\vec k',\lambda'} e^{-i(k+k')\cdot x} + a^\dagger_{\vec k,\lambda} a^\dagger_{\vec k',\lambda'} e^{i(k+k')\cdot x} \\ - a^\dagger_{\vec k,\lambda} a_{\vec k',\lambda'} e^{i(k-k')\cdot x} - a^\dagger_{\vec k',\lambda'} a_{\vec k,\lambda} e^{i(k'-k)\cdot x} \end{pmatrix} \times (-1)\times(\vec k\cdot \vec k')\Big\}$$

$$= \int d^3x \int \widetilde{d^3k} \int \widetilde{d^3k'} \sum_{\lambda=0}^{3}\sum_{\lambda'=0}^{3} \left(\sum_i e^i(k,\lambda)\, e^i(k',\lambda') - e^0(k,\lambda)\, e^0(k',\lambda')\right)$$

$$\times(-1)\times(k^0 k'^0 + \vec k\cdot\vec k')$$

$$\times \begin{pmatrix} +a_{\vec k,\lambda} a_{\vec k',\lambda'} e^{-i(k+k')\cdot x} + a^\dagger_{\vec k,\lambda} a^\dagger_{\vec k',\lambda'} e^{i(k+k')\cdot x} \\ - a^\dagger_{\vec k,\lambda} a_{\vec k',\lambda'} e^{i(k-k')\cdot x} - a^\dagger_{\vec k',\lambda'} a_{\vec k,\lambda} e^{i(k'-k)\cdot x} \end{pmatrix}$$

对生成湮灭算符的积分会产生 δ function. 但 $\vec k' = -\vec k$ 时, $k^0 k'^0 + \vec k\cdot\vec k' = 0$
(无质量条件, 故只有最后2项有贡献.

$$2H = \int \widetilde{d^3k} \int \widetilde{d^3k'} \sum_{\lambda=0}^{3}\sum_{\lambda'=0}^{3} e^\mu(k,\lambda)\, e_\mu(k,\lambda')\,(k^0k^0 + \vec k\cdot\vec k)$$

$$\left(-a^\dagger_{\vec k,\lambda} a_{\vec k,\lambda'} - a_{\vec k,\lambda'} a_{\vec k,\lambda}\right)(2\pi)^3\delta^{(3)}(\vec k' - \vec k)$$

$$\Big\}\ e^\mu(k,\lambda)\, e_\mu(k,\lambda') = g_{\lambda\lambda'}$$

$$= \int \widetilde{d^3k} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3} g_{\lambda\lambda'} \times (2k^{0\,2})$$

$$\left(-2a^\dagger_{k,\lambda} a_{\vec{k},\lambda'} \right) \frac{1}{2E_k}$$

$$:H: = \int \widetilde{d^3k} \left(\sum_{\lambda=1}^{3} a^\dagger_{k,\lambda} a_{k,\lambda} - a^\dagger_{k,0} a_{k,0} \right).$$

physical state.

○ 为什么是 creation & annihilation operator :

由于: Hamiltonian

$$H = \int d^3k\, \omega_k \left(\sum_{\lambda=1}^{3} a^{\dagger}_{k\lambda} a_{k\lambda} - a^{\dagger}_{k0} a_{k0}\right)$$

commutation relation :

$$[a_{k,\lambda}, a^{\dagger}_{k',\lambda'}] = -g_{\lambda\lambda'}\, \delta^{(3)}(k-k')$$

于是:

$$[a_{k,\lambda}, H] = [a_{k,\lambda}, -\int d^3k'\, \omega_{k'} (\sum_{\lambda'=0}^{3} a^{\dagger}_{k'\lambda'} a_{k'\lambda'} g_{\lambda'\lambda'})]$$

$$= -\sum_{\lambda'}\int d^3k'\, \omega_{k'} \cdot [a_{k,\lambda}, a^{\dagger}_{k'\lambda'} a_{k'\lambda'}]\, g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'}\int d^3k'\, \omega_{k'} (a_{k,\lambda} a^{\dagger}_{k'\lambda'} a_{k'\lambda'} - a^{\dagger}_{k'\lambda'} a_{k,\lambda} a_{k',\lambda'} + a^{\dagger}_{k',\lambda'} a_{k,\lambda} a_{k,\lambda'} - a^{\dagger}_{k',\lambda'} a_{k',\lambda'} a_{k,\lambda})\, g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'}\int d^3k'\, \omega_{k'}\, [a_{k,\lambda}, a^{\dagger}_{k',\lambda'}]\, a_{k',\lambda'}\, g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'}\int d^3k'\, \omega_{k'}\, (-g_{\lambda\lambda'})\, \delta^{(3)}(k-k')\, a_{k'\lambda'}\, g_{\lambda'\lambda'}$$

$$= \omega_k\, a_{k,\lambda}$$

$$[a^{\dagger}_{k,\lambda}, H] = [a^{\dagger}_{k,\lambda}, -\int d^3k'\, \omega_{k'} (\sum_{\lambda'=0}^{3} a^{\dagger}_{k'\lambda'} a_{k'\lambda'} g_{\lambda'\lambda'})]$$

$$= -\sum_{\lambda'}\int d^3k'\, \omega_{k'} \cdot [a^{\dagger}_{k,\lambda}, a^{\dagger}_{k'\lambda'} a_{k'\lambda'}]\, g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'}\int d^3k'\, \omega_{k'} (a^{\dagger}_{k,\lambda} a^{\dagger}_{k'\lambda'} a_{k'\lambda'} - a^{\dagger}_{k'\lambda'} a^{\dagger}_{k,\lambda} a_{k',\lambda'} + a^{\dagger}_{k',\lambda'} a^{\dagger}_{k,\lambda} a_{k,\lambda'} - a^{\dagger}_{k',\lambda'} a_{k',\lambda'} a^{\dagger}_{k,\lambda})\, g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'}\int d^3k'\, \omega_{k'} (-a^{\dagger}_{k',\lambda'})\, \underline{[a_{k',\lambda'}, a^{\dagger}_{k,\lambda}]}\, g_{\lambda'\lambda'}$$

$$= -\sum_{\lambda'}\int d^3k'\, \omega_{k'} (-a^{\dagger}_{k',\lambda'}) (-g_{\lambda\lambda'}\, \delta^{(3)}(k-k'))\, g_{\lambda'\lambda'}$$

$$= -\omega_k\, a^{\dagger}_{k,\lambda}$$

可见若有 H 的本征态:(则 $a^{\dagger}_{k,\lambda}$ 将其变为 $E+\omega_k$ 的本征态, $a_{k,\lambda}$ 将其变为 $E-\omega_k$ 的本征态)

$$H|E\rangle = E|E\rangle$$

则:

$$H a_{k,\lambda}|E\rangle = a_{k,\lambda} H|E\rangle + [H, a_{k,\lambda}]|E\rangle = E\, a_{k,\lambda}|E\rangle - \omega_k a_{k,\lambda}|E\rangle = (E-\omega_k)\, a_{k,\lambda}|E\rangle$$

$$H a^{\dagger}_{k,\lambda}|E\rangle = a^{\dagger}_{k,\lambda} H|E\rangle + [H, a^{\dagger}_{k,\lambda}]|E\rangle = (E+\omega_k)\, a^{\dagger}_{k,\lambda}|E\rangle$$

○ 定义 physical state.

—— 真空态定义，真空态定义为 $a_{k,\lambda}|0\rangle=0$！(能量最低态)

则，$a^{+}_{k,\lambda}|0\rangle$ 是有 $w_{k,\lambda}$ 能量的本征态！

激发态本征态内积：

$$\langle 0|a_{k,\lambda}a^{+}_{k,\lambda}|0\rangle=\langle 0|a^{+}_{k,\lambda}a_{k,\lambda}+[a_{k,\lambda},a^{+}_{k,\lambda}]|0\rangle$$

$$[a_{k,\lambda},a^{+}_{k',\lambda'}]=-g_{\lambda\lambda'}\delta^{(3)}(k-k')$$

$$=\langle 0|a^{+}_{k,\lambda}a_{k,\lambda}-g_{\lambda\lambda}\delta^{(3)}(0)|0\rangle$$

$$=-g_{\lambda\lambda}\delta^{(3)}(0)\langle 0|0\rangle$$

$$=-g_{\lambda\lambda}\delta^{(3)}(0)$$

—— 解决 $\delta^{(3)}(0)$ 的存在.

改变定义：

在k附近有非零值的 Function.

$$\begin{cases}|1_{k,\lambda}\rangle=\int d^3k'\,F_k(k')\,a^{+}_{k',\lambda}|0\rangle\\ \int d^3k'|F(k')|^2=1\end{cases}$$

$$\langle 1_{k,\lambda}|1_{k,\lambda}\rangle=\int d^3k''d^3k'\,F^{*}_k(k'')F_k(k')\left(-g_{\lambda\lambda}\delta^{(3)}(k'-k'')\right)$$

$$=-g_{\lambda\lambda}\langle 0|0\rangle$$

$$=-g_{\lambda\lambda}$$

—— 生成 $n_{k,\lambda}$ 个 k,λ 粒子.

$$\int d^3k'd^3k''d^3k'''d^3k''''\langle 0|F_k(k''')F_k(k'''')a_{k''',\lambda}a_{k'''',\lambda}F_k(k')F_k(k'')a^{+}_{k',\lambda}a^{+}_{k'',\lambda}|0\rangle$$

$$[a_{k,\lambda},a^{+}_{k',\lambda'}]=-g_{\lambda\lambda'}\delta^{(3)}(k-k')$$

$$=\int d^3k'd^3k''d^3k'''d^3k''''F_k(k')F_k(k'')F_k(k''')F_k(k'''')\cdot\langle 0|a_{k''',\lambda}a_{k'''',\lambda}a^{+}_{k',\lambda}a^{+}_{k'',\lambda}|0\rangle$$

$$=\int d^3k'd^3k''d^3k'''d^3k''''F_k(k')F_k(k'')F_k(k''')F_k(k'''')\cdot\left(\langle 0|a_{k''',\lambda}a^{+}_{k',\lambda}a_{k'''',\lambda}a^{+}_{k'',\lambda}|0\rangle-\langle 0|a_{k''',\lambda}a^{+}_{k'',\lambda}|0\rangle g_{\lambda\lambda}\delta^{(3)}(k'-k'''')\right)$$

$$=\int d^3k'd^3k''d^3k'''d^3k''''F_k(k')F_k(k'')F_k(k''')F_k(k'''')\cdot\Big(\langle 0|a_{k''',\lambda}a^{+}_{k',\lambda}a^{+}_{k'',\lambda}a_{k'''',\lambda}|0\rangle-\langle 0|a^{+}_{k'',\lambda}a_{k''',\lambda}|0\rangle g_{\lambda\lambda}\delta^{(3)}(k'-k'''')$$
$$-\langle 0|a_{k''',\lambda}a^{+}_{k',\lambda}\rangle g_{\lambda\lambda}\delta^{(3)}(k''-k'''')+\langle 0|0\rangle g_{\lambda\lambda}\cdot g_{\lambda\lambda}\cdot\delta^{(3)}(k'-k'''')\delta^{(3)}(k''-k''')\Big)$$

$$=\int d^3k'd^3k''d^3k'''d^3k''''F_k(k')F_k(k'')F_k(k''')F_k(k'''')\cdot\Big(\langle 0|a_{k''',\lambda}a^{+}_{k',\lambda}a^{+}_{k'',\lambda}a_{k'''',\lambda}|0\rangle-\langle 0|a^{+}_{k'',\lambda}a_{k''',\lambda}|0\rangle g_{\lambda\lambda}\delta^{(3)}(k'-k'''')$$
$$+\langle 0|0\rangle g_{\lambda\lambda}g_{\lambda\lambda}\delta^{(3)}(k''-k'''')\delta^{(3)}(k'''-k')+\langle 0|0\rangle g_{\lambda\lambda}\cdot g_{\lambda\lambda}\cdot\delta^{(3)}(k'-k'''')\delta^{(3)}(k''-k''')\Big)$$

$$=2g_{\lambda\lambda}\cdot g_{\lambda\lambda}$$

—— 定义.

$$|n_{k,\lambda}\rangle=\frac{1}{\sqrt{n_{k,\lambda}!}}\left(a^{+}_{k,\lambda}\right)^{n_{k,\lambda}}|0\rangle\;;\quad\longrightarrow\begin{cases}a_{k,\lambda}|n_{k,\lambda}\rangle=(-g_{\lambda\lambda})\sqrt{n_{k,\lambda}}\,|n_{k,\lambda}-1\rangle\\ a^{+}_{k,\lambda}|n_{k,\lambda}\rangle=\sqrt{n_{k,\lambda}+1}\cdot|n_{k,\lambda}+1\rangle\end{cases}$$

$$\langle n_{k,\lambda} | n_{k,\lambda} \rangle = (g_{\lambda\lambda})^{n_{k,\lambda}} \left(\delta^{(3)}(0)\right)^{n_{k,\lambda}}$$

—— 能量

$$\langle n_{k,\lambda} | H | n_{k,\lambda} \rangle = n_{k,\lambda} W_{k,\lambda} \langle n_{k,\lambda} | n_{k,\lambda} \rangle .$$

$$= n_{k,\lambda} W_{k,\lambda} \cdot (g_{\lambda\lambda})^{n_{k,\lambda}} \left(\delta^{(3)}(0)\right)^{n_{k,\lambda}}$$

—— 粒子数

$$\hat{n}_{k,\lambda} = a_{k,\lambda}^{+} a_{k,\lambda}$$

$$\hat{n}_{k,\lambda} | n_{k,\lambda} \rangle = a_{k,\lambda}^{+} a_{k,\lambda} | n_{k,\lambda} \rangle$$

$$= a_{k,\lambda}^{+} (-g_{\lambda\lambda}) \sqrt{n_{k,\lambda}} \, | n_{k,\lambda} - 1 \rangle$$

$$= (-g_{\lambda\lambda}) \cdot n_{k,\lambda} | n_{k,\lambda} - 1 \rangle$$

$$\langle n_{k,\lambda} | \hat{n}_{k,\lambda} | n_{k,\lambda} \rangle = (-g_{\lambda\lambda}) \, n_{k,\lambda} \langle n_{k,\lambda} | n_{k,\lambda} \rangle$$

$$= (-g_{\lambda\lambda})^{n_{k,\lambda}+1} \cdot n_{k,\lambda} \left(\delta^{(3)}(0)\right)^{n_{k,\lambda}}$$

Gupta – Bleuler Method.

- Require :

$$\langle \Phi | \partial_\mu A^\mu(x) | \Phi \rangle = 0$$

stronger :

(positive frequency field. annihilation op)

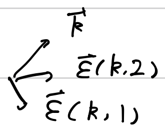
$$\partial_\mu A^{\mu(+)}(x) | \Phi \rangle = 0$$

Gurantees

$$\langle \Phi | \partial_\mu A^{\mu(-)}(x) = 0$$

$$\int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} e^{-ik\cdot x} \sum_\lambda a_{k,\lambda} k_\mu \varepsilon^\mu(k,\lambda) | \Phi \rangle = 0$$

由于 :



$$\varepsilon(k,1) = (0, \vec{\varepsilon}(k,1))$$

$$\varepsilon(k,2) = (0, \vec{\varepsilon}(k,2))$$

$$\varepsilon(k,3) = \frac{k - n(k\cdot n)}{[(k\cdot n)^2 - k^2]^{1/2}}$$

$(n = (1,0,0,0)$ In special frame!)

$$\varepsilon(k,0) = n$$

$(k^2=0)$ $k\cdot\varepsilon(k,1) = k\cdot\varepsilon(k,2) = 0$ $\quad k\cdot\varepsilon(k,3) = -(k\cdot n)$ $\quad k\cdot\varepsilon(k,0) = k\cdot n$

$$\int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} \cdot e^{-ik\cdot x} \left(a_{k,0}\, k\cdot\varepsilon(k,0) + a_{k,3}\, k\cdot\varepsilon(k,3) \right) | \Phi \rangle = 0$$

$$\int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} e^{-ik\cdot x} (n\cdot k)(a_{k,0} - a_{k,3}) | \Phi \rangle = 0$$

$$L_k | \Phi \rangle = 0$$

$$(a_{k,0} - a_{k,3}) | \Phi \rangle = 0$$

— Expectation values of numbers of longitudinal $(\lambda=3)$ and scalar $(\lambda=0)$ photons are equal.

$$\langle \Phi | a^\dagger_{k,0} a_{k,0} | \Phi \rangle = \langle \Phi | a^\dagger_{k,3} a_{k,3} | \Phi \rangle$$

$$\langle \Phi | a^\dagger_{k,0} = \langle \Phi | a^\dagger_{k,3}$$

$$a_{k,0} | \Phi \rangle = a_{k,3} | \Phi \rangle$$

— All states from transverse state

$$| \Phi_c \rangle = R_c | \Phi_T \rangle$$

Transverse photons only

$$R_c = 1 + \int d^3k\, c(k) L^\dagger_k + \int d^3k\, d^3k'\, c(k,k') L^\dagger_k L^\dagger_{k'} + \cdots$$

$$L_k = a_{k,0} - a_{k,3}$$

— state from transverse satisfies Gupta – Bleuler relation.

$$[L_k, L^\dagger_{k'}] = [a_{k,0} - a_{k,3}, a^\dagger_{k',0} - a^\dagger_{k',3}] = [a_{k,0}, a^\dagger_{k',0}] + [a_{k,3}, a^\dagger_{k',3}] = 0$$

$$[L_k, R_c] = 0$$

$$L_k R_c | \Phi_T \rangle = R_c L_k | \Phi_T \rangle = 0$$

— pseudo photons do not contributes to norm.

$$\langle \Phi_{c'} | \Phi_c \rangle = \langle \Phi_{T'} | R_{c'}^{\dagger} R_c | \Phi_T \rangle \underset{\uparrow}{=} \langle \Phi_{T'} | R_c R_{c'}^{\dagger} | \Phi_T \rangle = \langle \Phi_{T'} | \Phi_T \rangle$$

$$[R_c, R_{c'}^{\dagger}] = 0$$

Feynman propagator:

∘ 不分解的 propagator:

类比 proca field 的 propagator:

$$i D_F^{\mu\nu}(x-y) = \int \frac{d^3k}{2\omega_k (2\pi)^3} \left(\sum_{\lambda=0}^{3} (-g_{\lambda\lambda}) \varepsilon^{\mu}(k,\lambda)\, \varepsilon^{\nu}(k,\lambda) \right) \left(\Theta(x_0-y_0)\, e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\, e^{ik\cdot(x-y)} \right)$$

completeness:

$$\sum_{\lambda=0}^{3} g_{\lambda\lambda} \varepsilon^{\mu}(k,\lambda)\, \varepsilon^{\nu}(k,\lambda) = g^{\mu\nu}$$

$$= \int \frac{d^4k}{(2\pi)^4} (-g^{\mu\nu}) \left(\Theta(x_0-y_0)\, e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\, e^{ik\cdot(x-y)} \right)$$

$$= i \int \frac{d^4k}{(2\pi)^4} D_F^{\mu\nu}(k)\, e^{-ik\cdot(x-y)}$$

$$D_F^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2 + i\varepsilon}$$

∘ Decomposition propagator

$$D_{F\,\mu\nu}(k) = \frac{1}{k^2+i\varepsilon} \left(\sum_{\lambda=1}^{2} \varepsilon_\mu(k,\lambda)\, \varepsilon_\nu(k,\lambda) + \varepsilon_\mu(k,3)\, \varepsilon_\nu(k,3) - \varepsilon_\mu(k,0)\, \varepsilon_\nu(k,0) \right)$$

$$= \frac{1}{k^2+i\varepsilon} \left(\sum_{\lambda=1}^{2} \varepsilon_\mu(k,\lambda)\varepsilon_\nu(k,\lambda) + \frac{(k_\mu - n_\mu(k\cdot n))(k_\nu - n_\nu(k\cdot n))}{(k\cdot n)^2 - k^2} - n_\mu n_\nu \right)$$

$$= \frac{1}{k^2+i\varepsilon} \left(\underbrace{\sum_{\lambda=1}^{2} \varepsilon_\mu(k,\lambda)\varepsilon_\nu(k,\lambda)}_{\text{trans part}} + \underbrace{\frac{k^2 n_\mu n_\nu}{(k\cdot n)^2 - k^2}}_{\text{coul part}} + \underbrace{\frac{k_\mu k_\nu - (k_\mu n_\nu + k_\nu n_\mu)(k\cdot n)}{(k\cdot n)^2 - k^2}}_{\text{resid part!}} \right)$$

—— coul part: ($n = (1,0,0,0)$, In special Lorentz frame 时)

$$D_{F\,\mu\nu\,(coul)}(k) = \frac{\delta_{\mu 0}\delta_{\nu 0}}{|\vec{k}|^2}$$

$$D_{F\,\mu\nu\,(coul)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \cdot \frac{\delta_{\mu 0}\delta_{\nu 0}}{|\vec{k}|^2}$$

$$= \delta(x_0-y_0)\, \delta_{\mu 0}\, \delta_{\nu 0} \int \frac{d^3k}{(2\pi)^3} \frac{e^{+i\vec{k}\cdot(\vec{x}-\vec{y})}}{|\vec{k}|^2}$$

$$= \delta_{\mu 0}\delta_{\nu 0} \frac{\delta(x_0-y_0)}{4\pi|\vec{x}-\vec{y}|}$$ ← 用了奇点的性质, cauchy 积分.

Proca Equation

- Lagrangian:

real value A^μ $\quad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - j_\mu A^\mu$

charged field $\quad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu^* A^\mu - j_\mu A^\mu$

— Not Gauge Invariant:

$$A^\mu A_\mu \to (A^\mu + \partial^\mu \Lambda)(A_\mu + \partial_\mu \Lambda) = A^\mu A_\mu + 2(\partial^\mu \Lambda)A_\mu + (\partial^\mu \Lambda)(\partial_\mu \Lambda)$$

- Euler-lagrange equation

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - j_\mu A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}(2\cdot\partial^\mu A^\nu - 2\partial^\nu A^\mu)$$

$$= -F^{\mu\nu}$$

E-L Equation:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0$$

$$m^2 A^\nu - j^\nu + \partial_\mu F^{\mu\nu} = 0$$

$$m^2 A^\nu - j^\nu + \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$$

$$(\partial_\mu \partial^\mu)A^\nu + m^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu \longrightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\nu = j^\nu$$

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$$

Proca Equation $\quad \square A^\nu - \partial^\nu(\partial_\mu A^\mu) + m^2 A^\nu = j^\nu$

— 性质: Automatically satisfies Lorentz condition!

$$\partial_\nu(\square A^\nu) - \partial_\nu \partial^\nu(\partial_\mu A^\mu) + m^2 \partial_\nu A^\nu = \partial_\nu j^\nu \longrightarrow (\square + m^2)A^\nu = j^\nu$$

$$\partial_\nu A^\nu = \frac{1}{m^2}\partial_\nu j^\nu$$

conserve-current!

$$\partial_\nu A^\nu = 0$$

- Energy-momentum tensor:

$$\Theta^{\mu\nu} = -\mathcal{L}g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)}\cdot(\partial^\nu A_\sigma)$$

$$= -\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - j_\mu A^\mu\right)g^{\mu\nu} - F^{\mu\sigma}\cdot\partial^\nu A_\sigma$$

$$= \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\sigma}\cdot\partial^\nu A_\sigma - \frac{1}{2}m^2 g^{\mu\nu}A_\sigma A^\sigma + g^{\mu\nu}j_\sigma A^\sigma$$

—— Modified Momentum-Energy Tensor:

$$\tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma(\chi^{\sigma\mu\nu}) \leftarrow \text{其中 } \chi^{\sigma\mu\nu} = -\chi^{\mu\sigma\nu} \text{ (使得 } \int d^3x\, \Theta^{0\nu} \text{ 不变)}$$

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma(F^{\mu\sigma}A^\nu)$$

$$= \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\sigma}\partial^\nu A_\sigma - \frac{1}{2}m^2 g^{\mu\nu}A_\sigma A^\sigma + g^{\mu\nu}j_\sigma A^\sigma + \partial_\sigma(F^{\mu\sigma}A^\nu)$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\sigma} \partial^\nu A_\sigma - \frac{1}{2} m^2 g^{\mu\nu} A_\sigma A^\sigma + g^{\mu\nu} j_\sigma A^\sigma + \partial_\sigma (F^{\mu\sigma}) A^\nu + F^{\mu\sigma} \partial_\sigma (A^\nu)$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\sigma} F^\nu{}_\sigma - \frac{1}{2} m^2 g^{\mu\nu} A_\sigma A^\sigma + g^{\mu\nu} j_\sigma A^\sigma + \partial_\sigma (F^{\mu\sigma}) A^\nu$$

Proca Equation:

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\sigma} F_\sigma{}^\nu - \frac{1}{2} m^2 g^{\mu\nu} A_\sigma A^\sigma + g^{\mu\nu} j_\sigma A^\sigma - j^\mu A^\nu + m^2 A^\mu A^\nu$$

$$= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^{\mu\sigma} F_\sigma{}^\nu - \frac{1}{2} m^2 g^{\mu\nu} A_\sigma A^\sigma + m^2 A^\mu A^\nu + g^{\mu\nu} j_\sigma A^\sigma - j^\mu A^\nu$$

—— Modified Energy and momentum density

$$w = T^{00} = \frac{1}{2} (B^2 + E^2) + \frac{1}{2} m^2 (A_0^2 + \vec{A}^2) - \vec{j} \cdot \vec{A}$$

$$\vec{p} = E \times B - m^2 A^0 \vec{A} - j^0 \vec{A}$$

- Angular momentum : 和 Maxwell same!

plane wave Expansion — proca field!

- Proca Equation : $\begin{cases} (\Box + m^2) A^\nu = j^\nu \\ \partial_\nu A^\nu = 0 \end{cases} \Rightarrow j=0$时: $\begin{cases} (\Box + m^2) A^\nu = 0 \\ \partial_\nu A^\nu = 0 \end{cases}$

特定动量解: $k = (\sqrt{m^2 + \vec{k}^2}, \vec{k}) \rightarrow A^\mu(k, x, \lambda) = N_k \cdot \exp(-i k_\mu x^\mu) \cdot \varepsilon^\mu(k, \lambda)$

$$\begin{cases} ((-i)^2 k^\mu k_\mu + m^2) = 0 & \text{(自动满足)} \\ k_\mu \varepsilon^\mu = 0 \end{cases}$$

当然, $A^{\mu *}(k, x, \lambda)$ 也是解!

- General form of field mode.

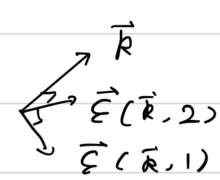
$$A^\mu(\vec{k}, x, \lambda) = N_{\vec{k}} \cdot \exp(-i(w_k t - \vec{k}\cdot\vec{x})) \cdot \varepsilon^\mu(\vec{k}, \lambda)$$

$$w_{\vec{k}} = \sqrt{m^2 + |\vec{k}|^2}$$

$$\varepsilon^\mu(\vec{k}, \lambda) \cdot \varepsilon_\mu(\vec{k}, \lambda') = g_{\lambda, \lambda'}$$

— $\lambda = 1$ spacelike $\varepsilon(\vec{k}, 1) = (0, \vec{\varepsilon}(\vec{k}, 1))$

— $\lambda = 2$ spacelike $\varepsilon(\vec{k}, 2) = (0, \vec{\varepsilon}(\vec{k}, 2))$



— $\lambda = 3$ spacelike $k^\mu \varepsilon_\mu(\vec{k}, 3) = 0$

Normalization condition: $\varepsilon^\mu(\vec{k}, \lambda) \varepsilon_\mu(\vec{k}, \lambda') = g_{\lambda\lambda'}$ → 空间指标和前两者正交 说明在空间指标上, 和 $\vec{k}$ 同方向!

$$\varepsilon(\vec{k}, 3) = \left(\frac{|\vec{k}|}{m}, \frac{\vec{k}}{|\vec{k}|} \frac{k_0}{m} \right)$$

↓ Normalization condition satisfied:

$$\varepsilon^\mu(\vec{k}, 3) \varepsilon_\mu(\vec{k}, 3) = \frac{|\vec{k}|^2}{m^2} - \frac{|\vec{k}|^2}{|\vec{k}|^2} \frac{k_0^2}{m^2}$$

$$= \frac{|\vec{k}|^2}{m^2} - \frac{k_0^2}{m^2} \qquad k^0 = \sqrt{|\vec{k}|^2 + m^2}$$

$$= -1 = g_{33}$$

— $\lambda = 0$ Timelike $\varepsilon(\vec{k}, 0) = \frac{k}{m} = \frac{1}{m}(k^0, \vec{k})$ ⟶ 但不满足 $k_\mu \varepsilon^\mu = 0$!

— Complete relation :

$$\sum_{\lambda=0}^{3} g_{\lambda\lambda} \varepsilon_\mu(\vec{k}, \lambda) \varepsilon_\nu(\vec{k}, \lambda) = g_{\mu\nu}$$

$$\sum_{\lambda=1}^{3} \varepsilon_\mu(k, \lambda) \varepsilon_\nu(k, \lambda) = -\left(g_{\mu\nu} - \frac{1}{m^2} k_\mu k_\nu\right)$$

Proca - field quantization

Hamiltonian formalism

- Hamiltonian formalism.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - j_\mu A^\mu$$

$L = \int d^3x \, \mathcal{L}(A^\mu, \partial_0 A^\mu)$

$L = L[A^\mu, \partial_0 A^\mu]$

$$\pi^\mu = \frac{\delta L}{\delta(\dot{A}_\mu)} = \frac{\delta L}{\delta(\partial_0 A_\mu)} = \frac{\partial \mathcal{L}}{\partial(\dot{A}_\mu)} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}$$

$$\frac{\partial(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta})}{\partial(\partial^\mu A^\nu)} = -F_{\mu\nu}$$

$$= -F^{0\mu}$$

$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$

$\partial_\mu F^{\mu 0} + m^2 A^0 = 0$

$-\nabla \cdot \vec{E} + m^2 A^0 = 0$

$A^0 = \frac{1}{m^2} \vec{\nabla} \cdot \vec{E}$

↑ dependent variable!

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \to F^{0\mu} = (0, -E^1, -E^2, -E^3)$$

$\pi^\mu = (0, E^1, E^2, E^3)$

— Hamiltonian density

$$\mathcal{H} = \pi_\mu \dot{A}^\mu - \mathcal{L} = -F_{0\mu} \dot{A}^\mu + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu + j_\mu A^\mu$$

$$= -\vec{E} \cdot \dot{\vec{A}} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu + j_\mu A^\mu$$

$F_{\mu\nu} F^{\mu\nu} = -2(E^2 - B^2)$ $\quad j = 0$

$$= -\vec{E} \cdot \dot{\vec{A}} - \frac{1}{2}(E^2 - B^2) - \frac{1}{2} m^2 (A_0^2 - \vec{A}^2)$$

$$\vec{E} \cdot \vec{\nabla} A_0 = \vec{\nabla} \cdot (\vec{E} A_0) - A_0 (\vec{\nabla} \cdot \vec{E})$$

$\vec{E} = -\vec{\nabla} A_0 - \partial_0 \vec{A}$

$$= \vec{\nabla} \cdot (\vec{E} A_0) + A_0 \vec{\nabla} \cdot (\vec{\nabla} A_0 + \partial_0 \vec{A})$$

$$= \vec{\nabla} \cdot (\vec{E} A_0) + A_0 (\nabla^2 A_0 + \partial_0 \vec{\nabla} \cdot \vec{A})$$

$\partial_\nu A^\nu = 0$

$$= \vec{\nabla} \cdot (\vec{E} A_0) + (\nabla^2 A_0 - \partial_0^2 A_0) A_0$$

$(\Box + m^2) A^\nu = j^\nu$

$$= \vec{\nabla} \cdot (\vec{E} A_0) + m^2 A_0^2$$

$\partial_0 \vec{A} = -\vec{\nabla} A_0 - \vec{E}$

$$= -\vec{E} \cdot (-\vec{\nabla} A_0 - \vec{E}) + \frac{1}{2}(B^2 - E^2 + m^2 \vec{A}^2) - \frac{1}{2} m^2 A_0^2$$

$$= \vec{\nabla} \cdot (\vec{E} A_0) + m^2 A_0^2 + \frac{1}{2}(B^2 + E^2 + m^2 \vec{A}^2) - \frac{1}{2} m^2 A_0^2$$

$$= \frac{1}{2}(B^2 + E^2 + m^2 \vec{A}^2 + m^2 A_0^2) + \vec{\nabla} \cdot (\vec{E} \cdot A_0)$$

$$= \frac{1}{2}(E^2 + (\nabla \times A)^2 + m^2 \vec{A}^2 + m^2 A_0^2) + \vec{\nabla} \cdot (\vec{E} \cdot A_0)$$

— Hamiltonian; H

$$= \int d^3x \, \mathcal{H} = \int d^3x \cdot \left(\frac{1}{2}(E^2 + (\nabla \times A)^2 + m^2 \vec{A}^2 + m^2 A_0^2) + \vec{\nabla} \cdot (\vec{E} \cdot A_0) \right)$$

$A_0 = \frac{1}{m^2} \nabla \cdot E$ $\quad$ leads to 0.

$\frac{1}{2} \frac{1}{m^2} (\nabla \cdot E)^2$

$$= \int d^3x \cdot \frac{1}{2} (E^2 + (\nabla \times \vec{A})^2 + m^2 \vec{A}^2 + \frac{1}{m^2} (\nabla \cdot E)^2)$$

$$= H[\vec{A}, \vec{E}]$$

Equal Time commutation relation.

- ETCR :
$$[A^i(\vec{x},t),\ E^j(\vec{x}',t)] = -i\delta_{ij}\,\delta^{(3)}(\vec{x}-\vec{x}')$$
$$[A^i(\vec{x},t),\ A^j(\vec{x}',t)] = 0$$
$$[E^i(\vec{x},t),\ E^j(\vec{x}',t)] = 0$$

- Heisenberg equation of motion :

— EOM of A
$$\frac{d\vec{A}}{dt} = \frac{1}{i\hbar}[\vec{A}, H]$$
$$i\frac{d\vec{A}}{dt} = [\vec{A}, H]$$
$$= \left[\vec{A}(x,t),\ \int d^3x'\,\frac{1}{2}\left(E^2 + (\nabla\times\vec{A})^2 + m^2\vec{A}^2 + \frac{1}{m^2}(\nabla\cdot E)^2\right)\right]$$
$$= \int d^3x'\,\frac{1}{2}[\vec{A}(x,t),\ E^2(x',t)] + \int d^3x'\,\frac{1}{2}\frac{1}{m^2}[\vec{A}(x,t),\ (\nabla\cdot\vec{E}(x',t))^2]$$

$$[A^i(\vec{x},t),\ E^j(\vec{x}',t)] = -i\delta_{ij}\,\delta^{(3)}(\vec{x}-\vec{x}')$$
$$\Downarrow$$
$$[A^i(\vec{x},t),\ E^2(x',t)] = A^i E^i(x',t)\,E^i(x',t) - E^i E^i A^i$$
$$= A^i E^i E^i - E^i A^i E^i + E^i A^i E^i - E^i E^i A^i$$
$$= [A^i(\vec{x},t),\ E^i(\vec{x}',t)]\,E^i(\vec{x}',t) + E^i(\vec{x},t)\,[A^i(\vec{x},t),\ E^i(\vec{x}',t)]$$
$$= -i2\delta^{(3)}(x-x')\,E^i(\vec{x}',t)$$
$$[\vec{A}(\vec{x},t),\ E^2(\vec{x}',t)] = -2i\,\delta^{(3)}(x-x')\,\vec{E}(\vec{x}',t)$$

$$[A^i(\vec{x},t),\ (\nabla\cdot\vec{E}(\vec{x}',t))^2] = A^i(\nabla\cdot E)(\nabla\cdot E) - (\nabla\cdot E)(\nabla\cdot E)A^i$$
$$= A^i(\nabla\cdot E)(\nabla\cdot E) - (\nabla\cdot E)A^i(\nabla\cdot E) + (\nabla\cdot E)A^i(\nabla\cdot E) - (\nabla\cdot E)(\nabla\cdot E)A^i$$
$$= [A^i(\vec{x},t),\ \nabla\cdot E(x',t)]\,\nabla\cdot E(x',t) + (\nabla\cdot E(x',t))\cdot[A^i(\vec{x},t),\ \nabla\cdot E(x',t)]$$
$$= \nabla\cdot[A^i(\vec{x},t),\ \vec{E}(x',t)]\,\nabla\cdot E(x',t) + (\nabla\cdot E(x',t))\,\nabla[A^i(x,t),\ \vec{E}(x',t)]$$
$$= 2\partial_i\left(-i\delta^{(3)}(x-x')\right)\nabla\cdot E(x',t)$$
$$= -2i\left[\partial_i\,\delta^{(3)}(x-x')\right]\nabla\cdot E(x',t)$$
$$[\vec{A}(x,t),\ (\nabla\cdot E(x',t))^2] = -2i\left(\vec{\nabla}\delta^{(3)}(x-x')\right)\nabla\cdot E(x',t)$$

$$= \int d^3x'\left(-i\delta^{(3)}(x-x')\,\vec{E}(\vec{x}',t) - \frac{1}{m^2}i\cdot\vec{\nabla}\cdot\delta^{(3)}(x-x')\ \nabla\cdot E(x',t)\right)$$
$$= -i\vec{E}(x,t) + \frac{i}{m^2}\vec{\nabla}(\nabla\cdot E(x,t))$$
$$\frac{d\vec{A}}{dt} = -\vec{E}(x,t) + \frac{1}{m^2}\vec{\nabla}(\nabla\cdot E(x,t))$$

— EOM of $\vec{E}$:

$$i\frac{d\vec{E}}{dt} = [\vec{E}, H]$$

$$= [\vec{E}(x,t), \int d^3x' \frac{1}{2}(E^2 + (\nabla\times\vec{A})^2 + m^2\vec{A}^2 + \frac{1}{m^2}(\nabla\cdot E)^2)]$$

$$= \frac{1}{2}\int d^3x' [\vec{E}(x,t), (\nabla\times\vec{A})^2] + \frac{m^2}{2}\int d^3x' [\vec{E}(x,t), \vec{A}^2(x',t)]$$

$$= \int d^3x' \left(\frac{1}{2}[\vec{E}(x,t), (\nabla\times\vec{A})^2] + \frac{m^2}{2}[\vec{E}(x,t), \vec{A}^2(x',t)]\right)$$

Consider: $[E^1(x,t), (\nabla\times\vec{A}(x',t))^2]$

$$= [E^1(x,t), (\nabla\times\vec{A}(x',t))_y^2 + (\nabla\times\vec{A}(x',t))_z^2]$$

$$= [E^1(x,t), (\partial_z A^1(x',t) - \partial_x A^3(x',t))^2 + (\partial_x A^2(x',t) - \partial_y A^1(x',t))^2]$$

$$= [E^1(x,t), \partial_3 A^1(x',t)\cdot\partial_3' A^1(x',t) - 2\partial_3 A^1(x',t)\partial_1' A^3(x',t)]$$

$$+ [E^1(x,t), \partial_2' A^1(x',t)\partial_2' A^1(x',t) - 2\partial_1' A^2(x',t)\partial_2' A^1(x',t)]$$

$$= 2\partial_3' A^1(x',t)\cdot\partial_3'\cdot[E^1(x,t), A^1(x',t)]$$

$$-2\partial_1' A^3(x',t)\,\partial_3'\,[E^1(x,t), A^1(x',t)]$$

$$+2\partial_2' A^1(x',t)\partial_2'\cdot[E^1(x,t), A^1(x',t)]$$

$$-2\partial_1' A^2(x',t)\partial_2'\,[E^1(x,t), A^1(x',t)]$$

$$= 2i\partial_3' A^1(x',t)\,\partial_3'\,\delta^{(3)}(x-x')\cdot$$

$$-2i\partial_1' A^3(x',t)\partial_3'\,\delta^{(3)}(x-x')$$

$$+2i\partial_2' A^1(x',t)\partial_2'\,\delta^{(3)}(x-x')$$

$$-2i\partial_1' A^2(x',t)\partial_2'\,\delta^{(3)}(x-x')$$

$$i\frac{dE^1}{dt} = \int d^3x' \Big(\frac{1}{2}\cdot(2i\partial_3' A^1(x',t)\partial_3'\delta^{(3)}(x-x') - 2i\partial_1' A^3(x',t)\partial_3'\delta^{(3)}(x-x')$$

$$+2i\partial_2' A^1(x',t)\partial_2'\delta^{(3)}(x-x') - 2i\partial_1' A^2(x',t)\partial_2'\delta^{(3)}(x-x'))$$

$$+\frac{m^2}{2}2i\cdot A^1(x',t)\cdot\delta^{(3)}(x-x')\Big)$$

$$= \int d^3x' \Big(-i(\partial_3'^2 A^1(x',t))\cdot\delta^{(3)}(x-x') - i(\partial_2'^2 A^1(x',t))\cdot\delta^{(3)}(x-x')$$

$$-i(\partial_1'^2 A^1(x',t))\delta^{(3)}(x-x') + i(\partial_1'^2 A^1(x',t))\delta^{(3)}(x-x')$$

$$+i(\partial_1'\partial_2' A^2(x',t))\delta^{(3)}(x-x') + i(\partial_1'\partial_3' A^3(x',t))\delta^{(3)}(x-x')$$

$$+ m^2\cdot i\cdot A^1(x',t)\delta^{(3)}(x-x')\Big)$$

$$= -i(\partial_1^2 + \partial_2^2 + \partial_3^2)A^1(x,t) + i\partial_1(\partial_1 A^1(x,t) + \partial_2 A^2(x,t) + \partial_3 A^3(x,t))$$

$$+ i m^2 A^1(x,t)$$

$$\frac{d\vec{E}}{dt} = -\nabla^2\vec{A}(x,t) + \nabla(\nabla\cdot\vec{A}(x,t)) + m^2\vec{A}(x,t)$$

Fourier Decomposition − Proca Field.

- fixed-momentum solution of proca field:

$$A^\mu(\vec{k}, x, \lambda) = N_{\vec{k}} \cdot \exp(-i(w_k t - \vec{k}\cdot\vec{x})) \cdot \varepsilon^\mu(\vec{k}, \lambda)$$

normalisation factor
$N_k = (2W_k(2\pi)^3)^{-1/2}$
$\lambda = 1, 2, 3$
$k = (\sqrt{|\vec{k}|^2 + m^2}, \vec{k})$

用 fixed-momentum solution linear Combine 为 解:(Neutral)

$$A^\mu(x) = \int d^3k \cdot \sum_{\lambda=1}^{3} (a_{k,\lambda} A^\mu(k,\lambda,x) + a^\dagger_{k,\lambda} A^{\mu *}(k,\lambda,x)) \longleftarrow \text{为了 Unitary}$$

$$= \int d^3k \sum_{\lambda=1}^{3} (a_{k,\lambda}\, \varepsilon^\mu(k,\lambda) e^{-ik\cdot x} + a^\dagger_{k,\lambda}\, \varepsilon^\mu(k,\lambda) e^{ik\cdot x}) N_k$$

For charged field (一般不考虑)

$$A^\mu(x) = \int d^3k \sum_{k=1}^{3} (a_{k,\lambda} \varepsilon^\mu(k,\lambda) e^{-ik\cdot x} + b^\dagger_{k,\lambda} \varepsilon^\mu(k,\lambda) e^{ik\cdot x}) N_k$$

— vector potential

$$\vec{A}(x) = \int d^3k \sum_{\lambda=1}^{3} (a_{k,\lambda} e^{-ik\cdot x} + a^\dagger_{k,\lambda} e^{ik\cdot x}) \vec{\varepsilon}(k,\lambda) N_k$$

— Electro field strength tensor

$$E = -\frac{\partial A}{\partial t} - \nabla A^0$$

$$\vec{E}(x) = -\int d^3k \sum_{\lambda=1}^{3} (-iW_k a_{k,\lambda} e^{-ik\cdot x} + iW_k a^\dagger_{k,\lambda} e^{ik\cdot x}) \vec{\varepsilon}(k,\lambda) N_k$$
$$- \int d^3k \sum_{\lambda=1}^{3} (a_{k,\lambda} e^{-ik\cdot x} - a^\dagger_{k,\lambda} e^{ik\cdot x}) i\vec{k}\cdot \varepsilon^0(k,\lambda) N_k$$

$$= \int d^3k \cdot i \frac{1}{\sqrt{2W_k(2\pi)^3}} \cdot \sum_{\lambda=1}^{3} W_k (\vec{\varepsilon}(k,\lambda) - \frac{\vec{k}}{W_k}\varepsilon^0(k,\lambda)) (a_{k,\lambda} e^{-ik\cdot x} - a^\dagger_{k,\lambda} e^{ik\cdot x})$$

Introducing:

$$\vec{\tilde{\varepsilon}}(k,\lambda) = \vec{\varepsilon}(k,\lambda) - \frac{\vec{k}}{W_k}\varepsilon^0(k,\lambda)$$

Transverserality relation:

$$W_k \varepsilon^0 - \vec{k}\cdot\varepsilon = 0$$

$$= \vec{\varepsilon}(k,\lambda) - \frac{\vec{k}}{W_k^2}(\vec{k}\cdot\vec{\varepsilon}(k,\lambda))$$

Property:

$$\vec{k}\cdot\vec{\tilde{\varepsilon}}(k,\lambda) = (1 - \frac{|\vec{k}|^2}{W_k^2})(\vec{k}\cdot\vec{\varepsilon}(k,\lambda)) = \frac{m^2}{W_k^2}\, \vec{k}\cdot\vec{\varepsilon}(k,\lambda)$$

$$\vec{\tilde{\varepsilon}}(k,\lambda) = f_\lambda \vec{\varepsilon}(k,\lambda) \qquad f_\lambda = \begin{cases} 1 & \lambda = 1,2 \\ m^2/W_k^2 & \lambda = 3 \end{cases}$$

∘ Project out creation & annihilation operator.

— scalar product of Two proca field:

definition $(A(x), A'(x)) = i\int d^3x\, A^{\mu *}(x) \overleftrightarrow{\partial_0} A'_\mu(x)$ $\quad \left(A \overleftrightarrow{\partial_0} B = A(\partial_0 B) - (\partial_0 A) B \right)$

Fixed-Momentum-solution:

$$A^\mu(\vec{k}, x, \lambda) = N_{\vec{k}} \cdot \exp(-i(w_k t - \vec{k}\cdot\vec{x})) \cdot \varepsilon^\mu(\vec{k}, \lambda)$$ 不是算符,是场.

$$(A(k',\lambda'), A(k,\lambda)) = i\int d^3x \cdot \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} \cdot \varepsilon^\mu(k',\lambda')\varepsilon_\mu(k,\lambda) \cdot \left(e^{ik'\cdot x} \overleftrightarrow{\partial_0} e^{-ik\cdot x}\right)$$

$$= i\int d^3x\, (e^{i(k'\cdot x - k\cdot x)})(-iW_k - iW_{k'}) \cdot \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} \varepsilon^\mu(k',\lambda')\varepsilon_\mu(k,\lambda)$$

$$= i\cdot(2\pi)^3 \cdot \delta^{(3)}(k'-k)\cdot(-2iW_k) \frac{1}{2W_k(2\pi)^3} \varepsilon^\mu(k',\lambda')\varepsilon_\mu(k,\lambda)$$

$$= \delta^{(3)}(k-k')\, \varepsilon^\mu(k,\lambda')\, \varepsilon_\mu(k,\lambda)$$

$$= \delta^{(3)}(k-k') \cdot g_{\lambda\lambda'}$$

$$(A^*(k',\lambda'), A^*(k,\lambda)) = i\int d^3x \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} \varepsilon^\mu(k',\lambda')\varepsilon_\mu(k,\lambda) \left(e^{-ik'\cdot x} \overleftrightarrow{\partial_0} e^{ik\cdot x}\right)$$

$$= i\int d^3x\, (iW_k + iW_{k'})\, e^{-ik'\cdot x + ik\cdot x} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} \varepsilon^\mu(k',\lambda')\varepsilon_\mu(k,\lambda)$$

$$= -\, 2W_k \cdot (2\pi)^3 \cdot \delta^{(3)}(k-k') \frac{1}{2W_k(2\pi)^3} \varepsilon^\mu(k,\lambda')\varepsilon_\mu(k,\lambda)$$

$$= -\,\delta^{(3)}(k-k')\, \varepsilon^\mu(k,\lambda')\, \varepsilon_\mu(k,\lambda)$$

$$= -\,\delta^{(3)}(k-k')\, g_{\lambda\lambda'}$$

$$(A^*(k',\lambda'), A(k,\lambda)) = i\int d^3x \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} \varepsilon^\mu(k',\lambda')\varepsilon_\mu(k,\lambda) \left(e^{-ik'\cdot x} \overleftrightarrow{\partial_0} e^{-ik\cdot x}\right)$$

$$= i\int d^3x \cdot (-iW_k + iW_{k'}) \cdot e^{-i(k\cdot x + k'\cdot x)} \cdots$$

$$= 0$$

$$= (A(k',\lambda'), A^*(k,\lambda))$$

— Project creation and annihilation op

$$(A(k,\lambda), A(x)) = \quad A^\mu(k,\lambda) = \frac{1}{\sqrt{2W_k(2\pi)^3}} \cdot \varepsilon^\mu(k,\lambda) e^{-ik\cdot x}$$

$$\int d^3k \frac{1}{\sqrt{2W_k(2\pi)^3}} \sum_{\lambda=1}^{3} \left(a_{k,\lambda}\, \varepsilon^\mu(k,\lambda) e^{-ik\cdot x} + a^\dagger_{k,\lambda}\, \varepsilon^\mu(k,\lambda) e^{ik\cdot x} \right)$$

$$= i \int d^3x \, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}} \varepsilon^{\mu}(k,\lambda) e^{+ik\cdot x} \overleftrightarrow{\partial_0} \cdot \frac{1}{\sqrt{2W_{k'}(2\pi)^3}}$$
$$\sum_{\lambda'=1}^{3} \left(a_{k',\lambda'} \varepsilon_{\mu}(k',\lambda') e^{-ik'\cdot x} + a^{\dagger}_{k',\lambda'} \varepsilon_{\mu}(k',\lambda') e^{ik'\cdot x} \right)$$

$$= i \sum_{\lambda'=1}^{3} \int d^3x \, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^{\mu}(k,\lambda) \varepsilon_{\mu}(k',\lambda')$$
$$\Big[e^{ik\cdot x} \cdot \left(-iW_{k'} a_{k',\lambda'} e^{-ik'\cdot x} + iW_{k'} a^{\dagger}_{k',\lambda'} e^{ik'\cdot x} \right)$$
$$- iW_k e^{ik\cdot x} \left(a_{k',\lambda'} e^{-ik'\cdot x} + a^{\dagger}_{k',\lambda'} e^{ik'\cdot x} \right) \Big]$$

$$= i \sum_{\lambda'=1}^{3} \int d^3x \, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^{\mu}(k,\lambda) \varepsilon_{\mu}(k',\lambda')$$
$$e^{ik\cdot x} \Big[a_{k',\lambda'} e^{-ik'\cdot x} (-iW_{k'} - iW_k) + a^{\dagger}_{k',\lambda'} e^{ik'\cdot x} (iW_{k'} - iW_k) \Big]$$

$$= i \sum_{\lambda'=1}^{3} \int d^3k' (2\pi)^3 \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^{\mu}(k,\lambda) \varepsilon_{\mu}(k',\lambda')$$
$$a_{k',\lambda'} \delta^{(3)}(k-k') (-2iW_k)$$
$$= \sum_{\lambda'=1}^{3} (2\pi)^3 \frac{1}{2W_k(2\pi)^3} \varepsilon^{\mu}(k,\lambda) \varepsilon_{\mu}(k,\lambda') a_{k,\lambda'} (2W_k)$$

$$= \sum_{\lambda'=1}^{3} g_{\lambda,\lambda'} a_{k,\lambda'}$$
$$= -a_{k,\lambda} \quad (g_{\lambda,\lambda} = -1 \text{ for } \lambda \neq 0)$$

这只是个场，但$A(x)$是operator!

$$(A^{*}(k,\lambda), A(x)) = \qquad A^{\mu *}(k,\lambda) = \frac{1}{\sqrt{2W_k(2\pi)^3}} \cdot \varepsilon^{\mu}(k,\lambda) e^{+ik\cdot x}$$

$$A(x) = \int d^3k \frac{1}{\sqrt{2W_k(2\pi)^3}} \sum_{\lambda=1}^{3} \left(a_{k,\lambda} \varepsilon^{\mu}(k,\lambda) e^{-ik\cdot x} + a^{\dagger}_{k,\lambda} \varepsilon^{\mu}(k,\lambda) e^{ik\cdot x} \right)$$

$$= i \int d^3x \, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}} \varepsilon^{\mu}(k,\lambda) e^{-ik\cdot x} \overleftrightarrow{\partial_0} \cdot \frac{1}{\sqrt{2W_{k'}(2\pi)^3}}$$
$$\sum_{\lambda'=1}^{3} \left(a_{k',\lambda'} \varepsilon_{\mu}(k',\lambda') e^{-ik'\cdot x} + a_{k',\lambda'} e^{ik'\cdot x} \right)$$

$$= i \sum_{\lambda'=1}^{3} \int d^3x \, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^{\mu}(k,\lambda) \varepsilon_{\mu}(k',\lambda')$$
$$\Big[e^{-ik\cdot x} \cdot \left(-iW_{k'} a_{k',\lambda'} e^{-ik'\cdot x} + iW_{k'} a^{\dagger}_{k',\lambda'} e^{ik'\cdot x} \right)$$
$$+ iW_k e^{-ik\cdot x} \left(a_{k',\lambda'} e^{+ik'\cdot x} + a^{\dagger}_{k',\lambda'} e^{-ik'\cdot x} \right) \Big]$$

$$= i \sum_{\lambda'=1}^{3} \int d^3x \, d^3k' \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^{\mu}(k,\lambda) \varepsilon_{\mu}(k',\lambda')$$
$$\Big(e^{-ik\cdot x} a_{k',\lambda'} e^{-ik'\cdot x} (-iW_{k'} + iW_k)$$
$$+ e^{-ik\cdot x} a^{\dagger}_{k',\lambda'} e^{ik\cdot x} (iW_{k'} + iW_k) \Big)$$

$$= i \sum_{\lambda'=1}^{3} \int d^3k' (2\pi)^3 \frac{1}{2W_k \cdot (2\pi)^3} \cdot \varepsilon^{\mu}(k,\lambda) \varepsilon_{\mu}(k,\lambda') \cdot (2iW_k) a^{\dagger}_{k,\lambda'} \delta^{(3)}(k-k')$$

$$= -\sum_{\lambda'=1}^{3} \varepsilon^{\mu}(k,\lambda) \varepsilon_{\mu}(k,\lambda') a^{\dagger}_{k,\lambda'} = a^{\dagger}_{k,\lambda}$$

Commutation of creation and annihilation operator.

- $a_{k,\lambda}$; $a^{\dagger}_{k,\lambda}$ 的另一种表达(不同于上页):

— $a_{k,\lambda} = -\left(A(k,\lambda),\ A(x)\right)$

$$= -i\int d^3x\ A^{\mu *}(k,\lambda)\ \overleftrightarrow{\partial_0}\ A_\mu(x)$$

$$A^\mu(k,\lambda) = \frac{1}{\sqrt{2W_k(2\pi)^3}}\ \varepsilon^\mu(k,\lambda)\ e^{-ik\cdot x}$$

$$= -i\int d^3x\left(A^{\mu *}(k,\lambda)\ \partial_0 A_\mu(x) - (\partial_0 A^{\mu *}(k,\lambda))\ A_\mu(x)\right)$$

$$= -i\int d^3x\left(A^{\mu *}(k,\lambda)\ \partial_0 A_\mu(x) - iW_k A^{\mu *}(k,\lambda)\ A_\mu(x)\right)$$

$$= -i\int d^3x \frac{1}{\sqrt{2W_k(2\pi)^3}}\ \varepsilon^\mu(k,\lambda)\ e^{ik\cdot x}\left(\partial_0 A_\mu(x) - iW_k A_\mu(x)\right)$$

proca Equation: $\partial_\mu A^\mu = 0 \rightarrow \partial_0 A^0 = -\nabla\cdot\vec{A}$

Definition of A: $\vec{E} = -\partial_0\vec{A} - \vec{\nabla}A_0$

$$= -i\int d^3x \frac{1}{\sqrt{2W_k(2\pi)^3}}\ e^{ik\cdot x}\left(\varepsilon^0\partial_0 A_0 - \vec{\varepsilon}\cdot\partial_0\vec{A} - iW_k\varepsilon^0 A_0 + iW_k\vec{\varepsilon}\cdot\vec{A}\right)$$

$$= -i\int d^3x \frac{1}{\sqrt{2W_k(2\pi)^3}}\ e^{ik\cdot x}\left(-\varepsilon^0\nabla\cdot\vec{A} + \vec{\varepsilon}\cdot\vec{E} + \vec{\varepsilon}\cdot\vec{\nabla}A_0 - iW_k\varepsilon^0 A_0 + iW_k\vec{\varepsilon}\cdot\vec{A}\right)$$

对橙色部分用 Integral by parts!

$$= -i\int d^3x \frac{e^{ik\cdot x}}{\sqrt{2W_k(2\pi)^3}}\left(-i\varepsilon^0\vec{k}\cdot\vec{A} + \vec{\varepsilon}\cdot\vec{E} + i\vec{\varepsilon}\cdot\vec{k}\,A_0 - iW_k\varepsilon^0 A_0 + iW_k\vec{\varepsilon}\cdot\vec{A}\right)$$

$k^\mu = (W_k, \vec{k})$, $W_k = \sqrt{|\vec{k}|^2 + m^2}$

$k^\mu\varepsilon_\mu = 0$ (Fixed momentum - proca solution, 性质)

$$= -i\int d^3x \frac{e^{ik\cdot x}}{\sqrt{2W_k(2\pi)^3}}\left(-i\varepsilon^0\vec{k}\cdot\vec{A} + \vec{\varepsilon}\cdot\vec{E} + iW_k\vec{\varepsilon}\cdot\vec{A}\right)$$

$$= \int d^3x \frac{e^{ik\cdot x}}{\sqrt{2W_k(2\pi)^3}}\left((W_k\vec{\varepsilon} - \varepsilon^0\vec{k})\cdot\vec{A} - i\,\vec{\varepsilon}\cdot\vec{E}\right)$$

$$\vec{\tilde{\varepsilon}}(k,\lambda) = \vec{\varepsilon}(k,\lambda) - \frac{\varepsilon^0}{W_k}\vec{k}$$

$$= \int d^3x \frac{e^{ik\cdot x}}{\sqrt{2W_k(2\pi)^3}}\left(W_k\,\vec{\tilde{\varepsilon}}(k,\lambda)\,\vec{A}(x) - i\,\vec{\varepsilon}(k,\lambda)\cdot\vec{E}(x)\right)$$

— $a^{\dagger}_{k,\lambda} = +\left(A^*(k,\lambda),\ A(x)\right)$

$$= +i\int d^3x\ A^\mu(k,\lambda)\ \overleftrightarrow{\partial_0}\ A_\mu(x)$$

$$A^\mu(k,\lambda) = \frac{1}{\sqrt{2W_k(2\pi)^3}}\ \varepsilon^\mu(k,\lambda)\ e^{-ik\cdot x}$$

$$= i\int d^3x\left(A^\mu(k,\lambda)\partial_0 A_\mu(x) - \partial_0 A^\mu(k,\lambda)\ A_\mu(x)\right)$$

$$= i\int d^3x\ A^\mu(k,\lambda)\left(\partial_0 A_\mu(x) + iW_k\ A_\mu(x)\right)$$

$$- = i\int d^3x \frac{1}{\sqrt{2W_k(2\pi)^3}} \varepsilon^{\mu}(k,\lambda)\, e^{-ik\cdot x}(\partial_0 A_\mu(x) + iW_k A_\mu(x))$$

} proca eq: $\partial_\mu A^\mu \to \partial_0 A_0 = -\vec{\nabla}\cdot\vec{A}$

} Definition relation A & E: $\partial_0\vec{A} = -\vec{E} - \nabla A_0$

$$= i\int d^3x \frac{1}{\sqrt{2W_k(2\pi)^3}} e^{-ik\cdot x}(-\varepsilon^0\nabla\cdot\vec{A} + \vec{\varepsilon}\cdot\vec{E} + \vec{\varepsilon}\cdot\nabla A_0$$
$$+ iW_k\varepsilon^0 A_0 - iW_k\vec{\varepsilon}\cdot\vec{A})$$

} 分部积分: $\varepsilon^0\nabla\cdot\vec{A} \to -i\vec{k}\cdot\vec{A}\ \varepsilon^0$

$\vec{\varepsilon}\cdot\vec{\nabla}A_0 \to -i\vec{k}\cdot\vec{\varepsilon}\ A_0$

$$= i\int d^3x \frac{1}{\sqrt{2W_k(2\pi)^3}} e^{-ik\cdot x}(+i\vec{k}\cdot\vec{A}\varepsilon^0 + \vec{\varepsilon}\cdot\vec{E} - i\vec{k}\cdot\vec{\varepsilon}A_0$$
$$+ iW_k\varepsilon^0 A_0 - iW_k\vec{\varepsilon}\cdot\vec{A})$$

} $k^\mu\varepsilon_\mu = 0$

$$= i\int d^3x \frac{1}{\sqrt{2W_k(2\pi)^3}} e^{-ik\cdot x}(i\vec{k}\cdot\vec{A}\varepsilon^0 + \vec{\varepsilon}\cdot\vec{E} - iW_k\vec{\varepsilon}\cdot\vec{A})$$

$$= \int d^3x \frac{1}{\sqrt{2W_k(2\pi)^3}} e^{-ik\cdot x}\Big(\underbrace{(W_k\vec{\varepsilon} - \vec{k}\varepsilon^0)}_{=W_k\tilde{\vec{\varepsilon}}}\cdot\vec{A} + i\,\vec{\varepsilon}\cdot\vec{E}\Big)$$

——— 综上:

$$a_{k,\lambda} = \int d^3x \frac{e^{ik\cdot x}}{\sqrt{2W_k(2\pi)^3}}\left(W_k\,\tilde{\vec{\varepsilon}}(k,\lambda)\,\vec{A}(x) - i\,\vec{\varepsilon}(k,\lambda)\cdot\vec{E}(x)\right)$$

$$a^\dagger_{k,\lambda} = \int d^3x \frac{e^{-ik\cdot x}}{\sqrt{2W_k(2\pi)^3}}\left(W_k\,\tilde{\vec{\varepsilon}}(k,\lambda)\,\vec{A}(x) + i\,\vec{\varepsilon}(k,\lambda)\cdot\vec{E}(x)\right)$$

$$\tilde{\vec{\varepsilon}}(k,\lambda) = \vec{\varepsilon}(k,\lambda) - \frac{\varepsilon^0(k,\lambda)}{W_k}\vec{k}$$

∘ Commutation of creation & anni

$$[a_{k',\lambda'}, a^\dagger_{k,\lambda}] = \int d^3x\, d^3x' \frac{1}{\sqrt{2W_{k'}(2\pi)^3}}\frac{1}{\sqrt{2W_k(2\pi)^3}} e^{ik'\cdot x'} e^{-ik\cdot x}$$

$$[W_{k'}\tilde{\vec{\varepsilon}}(k',\lambda')\cdot\vec{A}(x') - i\vec{\varepsilon}(k',\lambda')\cdot\vec{E}(x'),$$
$$W_k\tilde{\vec{\varepsilon}}(k,\lambda)\cdot\vec{A}(x) + i\vec{\varepsilon}(k,\lambda)\cdot\vec{E}(x)]$$

$$= \int d^3x\, d^3x' \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} e^{ik'\cdot x'} e^{-ik\cdot x}$$

$$\Big([W_{k'} \vec{\tilde{\varepsilon}}(k',\lambda')\cdot\vec{A}(x'), i\vec{\varepsilon}(k,\lambda)\cdot\vec{E}(x)] \to pt1$$

$$+ [-i\vec{\varepsilon}(k',\lambda')\cdot\vec{E}(x'), W_k \vec{\tilde{\varepsilon}}(k,\lambda)\cdot\vec{A}(x)]\Big) \to pt2$$

$$pt1 = iW_{k'} \vec{\tilde{\varepsilon}}(k',\lambda')\cdot\vec{\varepsilon}(k,\lambda)(-i\delta^{(3)}(x-x'))$$

$$pt2 = iW_k \vec{\tilde{\varepsilon}}(k,\lambda)\cdot\vec{\varepsilon}(k',\lambda')(-i\delta^{(3)}(x-x'))$$

$$= \int d^3x\, d^3x' \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} e^{ik'\cdot x'} e^{-ik\cdot x} \delta^{(3)}(x-x')$$

$$\Big(W_{k'} \vec{\tilde{\varepsilon}}(k',\lambda')\cdot\vec{\varepsilon}(k,\lambda) + W_k \vec{\tilde{\varepsilon}}(k,\lambda)\cdot\vec{\varepsilon}(k',\lambda')\Big)$$

$$= \int d^3x \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \frac{1}{\sqrt{2W_k(2\pi)^3}} e^{-ik'\cdot x} e^{-ik\cdot x}$$

$$\Big(W_{k'} \vec{\tilde{\varepsilon}}(k',\lambda')\cdot\vec{\varepsilon}(k,\lambda) + W_k \vec{\tilde{\varepsilon}}(k,\lambda)\cdot\vec{\varepsilon}(k',\lambda')\Big)$$

$$= \frac{1}{\sqrt{2W_k}\sqrt{2W_{k'}}} \cdot \delta^{(3)}(k-k')(\sim)$$

$$= \frac{1}{2W_k} W_k \Big(\vec{\tilde{\varepsilon}}(k,\lambda')\cdot\vec{\varepsilon}(k,\lambda) + \vec{\tilde{\varepsilon}}(k,\lambda)\cdot\vec{\varepsilon}(k,\lambda')\Big)\delta^{(3)}(k-k')$$

$$\vec{\tilde{\varepsilon}}(k,\lambda') = \vec{\varepsilon}(k,\lambda') - \frac{1}{W_k}\varepsilon^0(k,\lambda')\vec{k}$$

$$\vec{\tilde{\varepsilon}}(k,\lambda')\cdot\vec{\varepsilon}(k,\lambda) = \Big(\vec{\varepsilon}(k,\lambda') - \frac{\varepsilon^0(k,\lambda')}{W_k}\vec{k}\Big)\cdot\vec{\varepsilon}(k,\lambda)$$

$$= \vec{\varepsilon}(k,\lambda')\cdot\vec{\varepsilon}(k,\lambda) - \frac{\varepsilon^0(k,\lambda')}{W_k}\underbrace{\vec{k}\cdot\vec{\varepsilon}(k,\lambda)}_{=W_k\varepsilon^0(k,\lambda)}$$

$$= \vec{\varepsilon}(k,\lambda')\cdot\vec{\varepsilon}(k,\lambda) - \varepsilon^0(k,\lambda')\varepsilon^0(k,\lambda)$$

$$= -\varepsilon(k,\lambda')\cdot\varepsilon(k,\lambda)$$

$$= -g_{\lambda,\lambda'}$$

$$= \delta_{\lambda,\lambda'}$$

$$= \delta^{(3)}(k-k')\delta_{\lambda,\lambda'}$$

physical Quantities

- Hamiltonian：(12页 推导…)

$$\vec{A}(x) = \int \frac{d^3k}{\sqrt{2w_k(2\pi)^3}} \sum_{\lambda=1}^{3} \vec{\varepsilon}(k,\lambda)\left(a_{k,\lambda} e^{-ik\cdot x} + a^\dagger_{k,\lambda} e^{ik\cdot x}\right)$$

$$\vec{E}(x) = i\int \frac{d^3k}{\sqrt{2w_k(2\pi)^3}} \sum_{\lambda=1}^{3} w_k \vec{\tilde{\varepsilon}}(k,\lambda)\left(a_{k,\lambda} e^{-ik\cdot x} - a^\dagger_{k,\lambda} e^{ik\cdot x}\right)$$

$$\vec{\tilde{\varepsilon}}(k,\lambda) = \vec{\varepsilon}(k,\lambda) - \frac{1}{w_k}\varepsilon^0(k,\lambda)\vec{k}$$

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Normal order Hamiltonian：(creator stand to left of annihilators)(Commutation时不考虑负号. Anti-commutation 考虑负号)

$$H = :\int d^3x \frac{1}{2}\left(E^2 + m^2A^2 + (\nabla\times A)^2 + \frac{1}{m^2}(\nabla\cdot E)^2\right):$$

$$= \sum_{\lambda=1}^{3} \int d^3k\, w_k a^\dagger_{k,\lambda} a_{k,\lambda}$$

- Momentum：

$$P = :\int d^3x \; T^{0i}:$$

$$= \sum_{\lambda=1}^{3} \int d^3k\, \vec{k} a^\dagger_{k,\lambda} a_{k,\lambda}$$

- 自旋 spin.

由 Maxwell 场的 Noether theorem：

$$S = \int d^3x \; :E\times A:$$

— Helicity 螺旋态：$\Lambda = \vec{S}\cdot\vec{k}\frac{1}{|\vec{k}|}$

$$\Lambda = i\int d^3k\left(a^\dagger_{k2} a_{k1} - a^\dagger_{k1} a_{k2}\right)$$

— spherical Basis：

$$a_{k+} = \frac{1}{\sqrt{2}}\left(a_{k1} - i a_{k2}\right) \qquad a_{k-} = \frac{1}{\sqrt{2}}\left(a_{k1} + i a_{k2}\right)$$

$$a_{k0} = a_{k3}$$

$$\Lambda = \int d^3k\left(a^\dagger_{k+} a_{k+} - a^\dagger_{k-} a_{k-}\right)$$

commutation relation

$$[a_{k'+}, a^\dagger_{k+}] = [a_{k'-}, a^\dagger_{k-}] = [a_{k'0}, a^\dagger_{k0}] = \delta^{(3)}(k-k')$$

Hamiltonian under spherical basis：

$$H = \sum_{\sigma=-,0,+} \int d^3k\, w_k a^\dagger_{k,\sigma} a_{k,\sigma}$$

Commutator of Field A

$$A^{\mu}(x) = \sum_{\lambda=1}^{3} \int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} \varepsilon^{\mu}(k,\lambda) \cdot \left(a_{k,\lambda} e^{-ik\cdot x} + a_{k,\lambda}^{\dagger} e^{ik\cdot x} \right)$$

$$\Downarrow$$

$$[A^{\mu}(x), A^{\nu}(y)] = \int \frac{d^3k'}{\sqrt{2\omega_{k'}(2\pi)^3}} \int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} \sum_{\lambda,\lambda'=1}^{3} \varepsilon^{\mu}(k',\lambda')\varepsilon^{\nu}(k,\lambda) \Big([a_{k',\lambda'}, a_{k,\lambda}^{\dagger}] e^{-i(k'\cdot x - k\cdot y)} + [a_{k',\lambda'}^{\dagger}, a_{k,\lambda}] e^{i(k'\cdot x - k y)} \Big)$$

$$[a_{k,\lambda}, a_{k',\lambda'}^{\dagger}] = i\delta_{\lambda,\lambda'} \delta^{(3)}(k-k')$$

$$= \int \frac{d^3k}{2\omega_k (2\pi)^3} \sum_{\lambda,\lambda'=1}^{3} \varepsilon^{\mu}(k,\lambda')\varepsilon^{\nu}(k,\lambda) \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right)$$

$$\sum_{\lambda,\lambda'=1}^{3} \varepsilon^{\mu}(k,\lambda')\varepsilon^{\nu}(k,\lambda) = -g^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{m^2}$$

$$= \int \frac{d^3k}{2\omega_k (2\pi)^3} \left(-g^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{m^2}\right) \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right)$$

$$= \left(-g^{\mu\nu} - \frac{1}{m^2}\partial^{\mu}\partial^{\nu}\right) \int \frac{d^3k}{2\omega_k (2\pi)^3} \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right)$$

Invariant Pauli-Jordan Func:

$$i\Delta(x-y) = -i\int \frac{d^3k}{(2\pi)^3} \frac{\sin(k\cdot(x-y))}{\omega_k}$$

$$[A^{\mu}(x), A^{\nu}(y)] = -i\left(g^{\mu\nu} + \frac{1}{m^2}\partial^{\mu}\partial^{\nu}\right)\Delta(x-y)$$

Feynman Propogator

- Feynman propogator 定义为:

$$i\Delta_F^{\mu\nu}(x-y) := \langle 0|T(A^\mu(x) A^\nu(y))|0\rangle$$

$$\left\{ A^\mu(x) = \sum_{\lambda'=1}^{3} \int \frac{d^3k'}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^\mu(k',\lambda')\left(a_{k',\lambda'} e^{-ik'\cdot x} + a^\dagger_{k',\lambda'} e^{ik'\cdot x}\right) \right.$$

$$= \langle 0| \sum_{\lambda,\lambda'=1}^{3} \int d^3k' d^3k \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^\mu(k',\lambda')\varepsilon^\nu(k,\lambda)$$

$$(a_{k',\lambda'} e^{-ik'\cdot x} + a^\dagger_{k',\lambda'} e^{ik'\cdot x})(a_{k,\lambda} e^{-ik\cdot y} + a^\dagger_{k,\lambda} e^{ik\cdot y})|0\rangle$$

$$= \sum_{\lambda,\lambda'=1}^{3} \int d^3k' d^3k \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^\mu(k',\lambda')\varepsilon^\nu(k,\lambda)$$

$$\Big[e^{-ik'\cdot x} e^{ik\cdot y} \langle 0|a_{k',\lambda'} a^\dagger_{k,\lambda}|0\rangle \Theta(x_0-y_0)$$
$$+ e^{ik'\cdot x} e^{-ik\cdot y} \langle 0|a_{k,\lambda} a^\dagger_{k',\lambda'}|0\rangle \Theta(y_0-x_0) \Big]$$

Creation / annihilation commutation

$$[a_{k',\lambda'}, a^\dagger_{k,\lambda}] = \delta^{(3)}(k-k')\delta_{\lambda,\lambda'}$$

$$\langle 0|a_{k',\lambda'} a^\dagger_{k,\lambda}|0\rangle = \langle 0|\delta^{(3)}(k-k')\delta_{\lambda,\lambda'} + a^\dagger_{k,\lambda} a_{k',\lambda'}|0\rangle = \delta^{(3)}(k-k')\delta_{\lambda,\lambda'}$$

$$= \sum_{\lambda,\lambda'=1}^{3} \int d^3k' d^3k \frac{1}{\sqrt{2W_k(2\pi)^3}} \frac{1}{\sqrt{2W_{k'}(2\pi)^3}} \varepsilon^\mu(k',\lambda')\varepsilon^\nu(k,\lambda)$$

$$\Big[e^{-ik'\cdot x} e^{ik\cdot y} \Theta(x_0-y_0) \delta^{(3)}(k-k')$$
$$+ e^{ik'\cdot x} e^{-ik\cdot y} \Theta(y_0-x_0) \delta^{(3)}(k-k') \Big]\delta_{\lambda,\lambda'}$$

$$= \sum_{\lambda=1}^{3} \int d^3k \frac{1}{2W_k(2\pi)^3} \varepsilon^\mu(k,\lambda)\varepsilon^\nu(k,\lambda)$$
$$\left(e^{-ik\cdot(x-y)} \Theta(x_0-y_0) + e^{ik\cdot(x-y)} \Theta(y_0-x_0)\right)$$

$$\left\} \sum_{\lambda=1}^{3} \varepsilon^\mu(k,\lambda)\varepsilon^\nu(k,\lambda) = -\left(g^{\mu\nu} - \frac{1}{m^2} k^\mu k^\nu\right)\right.$$

$$= -\int d^3k \frac{1}{2W_k(2\pi)^3}\left(g^{\mu\nu} - \frac{1}{m^2}k^\mu k^\nu\right)\left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)}\right)$$

-

$$i\partial_x^\mu\, i\partial_x^\nu \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)}\right)$$
$$= k^\mu k^\nu \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)}\right)$$
$$+ g^{\mu 0} \cdot [i\partial_x^0 \Theta(x_0-y_0)][i\partial_x^\nu e^{-ik\cdot(x-y)}] + g^{\mu 0}[i\partial_x^0 \Theta(y_0-x_0)][i\partial_x^\nu e^{ik(x-y)}]$$
$$+ g^{\nu 0} [i\partial_x^0 \Theta(x_0-y_0)][i\partial_x^\mu e^{-ik\cdot(x-y)}] + g^{\nu 0}[i\partial_x^0 \Theta(y_0-x_0)][i\partial_x^\mu e^{ik\cdot(x-y)}]$$
$$+ g^{\mu 0} g^{\nu 0}[i\partial_x^0 i\partial_x^0 \Theta(x_0-y_0)] e^{-ik\cdot(x-y)} + g^{\mu 0} g^{\nu 0}[i\partial_x^0 i\partial_x^0 \Theta(y_0-x_0)] e^{ik\cdot(x-y)}$$