$-\partial_x^i \delta$ $\qquad\qquad$ $-\partial_x^0$

$$= k^\mu k^\nu \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$
$$+ g^{\mu 0} [i\delta(x_0-y_0)][k^\nu e^{-ik\cdot(x-y)}] + g^{\mu 0}[-i\delta(x_0-y_0)][-k^\nu e^{ik(x-y)}]$$
$$+ g^{\nu 0} [i\delta(x_0-y_0)][k^\mu e^{-ik\cdot(x-y)}] + g^{\nu 0}[-i\delta(x_0-y_0)][-k^\mu e^{ik\cdot(x-y)}]$$
$$+ g^{\mu 0} g^{\nu 0} [-\delta'(x_0-y_0)] e^{-ik\cdot(x-y)} + g^{\mu 0} g^{\nu 0} [+\delta'(x_0-y_0)] e^{ik\cdot(x-y)}$$

$$= k^\mu k^\nu \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$
$$+ ik^\nu g^{\mu 0} \delta(x_0-y_0) e^{-ik\cdot(x-y)} + ik^\nu g^{\mu 0} \delta(x_0-y_0) e^{ik(x-y)}$$
$$+ ik^\mu g^{\nu 0} \delta(x_0-y_0) e^{-ik\cdot(x-y)} + ik^\mu g^{\nu 0} \delta(x_0-y_0) e^{ik\cdot(x-y)}$$
$$- g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{-ik\cdot(x-y)} + g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{ik(x-y)}$$

于是:

$$k^\mu k^\nu \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$
$$= i\partial_x^\mu\, i\partial_x^\nu \left(\Theta(x_0-y_0) e^{-ik(x-y)} + \Theta(y_0-x_0) e^{ik(x-y)} \right)$$
$$- i(k^\nu g^{\mu 0} + k^\mu g^{\nu 0}) \delta(x_0-y_0) \left(e^{-ik\cdot(x-y)} + e^{ik\cdot(x-y)} \right)$$
$$+ g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{-ik\cdot(x-y)} - g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{ik\cdot(x-y)}$$

$i\Delta_F^{\mu\nu}(x-y)$
$$= -\int d^3k \frac{1}{2\omega_k (2\pi)^3} \left(g^{\mu\nu} - \frac{1}{m^2} k^\mu k^\nu \right) \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$

$$= -\int d^3k \frac{1}{2\omega_k (2\pi)^3} \left(g^{\mu\nu} - \frac{1}{m^2} (i\partial_x^\mu)(i\partial_x^\nu) \right) \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$

$$+ \frac{1}{m^2} \int d^3k \frac{1}{2\omega_k (2\pi)^3} \Big[-i(k^\nu g^{\mu 0} + k^\mu g^{\nu 0}) \delta(x_0-y_0) \left(e^{-ik\cdot(x-y)} + e^{ik\cdot(x-y)} \right)$$
$$+ g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{-ik\cdot(x-y)} - g^{\mu 0} g^{\nu 0} \delta'(x_0-y_0) e^{ik\cdot(x-y)} \Big]$$

$\int d^3k \cdot f(k)$ (k的奇 func) $= 0$

$\delta'(x-x_0) f(x) \to -\delta(x-x_0) f'(x)$

$$= -\int d^3k \frac{1}{2\omega_k (2\pi)^3} \left(g^{\mu\nu} - \frac{1}{m^2} (i\partial_x^\mu)(i\partial_x^\nu) \right) \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$

$$+ \frac{1}{m^2} \int d^3k \frac{1}{2\omega_k (2\pi)^3} \Big[+i\omega_k\, g^{\mu 0} g^{\nu 0} \delta(x_0-y_0) e^{-ik\cdot(x-y)}$$
$$+ i\omega_k\, g^{\mu 0} g^{\nu 0} \delta(x_0-y_0) e^{ik(x-y)} \Big]$$

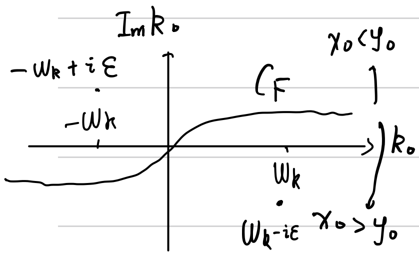
$$= -\left(g^{\mu\nu} + \frac{1}{m^2} \partial_x^\mu \partial_x^\nu \right) i\Delta_F(x-y) + \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x-y)$$

$$i\Delta_F(x-y) = \int \frac{d^3k}{2\omega_k (2\pi)^3} \left(\Theta(x_0-y_0) e^{-ik\cdot(x-y)} + \Theta(y_0-x_0) e^{ik\cdot(x-y)} \right)$$

• Momentum Feynman propogator :

$\to k=(W_k,\vec{k})$

$$i\Delta_F(x-y) = \int \frac{d^3k}{2W_k(2\pi)^3}\left(\Theta(x_0-y_0)\cdot e^{-ik\cdot(x-y)} + \Theta(y_0-x_0)\, e^{ik\cdot(x-y)}\right)$$



$$= \int \frac{d^3k}{2W_k(2\pi)^3}\, e^{-iW_k(x_0-y_0)}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\ (x_0>y_0) \to = \int d^3k\left[\int_{-\infty-i\epsilon}^{+\infty+i\epsilon} -\frac{1}{2\pi i}\frac{dk_0\cdot e^{-ik_0(x_0-y_0)}}{(k_0-W_k)(k_0+W_k)}\right]\frac{1}{(2\pi)^3}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$+\int \frac{d^3k}{2W_k(2\pi)^3}\, e^{iW_k(x_0-y_0)}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\ (y_0>x_0)$$

$\downarrow$

$$= \int d^3k\left[\int_{-\infty-i\epsilon}^{+\infty+i\epsilon}\frac{-1}{2\pi i}\frac{dk_0\, e^{ik_0(x_0-y_0)}}{(k_0-W_k)(k_0+W_k)}\right]\frac{1}{(2\pi)^3}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$= \frac{i}{(2\pi)^4}\int_{C_F}\frac{d^4k}{(k_0-W_k)(k_0+W_k)}\,e^{-ik\cdot(x-y)}$$

$\}\ W_k\to W_k-i\epsilon$

$\downarrow$

$$=\frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2-W_k^2+i\epsilon}\,e^{ik\cdot(x-y)}$$

$\}\ (k-W_k)(k+W_k)\to k^2-(W_k-i\epsilon)^2$

$\qquad\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\qquad k^2-W_k^2+\epsilon^2+2iW_k\epsilon$

$\qquad\qquad\qquad\qquad\downarrow$

$\qquad\qquad\qquad\qquad \underline{k^2-W_k^2+i\epsilon}$

• $i\Delta_F^{\mu\nu}(x-y) = -\left(g^{\mu\nu}+\frac{1}{m^2}\partial_x^\mu\partial_x^\nu\right)i\Delta_F(x-y)+\frac{i}{m^2}g^{\mu0}g^{\nu0}\delta^{(4)}(x-y)$

$$= -\left(g^{\mu\nu}+\frac{1}{m^2}\partial^\mu_x\partial^\nu_x\right)\frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2-W_k^2+i\epsilon}\cdot e^{-ik\cdot(x-y)}+\frac{i}{m^2}g^{\mu0}g^{\nu0}\int d^4k\frac{1}{(2\pi)^4}e^{-ik\cdot(x-y)}$$

$$=\int\frac{i\,d^4k}{(2\pi)^4}\cdot\left[-\frac{\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{m^2}\right)}{k^2-W_k^2+i\epsilon}+\frac{1}{m^2}g^{\mu0}g^{\nu0}\right]$$

$$\Delta_F^{\mu\nu}(k) = -\frac{\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{m^2}\right)}{k^2-W_k^2+i\epsilon}+\frac{1}{m^2}g^{\mu0}g^{\nu0}$$

Interacting quantum fields.

Interacting picture.

- From schrodinger picture to Interacting picture.

—— Schrodinger picture with split Hamiltonian

Involution: $|\alpha,t\rangle^S = U(t,t_0)|\alpha,t_0\rangle^S$

Equation of motion: $i\frac{\partial}{\partial t}|\alpha,t\rangle^S = (H_0^S + H_i^S)|\alpha,t\rangle^S$ $\rightarrow \begin{cases} i\frac{\partial}{\partial t}U = (H_0^S + H_i^S)U \\ i\frac{\partial}{\partial t}U^\dagger = -U^\dagger H^S \end{cases}$

Operator Involution: $O^S(t) = O^S(t_0) \equiv O^S$

—— Heisenberg Picture.

State: $|\alpha,t\rangle^H = U^\dagger(t,t_0)|\alpha,t\rangle^S = |\alpha,t_0\rangle^H = |\alpha,t_0\rangle^S$

Operator: $O^H(t) = U^\dagger(t,t_0)\, O^S\, U(t,t_0)$

EoM: $i\frac{\partial}{\partial t}O^H = (i\frac{\partial}{\partial t}U^\dagger)O^S U + U^\dagger O^S(i\frac{\partial}{\partial t}U)$

$= -U^\dagger H^S O^S U + U^\dagger O^S H^S U$

$= -U^\dagger H^S U U^\dagger O^S U + U^\dagger O^S U U^\dagger H^S U$

$= -H^H O^H + O^H H^H$

$= [O^H, H^H]$

—— Interaction picture.

State: $|\alpha,t\rangle^I = U_0^\dagger(t,t_0)|\alpha,t\rangle^S = U_0^\dagger(t,t_0)U(t,t_0)|\alpha,t_0\rangle^S$

operator: $O^I(t) = U_0^\dagger(t,t_0)O^S U_0(t,t_0)$

其中，$U_0(t,t_0)$ 是如下方程的解：

$i\frac{\partial}{\partial t}U_0(t,t_0) = H_0^S U_0(t,t_0) \longrightarrow i\frac{\partial}{\partial t}U_0^\dagger(t,t_0) = -U_0^\dagger(t,t_0)H_0^S$

Equation of motion — State:

$i\frac{\partial}{\partial t}|\alpha,t\rangle^I = i\frac{\partial}{\partial t}(U_0^\dagger(t,t_0))\cdot|\alpha,t\rangle^S + U_0^\dagger(t,t_0)\cdot i\frac{\partial}{\partial t}|\alpha,t\rangle^S$

$= -U_0^\dagger H_0^S|\alpha,t\rangle^S + U_0^\dagger(t,t_0)\cdot H^S|\alpha,t\rangle^S$

$= U_0^\dagger H_i^S|\alpha,t\rangle^S = U_0^\dagger H_i^S U_0 U_0^\dagger|\alpha,t\rangle^S$

$= H_i^I|\alpha,t\rangle^I$

Equation of motion —— Operator:

$i\frac{\partial}{\partial t}O^I(t) = (i\frac{\partial}{\partial t}U_0^\dagger(t,t_0))O^S U_0 + U_0^\dagger O^S(i\frac{\partial}{\partial t}U_0)$

$= -U_0^\dagger H_0^S O^S U_0 + U_0^\dagger O^S H_0^S U_0$

$= U_0^\dagger O^S U_0 U_0^\dagger H_0^S U_0 - U_0^\dagger H_0^S U_0 U_0^\dagger O^S U_0$

$= [O^I, H_0^I]$

—— H_0^S 不含时间时的简化：

此时：$i\frac{\partial}{\partial t}U_0(t,t_0) = H_0^S U_0(t,t_0)$ 的解的形式较简单.

$U_0(t,t_0) = e^{-iH_0^S t}$

$H_0^I = U_0^\dagger(t,t_0)H_0^S U_0(t,t_0) = H_0^S$ $\left(H_i^I = U_0^\dagger H_i^S U_0 = e^{iH_0^S t}H_i^S e^{-iH_0^S t}\right)$

$\begin{cases} i\frac{\partial}{\partial t}O^I(t) = [O^I(t), H_0^S] \\ i\frac{\partial}{\partial t}|\alpha,t\rangle^I = H_i^I|\alpha,t\rangle^I \end{cases}$ ⟶ 播算符的演化和只有 H_0 的情况相同.

- 定态（Quantum Mechanics 中的概念），review.

Schrodinger pic, H_0^S 中不含时间.

$$U(t,t_0) = e^{-iH_0^S(t-t_0)}$$

H_0^S 的本征态在经过时间演化后也是时间本征态.

$$H_0^S|E\rangle = E|E\rangle$$

$$U(t,t_0)|E\rangle = e^{-iE(t-t_0)}|E\rangle$$

- 若 $|a\rangle$ 是 operator A^S 的本征态，则在不同的 picture 下，相应的态矢量也是相应算符的本征态.

Schrodinger: $A^S|a\rangle^S = a\,|a\rangle^S$

Heisenberg: $A^H = U^\dagger(t,t_0)A^S U(t,t_0)$ $\quad |a\rangle^H = U^\dagger(t,t_0)|a\rangle^S \rightarrow A^H|a\rangle^H = a\,|a\rangle^H$

Interacting: $A^I = U_0^\dagger(t,t_0)A^S U_0(t,t_0)$ $\quad |a\rangle^I = U_0^\dagger(t,t_0)|a\rangle^S \longrightarrow A^I|a\rangle^I = a\,|a\rangle^I$

Time-Evolution operator and Dyson series.

- Interaction picture 中的 time-evolution picture：

  Define：
  $$|\alpha,t\rangle^I = U_I(t,t_0)\,|\alpha,t_0\rangle^I$$
  Satisfies：
  $$i\frac{\partial}{\partial t}U_I(t,t_0) = H_I^I\, U_I(t,t_0)$$

- 积分解 Time-Evolution op：
  $$U_I(t,t_0) = \mathbb{I} + (-i)\int_{t_0}^{t} dt'\, H_I^I(t')\, U(t',t_0)$$

- Neumann series：
  $$\begin{aligned} U_I(t,t_0) = \mathbb{I} &+ (-i)\int_{t_0}^{t} dt_1\, H_I^I(t_1) \\ &+ (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I^I(t_1)\, H_I^I(t_2) \\ &+ \cdots \\ &+ (-i)^n \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_I^I(t_1)\cdots H_I^I(t_n) \\ &+ \cdots \end{aligned}$$

- Dyson product.

  —— Time order：(should have minus when handling fermion particle; But H has even number of fermion op)
  $$T(H_I(t_1)H_I(t_2)\cdots H_I(t_n)) = H_I(t_{i_1})\cdots H(t_{i_n})$$
  $$(t_{i_1} \geq t_{i_2} \geq t_{i_3} \cdots)$$

  —— Dyson series
  $$\begin{aligned} U_I(t,t_0) &= \sum_{n=0}^{\infty}\frac{1}{n!}(-i)^n \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n\, T(H_I^I(t_1)\cdots H_I^I(t_n)) \\ &= T\exp\left(-i\int_{t_0}^{t} dt'\, H_I^I(t')\right) \\ &= T\exp\left(-i\int d^4x'\, \mathcal{H}_I^I(x')\right) \end{aligned}$$

The Gell-Mann-Low Theorem.

目的：求 $H_I(t=0)$ 的本征 state/value，

○ $H^S = H_0^S + H_i^S$ 不含时间时，$U_0(t,t_0) = e^{-iH_0^S(t-t_0)}$

$$H_0^I = H_0^S \ ; \quad H_i^I(t) = e^{iH_0^S(t-t_0)} H_i^S \cdot e^{-iH_0^S(t-t_0)} \qquad H^I = H_0^I + H_i^I$$

由 Interacting picture 部分的知识，找到 Interacting picture 下的 $H^I(t)$ 的本征态，经过变化，就是 Sch picture 下的本征态！ $H^I(t)|\Omega,t\rangle^I = E_0^I|\Omega,t\rangle^I \Rightarrow$

$\left\{ U_0^\dagger(t,t_0) H^S(t) U_0(t,t_0) = H^I(t) \right.$

$U_0^\dagger(t,t_0) H^S U_0(t,t_0) \cdot |\Omega,t\rangle^I = E_0^I |\Omega,t\rangle^I$

$H^S \cdot \underbrace{U_0(t,t_0)|\Omega,t\rangle^I}_{\Downarrow\ |\Omega,t\rangle^S} = E_0^I \cdot U_0(t,t_0)|\Omega,t\rangle^I$

求出 $|\Omega,t\rangle^I \longrightarrow |\Omega,t\rangle^S$ 是 $H^S(t)$ 的本征态！

$\downarrow$ 由 $H^S(t)$ 不含时

$|\Omega,t\rangle^S \sim e^{-iE_0^I t}|\Omega\rangle$

$\downarrow$ Heisenberg 下

$|\Omega,t\rangle^H$ 不随时变

$|\Omega,t\rangle^H = U^\dagger(t,t_0) U_0(t,t_0)|\Omega,t\rangle^I$

下面的讨论都在 Interacting Picture 下进行！

已知：

$$H = H_0 + \lambda H_1(t) \quad \left(H_1(t) = e^{iH_0 t} H_1(0) e^{-iH_0 t} \right)$$

H_0 的本征值已知！

$$H_0|\Phi\rangle = E_0|\Phi\rangle$$

目标：求 $H_0 + \lambda H_1(0)$ 的本征值

在 Interacting Picture 下改变 Hamiltonian：

$$H = H_0 + \lambda e^{-\epsilon|t|} H_1(t)$$

取态 $|\Phi\rangle$，$H_0|\Phi\rangle = E_0|\Phi\rangle$，让其从 $t=-\infty$ 开始演化！

$$|\Phi_\epsilon(-\infty)\rangle = |\Phi\rangle$$

$$|\Phi_\epsilon(t)\rangle = U_\epsilon(t,-\infty)|\Phi\rangle$$

U_ϵ satisfies Equation of Motion：

$$i\partial_t U_\epsilon(t,-\infty) = \lambda e^{-\epsilon|t|} H_1(t) U_\epsilon(t,-\infty)$$

定义：$|\Phi_\epsilon\rangle = |\Phi_\epsilon(0)\rangle$

—— 计算：$(H_0 - E_0)|\Phi_\epsilon\rangle$，gain commutator

$$\begin{aligned}(H_0 - E_0)|\Phi_\epsilon\rangle &= (H_0 - E_0)U_\epsilon(0,-\infty)|\Phi\rangle \\ &= H_0 U_\epsilon(0,-\infty)|\Phi\rangle - U_\epsilon(0,-\infty)H_0|\Phi\rangle \\ &= [H_0, U_\epsilon(0,-\infty)]|\Phi\rangle\end{aligned}$$

—— Calculate commutator's specific form：

$[H_0, U_\epsilon(0,-\infty)] =$

$$\left\{\begin{aligned} &H = H_0 + \lambda e^{-\epsilon|t|} \cdot H_1(t) \\ &U_\epsilon(t,t_0) = \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \lambda^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \, e^{-\epsilon(|t_1| + \cdots |t_n|)} T(H_1(t_1)\cdots H_1(t_n)) \end{aligned}\right.$$

$$= \left[H_0 ,\ \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \cdot \lambda^n \cdot \int_{-\infty}^{0} dt_1 \cdots \int_{-\infty}^{0} dt_n \cdot e^{\epsilon(t_1 + \cdots + t_n)} \cdot T(H_1(t_1)\cdots H_1(t_n)) \right]$$

$$\begin{aligned} &[H_0, T(H_1(t_1)\cdots H_1(t_n))] \\ &= [H_0, H_1(t_{i1})\cdots H_1(t_{in})] \qquad (t_{i1} \geq t_{i2} \cdots \geq t_{in}) \\ &= H_0 H_1(t_{i1})\cdots H_1(t_{in}) - H_1(t_{i1}) H_0 \cdot H_1(t_{i2}) \cdots H_1(t_{in}) \\ &\quad + H_1(t_{i1}) H_0 H_1(t_{i2}) \cdots H_1(t_{in}) - H_1(t_{i1}) H_1(t_{i2}) H_0 \cdots H_1(t_{in}) \\ &\quad \vdots \\ &= [H_0, H_1(t_{i1})] H_1(t_{i2}) \cdots H_1(t_{in}) + \cdots + H_1(t_{i1}) \cdots H_1(t_{in-1}) H_1(t_{in}) \end{aligned}$$

$$\leftarrow\quad [O^I(t), H_0^I] = i\partial_t O^I(t)$$

$$= -i\left(\partial_{t_{i1}} H_1(t_{i1})\right)\cdot H_1(t_{i2})\cdots + \cdots + H_1(t_{i1})\cdots\left(-i\partial_{t_{in}} H_1(t_{in})\right)$$

$$= (-i)\sum_{i=i_1}^{i_n}\frac{\partial}{\partial t_i}\left(H_1(t_{i1})\cdots H_1(t_{in})\right)$$

$$= (-i)\sum_{i=1}^{n}\frac{\partial}{\partial t_i}\left(H_1(t_{i1})\cdots H_1(t_{in})\right)$$

$$= \sum_{n=1}^{+\infty}\frac{(-i)^n}{n!}\cdot\lambda^n\cdot\int_{-\infty}^0 dt_1\cdots\int_{-\infty}^0 dt_n\cdot e^{\epsilon(t_1+\cdots+t_n)}\left[H_0, T(H_1(t_1)\cdots H_1(t_n))\right]$$

$$= \sum_{n=1}^{+\infty}\frac{(-i)^n}{n!}\cdot\lambda^n\cdot\int_{-\infty}^0 dt_1\cdots\int_{-\infty}^0 dt_n\cdot e^{\epsilon(t_1+\cdots+t_n)}(-i)\sum_{i=1}^{n}\frac{\partial}{\partial t_i}T\left(H_1(t_1)\cdots H_1(t_n)\right)$$

$$= (-1)\cdot\lambda\cdot\sum_{n=1}^{\infty}\frac{(-i)^{n-1}}{(n-1)!}\cdot\frac{1}{n}\cdot\lambda^{n-1}\int_{-\infty}^0 dt_1\cdots\int_{-\infty}^0 dt_n\; e^{\epsilon(t_1+\cdots+t_n)}\sum_{i=1}^{n}\frac{\partial}{\partial t_i}T\left(H_1(t_1)\cdots H_1(t_n)\right)$$

$$= (-1)\cdot\lambda\cdot\sum_{n=1}^{+\infty}\frac{(-i)^{n-1}}{(n-1)!}\cdot\lambda^{n-1}\cdot\int_{-\infty}^0 dt_1\cdots\int_{-\infty}^0 dt_n\cdot e^{\epsilon(t_1+\cdots+t_n)}\cdot\frac{\partial}{\partial t_1}T\left(H_1(t_1)\cdots H_1(t_n)\right)$$

$$\int_{-\infty}^0 dt_1\cdot e^{\epsilon t_1}\frac{\partial}{\partial t_1}T\left(H_1(t_1)\cdots H_1(t_n)\right)$$

$$= e^{\epsilon t_1}\cdot T\left(H_1(t_1)\cdots H_1(t_n)\right)\Big|_{-\infty}^{0} - \int_{-\infty}^0 dt_1\cdot\epsilon\cdot e^{\epsilon t_1}\cdot T\left(H_1(t_1)\cdots H_1(t_n)\right)$$

$$= H_1(0)\,T\left(H_1(t_2)\cdots H_1(t_n)\right) - \epsilon\cdot\int_{-\infty}^0 dt_1\cdot e^{\epsilon t_1}\cdot T\left(H_1(t_1)\cdots H_1(t_n)\right)$$

$$= (-1)\lambda\cdot\sum_{n=1}^{\infty}\frac{(-i)^{n-1}}{(n-1)!}\cdot\lambda^{n-1}\cdot\int_{-\infty}^0 dt_2\cdots\int_{-\infty}^0 dt_n\cdot e^{\epsilon(t_2+\cdots+t_n)}\cdot H_1(0)\,T\left(H_1(t_2)\cdots H_1(t_n)\right)$$

$$+\epsilon\lambda\sum_{n=1}^{\infty}\frac{(-i)^{n-1}}{(n-1)!}\cdot\lambda^{n-1}\cdot\int_{-\infty}^0 dt_1\cdots\int_{-\infty}^0 dt_n\cdot e^{\epsilon(t_1+\cdots+t_n)}\cdot T\left(H_1(t_1)\cdots H_1(t_n)\right)$$

$$= -\lambda\, H_1(0)\cdot U_\epsilon(0,-\infty) + \epsilon\lambda\,\frac{1}{-i}\cdot\frac{\partial}{\partial\lambda}\left(\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\cdot\lambda^n\int_{-\infty}^0 dt_1\cdots\int_{-\infty}^0 dt_n\; e^{\epsilon(t_1+\cdots+t_n)}\cdot T\left(H_1(t_1)\cdots H_1(t_n)\right)\right)$$

$$= -\lambda H_1(0)\, U_\epsilon(0,-\infty) + i\epsilon\lambda\frac{\partial}{\partial\lambda}\left(U_\epsilon(0,-\infty)\right).$$

—— Results!

$$(H_0 - E_0)|\Psi_\epsilon\rangle = \left(-\lambda H_1(0)\, U_\epsilon(0,-\infty) + i\epsilon\lambda\frac{\partial}{\partial\lambda}\left(U_\epsilon(0,-\infty)\right)\right)|\Phi\rangle$$

$$= -\lambda H_1(0)|\Psi_\epsilon\rangle + i\epsilon\lambda\frac{\partial}{\partial\lambda}|\Psi_\epsilon\rangle$$

—— 计算：$(H_0 - E_0)\dfrac{|\Psi_\epsilon\rangle}{\langle\Phi|\Psi_\epsilon\rangle}$

$$(H_0 - E_0)\frac{|\Psi_\epsilon\rangle}{\langle\Phi|\Psi_\epsilon\rangle} = -\lambda H_1(0)\frac{|\Psi_\epsilon\rangle}{\langle\Phi|\Psi_\epsilon\rangle} + i\epsilon\lambda\frac{\frac{\partial}{\partial\lambda}|\Psi_\epsilon\rangle}{\langle\Phi|\Psi_\epsilon\rangle}$$

$$(H_0 + \lambda H_1(0) - E_0)\frac{|\Psi_\epsilon\rangle}{\langle\Phi|\Psi_\epsilon\rangle} = +i\epsilon\cdot\lambda\cdot\left(\frac{\partial}{\partial\lambda}\left(\frac{|\Psi_\epsilon\rangle}{\langle\Phi|\Psi_\epsilon\rangle}\right) - \frac{|\Psi_\epsilon\rangle}{\langle\Phi|\Psi_\epsilon\rangle^2}\frac{\partial}{\partial\lambda}\left(\langle\Phi|\Psi_\epsilon\rangle\right)\right)$$

$$\Big(H - \underbrace{E_0 - i\epsilon\lambda\frac{\partial}{\partial\lambda}\ln\langle\Phi|\Psi_\epsilon\rangle}_{\Downarrow}\Big)\frac{|\Psi_\epsilon\rangle}{\langle\Phi|\Psi_\epsilon\rangle} = i\epsilon\lambda\cdot\frac{\partial}{\partial\lambda}\left(\frac{|\Psi_\epsilon\rangle}{\langle\Phi|\Psi_\epsilon\rangle}\right)$$

$$E = i\epsilon\lambda\frac{\partial}{\partial\lambda}\ln\langle\Phi|\Psi_\epsilon\rangle + E_0\ !$$

—— Conclusion — Eigen state and Eigen value

$$|\Psi\rangle = \lim_{\epsilon\to 0} \frac{U_\epsilon(0,-\infty)|\Phi\rangle}{\langle\Phi|U_\epsilon(0,-\infty)|\Phi\rangle}$$

$$\Delta E = \lim_{\epsilon\to 0} (+i\epsilon\lambda)\frac{\partial}{\partial\lambda}\cdot \ln\langle\Phi|U_\epsilon(0,-\infty)|\Phi\rangle$$

Write in symmetric fashion

$$\Delta E = \lim_{\epsilon\to 0} \frac{1}{2} i\epsilon\lambda\frac{\partial}{\partial\lambda} \ln\langle\Phi|U_\epsilon(+\infty,-\infty)|\Phi\rangle$$

- Quantum field (Interaction pic)

$$|\Omega\rangle = \frac{U(0,-\infty)|0\rangle}{\langle 0|U(0,-\infty)|0\rangle} = \frac{U(0,+\infty)|0\rangle}{\langle 0|U(0,+\infty)|0\rangle}$$

Heisenberg pic

完全演化 op　　$H_I(0)$ 的本征态.

$$|\Omega\rangle^H = U_F^\dagger(0,t_0)\cdot U_0(0,t_0)|\Omega\rangle$$

$$\varphi^H(x_i) = U_F^\dagger(t_i,t_0)U_0(t_i,t_0)\varphi^I(x_i)\cdot U_0^\dagger(t_i,t_0)U_F(t_i,t_0)$$

Interact Pic 下传播子

$$|\alpha,t\rangle = \underbrace{U_0^\dagger(t,t_0)U_F(t,t_0)U_F^\dagger(t',t_0)U_0(t',t_0)}_{U(t,t')}|\alpha,t'\rangle$$

- Heisenberg Pic 下计算

$$\langle\Omega|^H \varphi^H(x_1)\varphi^H(x_2)\cdots\varphi^H(x_N)|\Omega\rangle^H \leftarrow \text{Time ordered} (t_1 > \cdots)$$

$$= \langle\Omega|^I U_0^\dagger(0,t_0)\cdot U_F(0,t_0)\cdot U_F^\dagger(t_1,t_0)U_0(t_1,t_0)\varphi^I(x_1)U_0^\dagger(t_1,t_0)U_F(t_1,t_0)\cdots$$
$$\cdots U_F^\dagger(t_N,t_0)U_0(t_N,t_0)\varphi^I(x_N)U_0^\dagger(t_N,t_0)U_F(t_N,t_0)\cdot U_0^\dagger(0,t_0)U(0,t_0)|\Omega\rangle^I$$

$$= \langle\Omega|^I\cdot U(0,t_1)\cdot\varphi^I(x_1)U(t_1,t_2)\varphi^I(x_2)\cdots\varphi^I(x_N)U(t_N,0)\cdot|\Omega\rangle^I$$

$$= \langle 0|U(\infty,0)U(0,t_1)\varphi^I(x_1)\cdots\varphi^I(x_N)U(t_N,0)\cdot U(0,-\infty)|0\rangle \Big/ \underbrace{\langle 0|U(+\infty,-\infty)|0\rangle}_{\Downarrow\ \text{リヨー化!}}$$

$$= \langle 0|T\{\varphi^I(x_1)\cdots\varphi^I(x_N)\cdot U_{+\infty,-\infty}\}|0\rangle \Big/ \langle 0|U(+\infty,-\infty)|0\rangle$$

$$\underset{\Uparrow\ \text{Heisenberg Pic,}}{\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle} = \frac{\langle 0|U(+\infty,0)\overbrace{U(0,t_1)\phi_0(x_1)U(t_1,0)}^{\text{Heisenber pic 中 op.}}\cdots\phi_n(x_n)U(t_n,0)U(0,-\infty)|0\rangle}{\langle 0|U(+\infty,-\infty)|0\rangle}$$

Normalize to $\langle\Omega|\Omega\rangle = 1$

$$= \frac{\langle 0|T\{\phi_0(x_1)\cdots\phi_0(x_n)U_{\infty,-\infty}\}|0\rangle}{\langle 0|U_{+\infty,-\infty}|0\rangle}$$

} 都是 Interact Pic.

Wick's Theorem.

- Normal-order中的交换 operator 顺序性质:

$$:\Phi_A(x_1)\Phi_B(x_2): = \Phi_A^{(-)}(x_1)\Phi_B^{(-)}(x_2) + \Phi_A^{(+)}(x_1)\Phi_B^{(+)}(x_2) + \Phi_A^{(-)}(x_1)\Phi_B^{(+)}(x_2) + \varepsilon_{AB}\cdot\Phi_B^{(-)}(x_2)\Phi_A^{(+)}(x_1)$$

$$= \varepsilon_{AB}\left(\Phi_B^{(-)}(x_2)\Phi_A^{(-)}(x_1) + \Phi_B^{(+)}(x_2)\Phi_A^{(+)}(x_1) + \varepsilon_{AB}\Phi_A^{(-)}(x_1)\Phi_B^{(+)}(x_2) + \Phi_B^{(-)}(x_2)\Phi_A^{(+)}(x_1)\right)$$

$$= \varepsilon_{AB}\cdot :\Phi_B(x_2)\Phi_A(x_1):$$

- Time-order T 的交换 operator 顺序性质:

$$T(\Phi_A(x_1)\Phi_B(x_2)) = \varepsilon_{AB}\cdot T(\Phi_B(x_2)\Phi_A(x_1))$$

- Two-operator; Time order decompose to

— $t_1 > t_2$

$$T(\Phi_A(x_1)\Phi_B(x_2)) = \Phi_A(x_1)\Phi_B(x_2)$$

$$= \Phi_A^{(+)}(x_1)\Phi_B^{(+)}(x_2) + \Phi_A^{(-)}(x_1)\Phi_B^{(-)}(x_2) + \Phi_A^{(-)}(x_1)\Phi_B^{(+)}(x_2) + \Phi_A^{(+)}(x_1)\Phi_B^{(-)}(x_2)$$

$$= \Phi_A^{(+)}(x_1)\Phi_B^{(+)}(x_2) + \Phi_A^{(-)}(x_1)\Phi_B^{(-)}(x_2) + \Phi_A^{(-)}(x_1)\Phi_B^{(+)}(x_2) + \varepsilon_{AB}\Phi_B^{(-)}(x_2)\Phi_A^{(+)}(x_1) + [\Phi_A^{(+)}(x_1), \Phi_B^{(-)}(x_2)]_{\mp}$$

+: Anti-commutator!
−: commutator!

$$= :\Phi_A(x_1)\Phi_B(x_2): + [\Phi_A^{(+)}(x_1), \Phi_B^{(-)}(x_2)]_{\mp}$$ ← c number!

$$= :\Phi_A(x_1)\Phi_B(x_2): + \langle 0|\,[\Phi_A^{(+)}(x_1), \Phi_B^{(-)}(x_2)]_{\mp}|0\rangle$$

$$= :\Phi_A\Phi_B: + \langle 0|\Phi_A^{(+)}\Phi_B^{(-)}|0\rangle$$

↓ ← $t_1 > t_2$

$$= :\Phi_A\Phi_B: + \langle 0|T(\Phi_A(x_1)\Phi_B(x_2))|0\rangle$$

— $t_2 > t_1$

$$T(\Phi_A(x_1)\Phi_B(x_2)) = \varepsilon_{AB}\cdot T(\Phi_B(x_2)\Phi_A(x_1))$$

$$= \varepsilon_{AB}\left(:\Phi_B\Phi_A: + \langle 0|T(\Phi_B(x_2)\Phi_A(x_1))|0\rangle\right)$$

$$= :\Phi_A\Phi_B: + \langle 0|T(\Phi_A\Phi_B)|0\rangle$$

— In all:

$$T(\Phi_A\Phi_B) = :\Phi_A\Phi_B: + :\underbrace{\Phi_A\Phi_B}:$$

- Contraction:

$$\underbrace{\Phi_A(x_1)\Phi_B(x_2)} = \langle 0|T(\Phi_A(x_1)\Phi_B(x_2))|0\rangle$$

$$:ABCDEF\cdots KLM: = \varepsilon_p :ABF\cdots KM:\underbrace{CE}\,\underbrace{DL}$$

↑ $\varepsilon_p = \pm 1$ is the parity of permutation of fermionic operators.

- Three-operator Time order Decompose Normal ordering & Contraction.

—— $t_1, t_2 > t_3$

$$T(\Phi_A(x_1)\Phi_B(x_2)\Phi_C(x_3)) = T(\Phi_A(x_1)\Phi_B(x_2))\Phi_C(x_3)$$

$$= :\Phi_A(x_1)\Phi_B(x_2):\Phi_C(x_3) + :\underbrace{\Phi_A(x_1)\Phi_B(x_2)}:\Phi_C(x_3)$$

Consider the term:

$$:\Phi_A(x_1)\Phi_B(x_2):\Phi_C(x_3) = \left(\Phi_A^{(+)}(x_1)\Phi_B^{(+)}(x_2) + \Phi_A^{(-)}(x_1)\Phi_B^{(-)}(x_2) + \Phi_A^{(-)}(x_1)\Phi_B^{(+)}(x_2) + \epsilon_{AB}\Phi_B^{(-)}(x_2)\Phi_A^{(+)}(x_1)\right)$$
$$\left(\Phi_C^{(+)}(x_3)+\Phi_C^{(-)}(x_3)\right)$$

$$\Phi_A^{(+)}(x_1)\Phi_B^{(+)}(x_2)\Phi_C^{(-)}(x_3) = \epsilon_{BC}\cdot\Phi_A^{(+)}(x_1)\Phi_C^{(-)}(x_3)\Phi_B^{(+)}(x_2) + \Phi_A^{(+)}(x_1)[\Phi_B^{(+)}(x_2),\Phi_C^{(-)}(x_3)]_\mp$$

$$= \epsilon_{BC}\epsilon_{AC}\Phi_C^{(-)}(x_3)\Phi_A^{(+)}(x_1)\Phi_B^{(+)}(x_2) + \epsilon_{BC}\Phi_B^{(+)}(x_2)\cdot[\Phi_A^{(+)}(x_1),\Phi_C^{(-)}(x_3)]_\mp$$
$$+ \Phi_A^{(+)}(x_1)[\Phi_B^{(+)}(x_2),\Phi_C^{(-)}(x_3)]_\mp$$

$$\Phi_A^{(-)}(x_1)\Phi_B^{(+)}(x_2)\Phi_C^{(-)}(x_3) = \epsilon_{BC}\Phi_A^{(-)}(x_1)\Phi_C^{(-)}(x_3)\Phi_B^{(+)}(x_2) + \Phi_A^{(-)}(x_1)[\Phi_B^{(+)}(x_2),\Phi_C^{(-)}(x_3)]_\mp$$

$$\epsilon_{AB}\Phi_B^{(-)}(x_2)\Phi_A^{(+)}(x_1)\Phi_C^{(-)}(x_3) = \epsilon_{AB}\cdot\epsilon_{AC}\Phi_B^{(-)}(x_2)\Phi_C^{(-)}(x_3)\Phi_A^{(+)}(x_1) + \epsilon_{AB}\Phi_B^{(-)}(x_2)[\Phi_A^{(+)}(x_1),\Phi_C^{(-)}(x_3)]_\mp$$

$$= :\Phi_A(x_1)\Phi_B(x_2)\Phi_C(x_3): + \epsilon_{BC}\Phi_B^{(+)}(x_2)[\Phi_A^{(+)}(x_1),\Phi_C^{(-)}(x_3)]_\mp$$
$$+ \Phi_A^{(+)}(x_1)[\Phi_B^{(+)}(x_2),\Phi_C^{(-)}(x_3)]_\mp$$
$$+ \Phi_A^{(-)}(x_1)[\Phi_B^{(+)}(x_2),\Phi_C^{(-)}(x_3)]_\mp$$
$$+ \epsilon_{AB}\Phi_B^{(-)}(x_2)[\Phi_A^{(+)}(x_1),\Phi_C^{(-)}(x_3)]_\mp$$
$$= :\Phi_A(x_1)\Phi_B(x_2)\Phi_C(x_3): + :\underbrace{\Phi_A(x_1)\Phi_B(x_1)\Phi_C}(x_3): + :\Phi_A(x_1)\underbrace{\Phi_B(x_2)\Phi_C}(x_3):$$

In all:

$$T(\Phi_A(x_1)\Phi_B(x_2)\Phi_C(x_3)) = :\Phi_A(x_1)\Phi_B(x_2)\Phi_C(x_3): + :\underbrace{\Phi_A(x_1)\Phi_B(x_2)\Phi_C}(x_3): + :\Phi_A(x_1)\underbrace{\Phi_B(x_2)\Phi_C}(x_3):$$
$$+ :\underbrace{\Phi_A(x_1)\Phi_B}(x_2)\Phi_C(x_3):$$

—— t_3 不是最大，如 t_1 最大.

BCA order!

$$T(\Phi_A(x_1)\Phi_B(x_2)\Phi_C(x_3)) = \epsilon_P\cdot T(\Phi_B(x_2)\Phi_C(x_3)\Phi_A(x_1)) = \epsilon_P(:\cdots: + \text{contraction terms})$$
$$= :\cdots: + \text{Contraction term} \leftarrow ABC \text{ order!}$$

○ Proof of Wick's Theorem:

—— $t_Z < t_A \cdots t_Y$ 时证明:

$$:AB\cdots XY:Z = :AB\cdots XYZ:$$
$$+ :\underbrace{AB\cdots XYZ}: + :A\underbrace{B\cdots XYZ}: + \cdots + :AB\cdots X\underbrace{YZ}:$$

1° 只用 consider $Z^{(-)}$! 要证明:

$$:AB\cdots XY:(Z^{(+)}+Z^{(-)}) = :AB\cdots XYZ^{(+)}: + :AB\cdots XY:Z^{(-)}$$
$$= :AB\cdots XYZ: + :\underbrace{AB\cdots XYZ}: + \cdots + \cdots$$

引用: proof ↓ ($\underbrace{OZ^{(-)}} = \underbrace{O^{(+)}Z^{(-)}} = \underbrace{OZ}$; $t_z < t_0$)

$$:AB\cdots XY:Z^{(-)} = :AB\cdots XYZ^{(-)}: + :\underbrace{AB\cdots XYZ^{(-)}}: + \cdots + :AB\cdots X\underbrace{YZ^{(-)}}:$$

2° 引用 Consider $O^{(+)}$:

若: $:A^{(+)}B^{(+)}\cdots Y^{(+)}:Z^{(-)} = :A^{(+)}B^{(+)}\cdots Z^{(-)}: + :\underbrace{A^{(+)}\cdots Z^{(-)}}: +$ other contraction.

则:

$$:A^{(+)}B^{(+)}\cdots Y:Z^{(-)} = :A^{(+)}\cdots Y^{(+)}Z^{(-)}: + :\underbrace{A^{(+)}\cdots Y^{(+)}Z^{(-)}}: + :A^{(+)}\underbrace{B^{(+)}\cdots Y^{(+)}Z^{(-)}}: + \cdots + :A^{(+)}\cdots \underbrace{Y^{(+)}Z^{(-)}}:$$

$$+ :A^{(+)}\cdots Y^{(-)}:Z^{(-)}$$

$$= \Big\{ :A^{(+)}\cdots Y^{(-)}:Z^{(-)} = \varepsilon_Y\, Y^{(-)} A^{(+)}\cdots X^{(+)}Z^{(-)}$$

$$= \varepsilon_Y\cdot Y^{(-)}\left(:A^{(+)}\cdots X^{(+)}Z^{(-)}: + :\underbrace{A^{(+)}\cdots X^{(+)}Z^{(-)}}: + \cdots\right)$$

$$= :A^{(+)}\cdots X^{(+)}Y^{(-)}Z^{(-)}: + :\underbrace{A^{(+)}\cdots X^{(+)}Y^{(-)}Z^{(-)}}: + \cdots + :A^{(+)}\cdots \underbrace{X^{(+)}Y^{(-)}Z^{(-)}}:$$

$$\Big\}\ \underbrace{Y^{(+)}Z^{(-)}} = \underbrace{YZ^{(-)}}$$

$$= :A^{(+)}\cdots YZ^{(-)}: + :\underbrace{A^{(+)}\cdots YZ^{(-)}}: + \cdots + :A^{(+)}\cdots \underbrace{YZ^{(-)}}:$$

$$:A^{(+)}\cdots XY:Z^{(-)} = :A^{(+)}\cdots X^{(+)}YZ^{(-)}: + :\underbrace{A^{(+)}\cdots X^{(+)}YZ^{(-)}}: + \cdots + :A^{(+)}\cdots X^{(+)}\underbrace{YZ^{(-)}}:$$

$$+ :A^{(+)}\cdots X^{(-)}Y:Z^{(-)}$$

$$:A^{(+)}\cdots X^{(-)}Y:Z^{(-)}$$

$$= \varepsilon_X\cdot X^{(-)}:A^{(+)}\cdots W^{(+)}Y:Z^{(-)}$$

$$= \varepsilon_X\cdot X^{(-)}\left(:A^{(+)}\cdots W^{(+)}YZ^{(-)}: + :\underbrace{A^{(+)}\cdots W^{(+)}YZ^{(-)}}: + \cdots\right)$$

$$= :A^{(+)}\cdots X^{(-)}YZ^{(-)}: + :\underbrace{A^{(+)}\cdots X^{(-)}YZ^{(-)}}: + \cdots$$

$$\underbrace{X^{(+)}Z^{(-)}} = \underbrace{XZ^{(-)}}$$

$$= :A^{(+)}\cdots XYZ^{(-)}: + :\underbrace{A^{(+)}\cdots XYZ^{(-)}}: + \cdots$$

经过归纳，可得

$$:A\cdots Y:Z^{(-)} = :A\cdots Z^{(-)}: + :\underbrace{A\cdots Z^{(-)}}: + \cdots$$

上面的证明可用于证明 Wick's Theorem：

$$T(AB) = :AB: + \underbrace{AB}$$

$$T(A\cdots Y) = :A\cdots Y: + :\underbrace{A\cdots Y}: + \cdots$$

则：$T(A\cdots Y)Z = T(AB\cdots YZ) = :A\cdots Z: + :\underbrace{A\cdots Z}: +$ other contractions!

- In all:

Wick's Theorem

↑ Generalize to (t_z) not minimal.

Wick's Theorem, t_z minimal

↑

$$:A\cdots Y:Z = :A\cdots Z: + :\underbrace{A\cdots YZ}: + \cdots + :A\cdots \underbrace{YZ}: \quad (t_z \text{ minimal})$$

↑

$$:A\cdots Y:Z^{(-)} = :A\cdots Z^{(-)}: + :\underbrace{A\cdots YZ^{(-)}}: + \cdots + :A\cdots \underbrace{YZ^{(-)}}: \quad (t_z \text{ minimal})$$

↑ 归纳法

$$:A^{(+)}\cdots Y^{(+)}:Z^{(-)} = :A^{(+)}\cdots Z^{(-)}: + :\underbrace{A^{(+)}\cdots Z^{(-)}}: + \cdots$$

Quantum - Electrodynamics and Feynman Rules.

- Hamiltonian and Lagrangian

$$\mathcal{L} = \mathcal{L}_0^{Dirac} + \mathcal{L}_0^{e.m} + \mathcal{L}_1$$

Dirac Field　Massless electro-dynamic field.

—— Dirac $\mathcal{L}_0^{Dirac} = \bar{\psi}\left(\frac{i}{2}\gamma_\mu \overleftrightarrow{\partial}^\mu - m\right)\psi$

—— Electro-magnetic $\mathcal{L}_0^{e.m.} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\zeta(\partial_\mu A^\mu)^2$

—— Interaction $\mathcal{L}_1 = -e\bar{\psi}\gamma_\mu\psi A^\mu$

—— Hamiltonian 中的 Interaction part:

$$\mathcal{H}_1 = e\bar{\psi}\gamma_\mu\psi A^\mu$$

- S operator, 用 Dyson 级数表示.

$$S = I + \sum_{n=1}^{+\infty} S^{(n)}$$

$$= \sum_{n=0}^{+\infty}\frac{1}{n!}(-ie)^n\int d^4x_1 d^4x_2\cdots d^4x_n\, T[:\bar{\psi}(x_1)\gamma_{\mu_1}\psi(x_1)A^{\mu_1}:\cdots:\bar{\psi}(x_n)\gamma_{\mu_n}\psi(x_n)A^{\mu_n}:]$$

$$S^{(n)} = \frac{1}{n!}\int d^4x_1\cdots d^4x_n \cdot S(x_1\cdots x_n)$$

- S operator 的矩阵元:(S op 中要有相应的湮灭算符在右侧，生成算符在左)

$$S_{fi} = \langle k_1'\lambda_1', \cdots, \bar{p}_1'\bar{s}_1', \cdots, p_1's_1', \cdots | S | p_1 s_1, \cdots, \bar{p}_1\bar{s}_1, \cdots, k_1\lambda_1, \cdots\rangle$$

光子　positron　electron　electron　position　photon

$$| p_1 s_1, \cdots, \bar{p}_1\bar{s}_1\cdots, k_1\lambda_1, \cdots\rangle = b^\dagger_{p_1s_1}\cdots d^\dagger_{\bar{p}_1\bar{s}_1}\cdots a^\dagger_{k_1,\lambda_1}\cdots|0\rangle$$

$$\langle k_1'\lambda_1', \cdots, \bar{p}_1'\bar{s}_1', \cdots, p_1's_1', \cdots | = \langle 0| a_{k_1'\lambda_1'}\cdots d_{\bar{p}_1'\bar{s}_1'}\cdots b_{p_1's_1'}\cdots$$

- 场算符的基底展开形式:

$$\psi(x) = \int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_s\left(b_{ps}u(p,s)e^{-ip\cdot x} + d^\dagger_{ps}v(p,s)e^{ip\cdot x}\right)$$ (Incoming electron & outgoing positron)

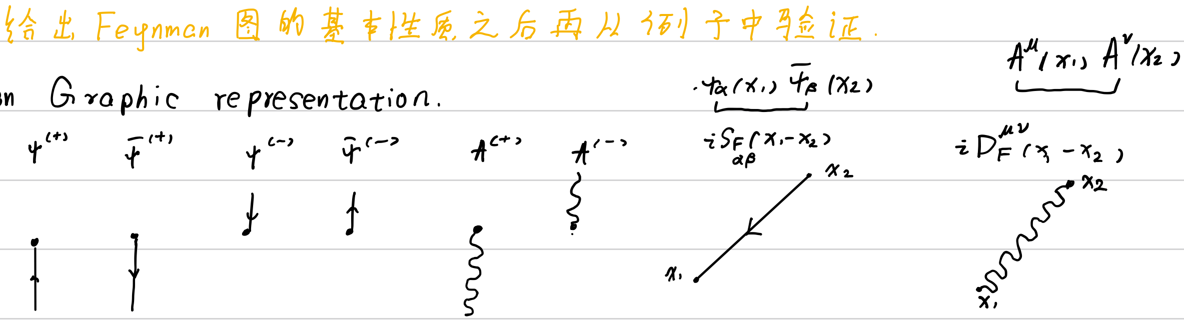
$$\bar{\psi}(x) = \int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_s\left(d_{ps}\bar{v}(p,s)e^{-ip\cdot x} + b^\dagger_{ps}\bar{u}(p,s)e^{ip\cdot x}\right)$$ (Incoming positron & outgoing electron)

$$A_\mu(x) = \int\frac{d^3k}{\sqrt{(2\pi)^3\cdot 2\omega_k}}\sum_\lambda\left(a_{k,\lambda}\varepsilon^\mu(k,\lambda)e^{-ik\cdot x} + a^\dagger_{k,\lambda}\varepsilon^{*\mu}(k,\lambda)e^{ik\cdot x}\right)$$ (Incoming photon & outgoing photon)

- Effects of Wick's Theorem of normal ordering inside (Contractions in Same Normal Ordering Not enter)

先给出 Feynman 图的基本性质之后再从例子中验证.

- Feynman Graphic representation.



—— propagator 定义为:

$$iD_F^{\mu\nu}(x-y) := \langle 0|T(A^\mu(x)A^\nu(y))|0\rangle = iD_F^{\mu\nu}(y-x)$$

$$iS_{F\alpha\beta}(x-y) = \langle 0|T(\hat{\psi}_\alpha(x)\hat{\bar{\psi}}_\beta(y))|0\rangle$$

且 propagator 的动量展开为:

$$iS_{F\alpha\beta}(x-y) = i(i\cancel{\partial} + m)_{\alpha\beta}\Delta_F(x-y) = i\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{\cancel{p}+m}{p^2-m^2+i\varepsilon} \quad (\cancel{p} = p^\mu\gamma_\mu)$$

$$iD_F^{\mu\nu}(x-y) = i\int\frac{d^4k}{(2\pi)^4}D_F^{\mu\nu}(k) \quad \left(D_F^{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2+i\varepsilon}\right)$$

- Feynman Rules in coordinate space.
  - vertex: $-ie\,(\gamma_\mu)_{\alpha\beta}$
  - Internal photon line $\quad -i D_F^{\mu\nu}(x_k - x_\ell)$
  - Internal fermion line $\quad -i S_{F\alpha\beta}(x_k - x_\ell)$
  - Exerternal fermion line
    - $N_p\, u(p,s)\, e^{-ip\cdot x}$ Incoming electron
    - $N_p\, \bar{v}(p,s)\, e^{-ip\cdot x}$ Incoming positron
    - $N_p\, \bar{u}(p,s)\, e^{ip\cdot x}$ outgoing electron
    - $N_p\, v(p,s)\, e^{ip\cdot x}$ outgoing positron
  - External photon line
    - $N_k\, \varepsilon^\mu(k,\lambda)\, e^{-ik\cdot x}$ incoming photon
    - $N_k\, \varepsilon^{\mu *}(k,\lambda)\, e^{ik\cdot x}$ outgoing photon.

    Normalization factor for Dirac & photon wave function.

    $N_p = \sqrt{\frac{m}{(2\pi)^3 E_p}} \qquad N_k = \sqrt{\frac{1}{(2\pi)^3\, 2\omega_k}}$ (If use continum normalization, replace $(2\pi)^3$ with V)
  - All coordinates x_i are integrated over
  - Each closed fermion loop leads to factor -1

- Feynman relues of QED in momentum space.
  - Each internal line is assigned a momentum variable p_i or k_i
  - vertex $\quad -ie(\gamma_\mu)_{\alpha\beta}$
  - Internal photon line $\quad -i D_F^{\mu\nu}(k_i)$
  - Internal fermion line
    - $N_p\, u(p,s)$ incoming electron
    - $N_p\, \bar{v}(p,s)$ incoming positron.
    - $N_p\, \bar{u}(p,s)$ outgoing electron
    - $N_p\, v(p,s)$ outgoing positron.
  - Extemal photon line
    - $N_k\, \varepsilon^\mu(k,\lambda)$ incoming photon
    - $N_k\, \varepsilon^{\mu *}(k,\lambda)$ outgoing photon.

    $N_p = \sqrt{\frac{m}{(2\pi)^3 E_p}} \qquad N_k = \sqrt{\frac{1}{(2\pi)^3\, 2\omega_k}}$
  - All momenta of internal lines integrate over.

    $\int \frac{d^4 p}{(2\pi)^4}$
  - Each closed fermion loop leads to factor -1.
  - Each vertex is associated with factor $(2\pi)^4 \delta^{(4)}(p' - p \pm k)$

○ $S^{(1)}$ 的图表示:

$$S^{(1)} = \frac{1}{1!}\int d^4x \cdot (-ie)\cdot T[:\bar\psi(x)\gamma_\mu\psi(x)A^\mu(x):]$$

$$= (-ie)\cdot\int d^4x \cdot :\bar\psi(x)\gamma_\mu\psi(x)A^\mu(x):$$

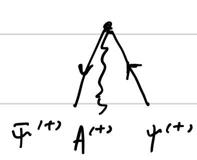
$$= (-ie)\int d^4x :(\bar\psi^{(+)}_\alpha(x) + \bar\psi^{(-)}_\alpha(x))(\gamma_\mu)_{\alpha\beta}\cdot(\psi^{(+)}_\beta(x) + \psi^{(-)}_\beta(x))\cdot(A^{\mu(+)}(x) + A^{\mu(-)}(x)):$$

$$\begin{aligned}
= (-ie)\cdot\int d^4x \cdot (\ & \bar\psi^{(+)}_\alpha(x)(\gamma_\mu)_{\alpha\beta}\psi^{(+)}_\beta(x)\cdot A^{\mu(+)}(x) && (+)(+)(+) && (a)\\
&+ A^{\mu(-)}(x)\,\bar\psi^{(+)}_\alpha(x)(\gamma_\mu)_{\alpha\beta}\psi^{(+)}_\beta(x) && (+)(+)(-) && (b)\\
&- \psi^{(-)}_\beta(x)\cdot\bar\psi^{(+)}_\alpha(x)(\gamma_\mu)_{\alpha\beta}A^{\mu(+)}(x) && (+)(-)(+) && (c)\\
&- A^{\mu(-)}(x)\psi^{(-)}_\beta(x)\bar\psi^{(+)}_\alpha(x)(\gamma_\mu)_{\alpha\beta} && (+)(-)(-) && (d)\\
&+ \bar\psi^{(-)}_\alpha(x)(\gamma_\mu)_{\alpha\beta}\psi^{(+)}_\beta(x)A^{\mu(+)}(x) && (-)(+)(+) && (e)\\
&+ \bar\psi^{(-)}_\alpha(x)A^{\mu(-)}(x)\cdot(\gamma_\mu)_{\alpha\beta}\cdot\psi^{(+)}_\beta(x) && (-)(+)(-) && (f)\\
&+ \bar\psi^{(-)}_\alpha(x)(\gamma_\mu)_{\alpha\beta}\psi^{(-)}_\beta(x)A^{\mu(+)}(x) && (-)(-)(+) && (g)\\
&+ \bar\psi^{(-)}_\alpha(x)(\gamma_\mu)_{\alpha\beta}\psi^{(-)}_\beta(x)A^{\mu(-)}(x)\ ) && (-)(-)(-) && (h)
\end{aligned}$$

图表示

$$\begin{cases}
\psi(x) = \int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_s(b_{ps}u(p,s)e^{-ip\cdot x} + d^+_{ps}V(p,s)e^{ip\cdot x}) & \text{(Incoming electron \& outgoing positron)}\\
\bar\psi(x) = \int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_s(d_{ps}\bar V(p,s)e^{-ip\cdot x} + b^+_{p,s}\bar u(p,s)e^{ip\cdot x}) & \text{(Incoming positron \& outgoing electron)}\\
A_\mu(x) = \int\frac{d^3k}{\sqrt{(2\pi)^3\cdot 2\omega_k}}\sum_\lambda(a_{k,\lambda}\varepsilon^\mu(k,\lambda)e^{-ik\cdot x} + a^+_{k,\lambda}\varepsilon^{*\mu}(k,\lambda)e^{ik\cdot x}) & \text{(Incoming photon \& outgoing photon)}
\end{cases}$$

—— (a):



$$(-ie)\cdot\int d^4x\,\bar\psi^{(+)}_\alpha(x)(\gamma_\mu)_{\alpha\beta}\psi^{(+)}_\beta(x)\cdot A^{\mu(+)}(x) = (-ie)\iiiint d^4x\cdot d^3P_1\,d^3P_2\cdot d^3k\cdot\sum_{s_1}\sum_{s_2}\sum_\lambda\cdot N_{P_1}\cdot N_{P_2}\cdot N_k\cdot d_{P_1s_1}\bar V_\alpha(P_1,s_1)\cdot e^{-iP_1\cdot x}(\gamma_\mu)_{\alpha\beta}\cdot b_{P_2,s_2}u_\beta(P_2,s_2)e^{-iP_2\cdot x}\cdot a_{k,\lambda}\varepsilon^\mu(k,\lambda)e^{-ik\cdot x}$$

$$\begin{cases}
\text{对易 relation:}\\
[a_{k,\lambda}, a^+_{k',\lambda'}] = -g_{\lambda\lambda'}\delta^{(3)}(k-k')\\
\{\hat b(P,s), \hat b^+(P',s')\} = \delta^{(3)}(P-P')\delta_{ss'}\\
\{\hat d(P,s), \hat d^+(P',s')\} = \delta^{(3)}(P-P')\delta_{ss'}
\end{cases}$$

{ 取这一项 operator 与初末态的内积：左: $\langle 0|$ 右: $\underbrace{b^+_{P_2's_2'}}_{\text{外侧}}\,d^+_{P_1's_1'}\cdot\cdot\underbrace{a^+_{k'\lambda'}}_{\text{内侧}}|0\rangle$

注意这个顺序是个约定!

$$\langle 0|(-ie)\iiiint d^4x\cdot d^3P_1\,d^3P_2\cdot d^3k\cdot\sum_{s_1}\sum_{s_2}\sum_\lambda\cdot N_{P_1}\cdot N_{P_2}\cdot N_k\cdot d_{P_1s_1}\bar V_\alpha(P_1,s_1)\cdot e^{-iP_1\cdot x}(\gamma_\mu)_{\alpha\beta}\cdot b_{P_2,s_2}u_\beta(P_2,s_2)e^{-iP_2\cdot x}\cdot a_{k,\lambda}\varepsilon^\mu(k,\lambda)e^{-ik\cdot x}\;b^+_{P_2's_2'}\,d^+_{P_1's_1'}\,a^+_{k'\lambda'}|0\rangle$$

$$=$$

$$\langle 0|(-ie)\iiiint d^4x\cdot d^3P_1\,d^3P_2\cdot d^3k\cdot\sum_{s_1}\sum_{s_2}\sum_\lambda\cdot N_{P_1}\cdot N_{P_2}\cdot N_k\cdot d_{P_1s_1}\bar V_\alpha(P_1,s_1)\cdot e^{-iP_1\cdot x}(\gamma_\mu)_{\alpha\beta}\cdot b_{P_2,s_2}u_\beta(P_2,s_2)e^{-iP_2\cdot x}\cdot\varepsilon^\mu(k,\lambda)e^{-ik\cdot x}\;b^+_{P_2's_2'}\,d^+_{P_1's_1'}\,(-g_{\lambda\lambda'})\cdot\delta^{(3)}(k-k')|0\rangle$$

$$= \langle 0|(-ie)\iiiint d^4x\cdot d^3P_1\,d^3P_2\cdot d^3k\cdot\sum_{s_1}\sum_{s_2}\sum_\lambda\cdot N_{P_1}\cdot N_{P_2}\cdot N_k\cdot d_{P_1s_1}\bar V_\alpha(P_1,s_1)\cdot e^{-iP_1\cdot x}(\gamma_\mu)_{\alpha\beta}\cdot u_\beta(P_2,s_2)e^{-iP_2\cdot x}\cdot\varepsilon^\mu(k,\lambda)e^{-ik\cdot x}\;\delta_{s_2's_2}\cdot\delta^{(3)}(P_2-P_2')\cdot d^+_{P_1's_1'}\,(-g_{\lambda\lambda'})\cdot\delta^{(3)}(k-k')|0\rangle$$

$$= \langle 0|(-ie)\iiiint d^4x\cdot d^3P_1\,d^3P_2\cdot d^3k\cdot\sum_{s_1}\sum_{s_2}\sum_\lambda\cdot N_{P_1}\cdot N_{P_2}\cdot N_k\cdot\bar V_\alpha(P_1,s_1)\cdot e^{-iP_1\cdot x}(\gamma_\mu)_{\alpha\beta}\cdot u_\beta(P_2,s_2)e^{-iP_2\cdot x}\cdot\varepsilon^\mu(k,\lambda)e^{-ik\cdot x}\;\delta_{s_1,s_1'}\cdot\delta^{(3)}(P_1-P_1')\;\delta_{s_2's_2}\cdot\delta^{(3)}(P_2-P_2')\cdot(-g_{\lambda\lambda'})\cdot\delta^{(3)}(k-k')|0\rangle$$

$$= \langle 0|(-ie)\int d^4x \cdot N_{p_1'} N_{p_2'} N_{k'} \bar{v}_\alpha(p_1',s_1') e^{-ip_1'\cdot x} (\gamma_\mu)_{\alpha\beta} u_\beta(p_2',s_2') e^{-ip_2'\cdot x} \cdot \varepsilon^\mu(k',\lambda') e^{-ik'\cdot x} \underbrace{(-g_{\lambda'\lambda'})}_{(\lambda'=1\text{ or }2)} |0\rangle$$

$$= \int d^4x \cdot \underset{\uparrow}{(-ie)\cdot(\gamma_\mu)_{\alpha\beta}} \cdot \underline{N_{p_1'} \bar{v}_\alpha(p_1',s_1') e^{-ip_1'\cdot x}} \cdot \underline{N_{p_2'} \cdot u_\beta(p_2',s_2') e^{-ip_2'\cdot x}} \cdot \underline{N_{k'} \cdot \varepsilon^\mu(k',\lambda') e^{-ik'\cdot x}}$$

Notice! γ_μ 并不是对称矩阵. positron 左! electron 右!

可见，(a) 是满足 feynman 图规律的.

动量守恒与能量守恒，上式对 x 积分得：

$$\int d^4x \; e^{-ip_1'\cdot x} \cdot e^{-ip_2'\cdot x} e^{-ik'\cdot x} = (2\pi)^4 \delta^{(4)}(p_1'+p_2'+k')$$

$$\left(\begin{aligned} p_1' &= (\sqrt{|\vec{p}_1'|^2+m^2}, \vec{p}_1') \\ p_2' &= (\sqrt{|\vec{p}_2'|^2+m^2}, \vec{p}_2') \\ k' &= (|\vec{k}'|, \vec{k}') \end{aligned} \right)$$

$$\begin{cases} \vec{p}_1' + \vec{p}_2' + \vec{k}' = 0 \\ \sqrt{m^2+(\vec{p}_1')^2} + \sqrt{m^2+(\vec{p}_2')^2} + |\vec{k}| = 0 \end{cases}$$ ← 恒不成立!

—— (b)

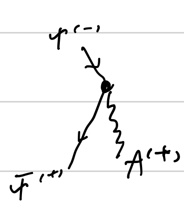
直接用 Feynman 规则写：

$$\int d^4x \; (-ie)(\gamma_\mu)_{\alpha\beta} \cdot N_{p_1} u_\beta(p_1,s_1) \cdot e^{-ip_1\cdot x} \; N_{p_2} \bar{v}_\alpha(p_2,s_2) e^{-ip_2\cdot x_2} \; N_k \cdot \varepsilon^{\mu *}(k,\lambda) \cdot e^{ik\cdot x}$$

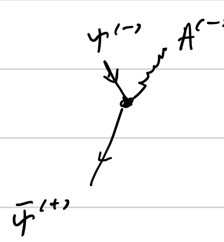
由于不满足能动量守恒关系，这一项也是 0!

—— 后面的不写具体式子了，但最终结果都是 0!

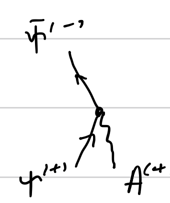
(c)



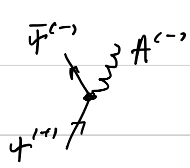
(d)



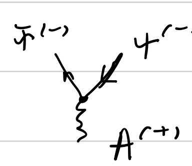
(e)



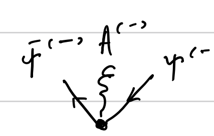
(f)



(g)

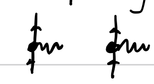


(h)

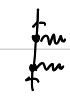


。$S^{(2)}$ 的计算与图表示： Topological figure:

$$S^{(2)} = \frac{(-ie)^2}{2!}\cdot\int d^4x_1\, d^4x_2 \; :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (a)$$



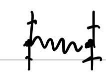
$$+ :\bar{\psi}(x_1)\gamma_\mu\underbrace{\psi(x_1)A^\mu(x_1)\;\bar{\psi}(x_2)}\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (b)$$



$$+ :\underbrace{\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar{\psi}(x_2)\gamma_\nu\psi(x_2)}A^\nu(x_2): \quad (c)$$

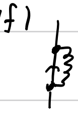
按定义，(b)=(c)

$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)\underbrace{A^\mu(x_1)\;\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2)}: \quad (d)$$



$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (e)$$

(e)=(f)



$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (f)$$

$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (g)$$

$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (h)$$

上面 Feynman 图表示对于的具体系数有待商榷.

- Ignore disconnected Feynman chart!

$$S = \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \cdot \int d^4x_1 \cdots d^4x_n \cdot T\left(\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)\right)$$

$$T(\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)) = :\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n): + \text{othercontract term}$$

$= (:\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n): + \cdots) \leftarrow$ 0 in disconnect part.

$(:\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n):\underbrace{\mathcal{H}_1(x_i)}_{\text{contract, with in.}} + \cdots) \leftarrow$ 1 in disconnect part.

m 个 vertex 在 dis connect part 里!

...

$\Downarrow$

T 不是时序, 单纯是真空项.

$$T(\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)) = \sum_{m=0}^{n}\sum_{\text{Combination}} T_{ext}\cdot(\mathcal{H}_1(x_1')\cdots\mathcal{H}_1(x_{n-m}'))\cdot T_{vac}(\mathcal{H}_1(x_{n-m+1}')\cdots\mathcal{H}_1(x_n'))\cdot$$

$$= \sum_{m=0}^{n} C_n^m \cdot T_{ext}\cdot(\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_{n-m}))\cdot T_{vac}(\mathcal{H}_1(x_{n-m+1})\cdots\mathcal{H}_1(x_n))$$

$$= \sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\cdot T_{ext}(\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_{n-m}))\cdot T_{vac}(\mathcal{H}_1(x_{n-m+1})\cdots\mathcal{H}_1(x_n))$$

$$= \int\sum_{n=0}^{+\infty}\sum_{m=0}^{n}(-i)^n\cdot\frac{1}{n!}\frac{n!}{m!(n-m)!}\cdot d^4x_1\cdots d^4x_n \quad T_{ext}(\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_{n-m}))\cdot T_{vac}\cdot(\mathcal{H}_1(x_{n-m+1})\cdots\mathcal{H}_1(x_n))$$

$$= \int\sum_{k=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{(-i)^m}{m!}\cdot\frac{(-i)^k}{k!}\cdot d^4x_1\cdots d^4x_{k+m} \quad T_{ext}(\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_k))\cdot T_{vac}(\mathcal{H}_1(x_{k+1})\cdots\mathcal{H}_1(x_{m+k}))$$

$$= \int\sum_{k=0}^{+\infty}\frac{(-i)^k}{k!}\cdot\int d^4x_1\cdots d^4x_k \cdot T_{ext}(\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_k))$$

$$\sum_{m=0}^{+\infty}\frac{(-i)^m}{m!}\int d^4x_{k+1}\cdots d^4x_{k+m}\cdot T_{vac}(\mathcal{H}_1(x_{k+1})\cdots\mathcal{H}_1(x_{k+m}))\cdot$$

$\langle 0|S|0\rangle$ = vacuum term!

$$= \sum_{k=0}^{+\infty}\frac{(-i)^k}{k!}\cdot\int d^4x_1\cdots d^4x_k\cdot T_{ext}(\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_k))\cdot$$

$\langle 0|S|0\rangle$

Exercise 8.4 – Feynman Diagram For QED process!

- Electron-electron scattering:

态的 defination: $|p_1 s_1; \cdots \bar{p}_1 \bar{s}_1; \cdots k_1 \lambda_1; \cdots\rangle = b^+_{p_1 s_1} \cdots d^+_{\bar{p}_1 \bar{s}_1} \cdots a^+_{k_1 \lambda_1} \cdots |0\rangle$

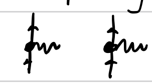
初态:(两个 electron): $b^+_{p_1 s_1}\, b^+_{p_2 s_2} |0\rangle = |i\rangle$

末态:(两个 electron): $\langle 0|\, b_{p_2' s_2'}\, b_{p_1' s_1'} = \langle f|$

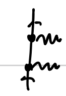
回顾 S operator 的 2 阶项: $\langle f|S|i\rangle$ (1阶项不做任何贡献):

Topological figure:

$$S^{(2)} = \frac{(-ie)^2}{2!} \int d^4x_1\, d^4x_2 \;\; :\bar{\psi}(x_1)\gamma_\mu \psi(x_1) A^\mu(x_1)\, \bar{\psi}(x_2)\gamma_\nu \psi(x_2) A^\nu(x_2): \quad (a)$$



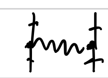
$$+ :\bar{\psi}(x_1)\gamma_\mu \underbrace{\psi(x_1) A^\mu(x_1)\, \bar{\psi}(x_2)}\gamma_\nu \psi(x_2) A^\nu(x_2): \quad (b)$$



$$+ :\underbrace{\bar{\psi}(x_1)\gamma_\mu \psi(x_1) A^\mu(x_1)\, \bar{\psi}(x_2)\gamma_\nu \psi(x_2)} A^\nu(x_2): \quad (c)$$

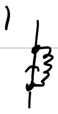
按定义,(b)=(c)

$$+ :\bar{\psi}(x_1)\gamma_\mu \psi(x_1) \underbrace{A^\mu(x_1)\, \bar{\psi}(x_2)\gamma_\nu \psi(x_2) A^\nu(x_2)}: \quad (d)$$



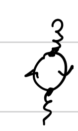
$$+ :\bar{\psi}(x_1)\gamma_\mu \psi(x_1) A^\mu(x_1)\, \bar{\psi}(x_2)\gamma_\nu \psi(x_2) A^\nu(x_2): \quad (e)$$

(e)=(f)



$$+ :\bar{\psi}(x_1)\gamma_\mu \psi(x_1) A^\mu(x_1)\, \bar{\psi}(x_2)\gamma_\nu \psi(x_2) A^\nu(x_2): \quad (f)$$

$$+ :\bar{\psi}(x_1)\gamma_\mu \psi(x_1) A^\mu(x_1)\, \bar{\psi}(x_2)\gamma_\nu \psi(x_2) A^\nu(x_2): \quad (g)$$



$$+ :\bar{\psi}(x_1)\gamma_\mu \psi(x_1) A^\mu(x_1)\, \bar{\psi}(x_2)\gamma_\nu \psi(x_2) A^\nu(x_2): \quad (h)$$

其中的 (d) 项对 $\langle f|S|i\rangle$ 有贡献!

$$S_{fi} = \frac{(-ie)^2}{2!} \int d^4x_1\, d^4x_2 \; \langle 0|\, b_{p_2' s_2'}\, b_{p_1' s_1'} \; : \bar{\psi}(x_1)\gamma_\mu \psi(x_1) \cdot \underbrace{A^\mu(x_1) \bar{\psi}(x_2)\gamma_\nu \psi(x_2) A^\nu(x_2)}: \; b^+_{p_1 s_1}\, b^+_{p_2 s_2} |0\rangle$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} \sum_s \left(b_{ps}\, u(p,s) e^{-ip\cdot x} + d^+_{ps}\, v(p,s) e^{ip\cdot x} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} \sum_s \left(d_{ps}\, \bar{v}(p,s) e^{-ip\cdot x} + b^+_{p,s}\, \bar{u}(p,s) e^{ip\cdot x} \right)$$

$$= \frac{(-ie)^2}{2!} \int d^4x_1\, d^4x_2 \cdot \underbrace{A^\mu(x_1) A^\nu(x_2)} \; \langle 0|\, b_{p_2' s_2'}\, b_{p_1' s_1'} \; : \bar{\psi}^{(-)}(x_1)\gamma_\mu \psi^{(+)}(x_1)\; \bar{\psi}^{(-)}(x_2)\gamma_\nu \psi^{(+)}(x_2) : \; b^+_{p_1 s_1}\, b^+_{p_2 s_2} |0\rangle$$

$$\underbrace{A^\mu(x_1) A^\nu(x_2)} = i D_F^{\mu\nu}(x_1 - x_2)$$

$$= \frac{(-ie)^2}{2!} \int d^4x_1\, d^4x_2 \cdot i D_F^{\mu\nu}(x_1 - x_2) \cdot \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}$$

$$\int \frac{d^3q_1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_{q_1}}} \cdot \int \frac{d^3q_2}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_{q_2}}} \cdot \int \frac{d^3q_3}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_{q_3}}} \cdot \int \frac{d^3q_4}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_{q_4}}}$$

$$\bar{u}(q_1, \sigma_1) e^{iq_1 \cdot x_1} \gamma_\mu u(q_2, \sigma_2) \cdot e^{-iq_2\cdot x_1} \; \bar{u}(q_3, \sigma_3) \cdot e^{iq_3 x_2} \gamma_\nu u(q_4, \sigma_4) e^{-iq_4 \cdot x_2}$$

$$\langle 0|\, b_{p_2' s_2'}\, b_{p_1' s_1'} : b^+_{q_1 \sigma_1} b_{q_2 \sigma_2} b^+_{q_3 \sigma_3} b_{q_4 \sigma_4} : b^+_{p_1 s_1}\, b^+_{p_2 s_2} |0\rangle$$

Vacuum expectation

$$-\langle 0|\, b_{p_2's_2'}\, b_{p_1's_1'}\, b^{+}_{q_1\sigma_1}\, b^{+}_{q_3\sigma_3}\, b_{q_2\sigma_2}\, b_{q_4\sigma_4}\, b^{+}_{p_1s_1}\, b^{+}_{p_2s_2}\, |0\rangle$$

$$\{b_{p,s},\, b^{+}_{p',s'}\} = \delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta_{s,s'}$$

$$= -\langle 0|\, b_{p_2's_2'}\left(-b^{+}_{q_1\sigma_1}\, b_{p_1's_1'} + \delta^{(3)}(p_1'-q_1)\,\delta_{\sigma_1 s_1'}\right) b^{+}_{q_3\sigma_3}$$

$$b_{q_2\sigma_2}\left(-b^{+}_{p_1s_1}\, b_{q_4\sigma_4} + \delta^{(3)}(p_1-q_4)\,\delta_{s_1\sigma_4}\right) b^{+}_{p_2s_2}\,|0\rangle$$

$$= -\langle 0|\Big(-\delta^{(3)}(p_2'-q_1)\,\delta_{s_2'\sigma_1}\,\delta^{(3)}(p_1'-q_3)\,\delta_{s_1'\sigma_3} + \delta^{(3)}(p_1'-q_1)\,\delta_{\sigma_1 s_1'}\,\delta^{(3)}(p_2'-q_3)\,\delta_{s_2'\sigma_3}\Big)$$

$$\Big(-\delta^{(3)}(p_1-q_2)\,\delta_{s_1\sigma_2}\,\delta^{(3)}(q_4-p_2)\,\delta_{\sigma_4 s_2} + \delta^{(3)}(p_1-q_4)\,\delta_{s_1\sigma_4}\,\delta^{(3)}(q_2-p_2)\,\delta_{\sigma_2 s_2}\Big)|0\rangle$$

$$= -\Big(\delta^{(3)}(p_2'-q_1)\,\delta^{(3)}(p_1'-q_3)\,\delta_{s_2'\sigma_1}\,\delta_{s_1'\sigma_3} - \delta^{(3)}(p_1'-q_1)\,\delta^{(3)}(p_2'-q_3)\,\delta_{\sigma_1 s_1'}\,\delta_{\sigma_3 s_2'}\Big)$$

$$\Big(\delta^{(3)}(p_1-q_2)\,\delta^{(3)}(q_4-p_2)\,\delta_{s_1\sigma_2}\,\delta_{\sigma_4 s_2} - \delta^{(3)}(p_1-q_4)\,\delta^{(3)}(q_2-p_2)\,\delta_{s_1\sigma_4}\,\delta_{\sigma_2 s_2}\Big)$$

$$= \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\; i D_F^{\mu\nu}(x_1-x_2)\sum_{\sigma_1\sigma_2\sigma_3\sigma_4}$$

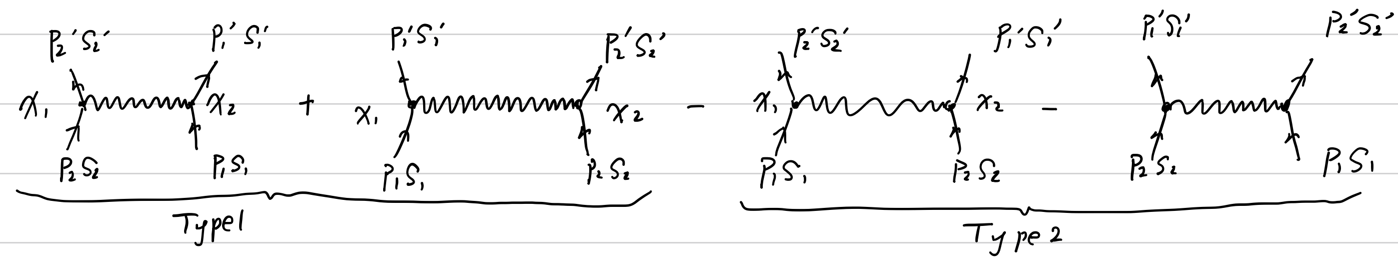
$$\int\frac{d^3q_1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{q_1}}}\int\frac{d^3q_2}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{q_2}}}\int\frac{d^3q_3}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{q_3}}}\int\frac{d^3q_4}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{q_4}}}$$

$$\bar{u}(q_1,\sigma_1)\,e^{iq_1\cdot x_1}\,\gamma_\mu\, u(q_2,\sigma_2)\,e^{-iq_2\cdot x_1}\;\bar{u}(q_3,\sigma_3)\,e^{iq_3\cdot x_2}\,\gamma_\nu\, u(q_4,\sigma_4)\,e^{-iq_4\cdot x_2}$$

$$(-1)\Big(\delta^{(3)}(p_2'-q_1)\,\delta^{(3)}(p_1'-q_3)\,\delta_{s_2'\sigma_1}\,\delta_{s_1'\sigma_3} - \delta^{(3)}(p_1'-q_1)\,\delta^{(3)}(p_2'-q_3)\,\delta_{\sigma_1 s_1'}\,\delta_{\sigma_3 s_2'}\Big)$$

$$\Big(\delta^{(3)}(p_1-q_2)\,\delta^{(3)}(q_4-p_2)\,\delta_{s_1\sigma_2}\,\delta_{\sigma_4 s_2} - \delta^{(3)}(p_1-q_4)\,\delta^{(3)}(q_2-p_2)\,\delta_{s_1\sigma_4}\,\delta_{\sigma_2 s_2}\Big)$$

$= \frac{1}{2!}\cdot\Big($



$\Big)$

$$= (\text{type1} - \text{type2})$$

$$= (-ie)^2\int d^4x_1\, d^4x_2\; i D_F^{\mu\nu}(x_1-x_2)\,\frac{1}{(2\pi)^6}\sqrt{\frac{m}{E_{p_1}}}\sqrt{\frac{m}{E_{p_2}}}\sqrt{\frac{m}{E_{p_1'}}}\sqrt{\frac{m}{E_{p_2'}}}$$

$$\Big(e^{i(p_2'-p_2)\cdot x_1}\,e^{i(p_1'-p_1)\cdot x_2}\;\bar{u}(p_2',s_2')\gamma_\mu u(p_2,s_2)\;\bar{u}(p_1',s_1')\gamma_\nu u(p_1,s_1)$$

$$- e^{i(p_2'-p_1)\cdot x_1}\,e^{i(p_1'-p_2)\cdot x_2}\;\bar{u}(p_2',s_2')\gamma_\mu u(p_1,s_1)\;\bar{u}(p_1',s_1')\gamma_\nu u(p_2,s_2)\Big).$$

可验证正确的

- Feynman Rules in coordinate space.
  - vertex: $-ie\,(\gamma_\mu)_{\alpha\beta}$
  - Internal photon line $\quad i D_F^{\mu\nu}(x_k - x_\ell)$
  - Internal fermion line $\quad i S_{F\alpha\beta}(x_k - x_\ell)$
  - Exerternal fermion line
    - $N_p\, u(p,s)\, e^{-ip\cdot x}$ Incoming electron
    - $N_p\, \bar{v}(p,s)\, e^{ip\cdot x}$ Incoming positron
    - $N_p\, \bar{u}(p,s)\, e^{ip\cdot x}$ outgoing electron
    - $N_p\, v(p,s)\, e^{ip\cdot x}$ outgoing positron

— External photon line $N_k \epsilon^{\mu}(k,\lambda) e^{-ik\cdot x}$ incoming photon

$N_k \epsilon^{\mu *}(k,\lambda) e^{ik\cdot x}$ outgoing photon.

Normalization factor for Dirac & photon wave function.

$N_p = \sqrt{\frac{m}{(2\pi)^3 E_p}}$ $N_k = \sqrt{\frac{1}{(2\pi)^3 2\omega_k}}$

— All coordinates x_i are integrated over (If use continum normalization, replace $(2\pi)^3$ with V)

— Each closed fermion loop leads to factor -1

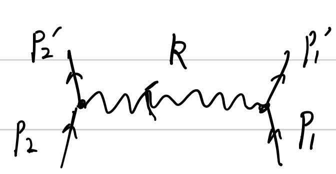
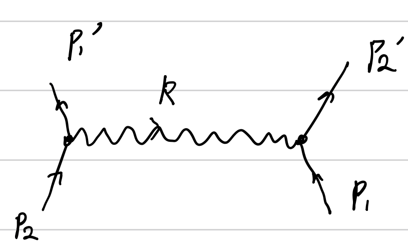
leads to Momentum Feynman Diagram.

$$i D_F^{\mu\nu}(x-y) = i\int \frac{d^4k}{(2\pi)^4} D_F^{\mu\nu}(k)\, e^{-ik(x-y)}$$

$$D_F^{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2+i\epsilon}$$

$$= (-ie)^2 \int d^4x_1 \cdot d^4x_2 \cdot \int \frac{d^4k}{(2\pi)^4}\, iD_F^{\mu\nu}(k)\, e^{-ik\cdot(x_1-x_2)} \frac{1}{(2\pi)^6} \sqrt{\frac{m}{E_{p_1}}}\sqrt{\frac{m}{E_{p_2}}}\sqrt{\frac{m}{E_{p_1'}}}\sqrt{\frac{m}{E_{p_2'}}}$$
$$\Big(e^{i(p_2'-p_2)\cdot x_1} e^{i(p_1'-p_1)\cdot x_2}\, \bar{u}(p_2',s_2')\gamma_\mu u(p_2,s_2)\, \bar{u}(p_1',s_1')\gamma_\nu u(p_1,s_1)$$
$$- e^{i(p_2'-p_1)\cdot x_1} e^{i(p_1'-p_2)\cdot x_2}\, \bar{u}(p_2',s_2')\gamma_\mu u(p_1,s_1)\, \bar{u}(p_1',s_1')\gamma_\nu u(p_2,s_2)\Big).$$

$$= \int \frac{d^4k}{(2\pi)^4} \Big\{ \sqrt{\frac{m}{(2\pi)^3E_{p_2'}}}\,\bar{u}(p_2',s_2')(-ie)\gamma_\mu (2\pi)^4\delta^{(4)}(p_2'-p_2-k)\sqrt{\frac{m}{(2\pi)^3E_{p_2}}}\, u(p_2,s_2)\cdot iD_F^{\mu\nu}(k)$$
$$\sqrt{\frac{m}{(2\pi)^3E_{p_1'}}}\,\bar{u}(p_1',s_1')(-ie)\gamma_\nu (2\pi)^4\delta^{(4)}(p_1'-p_1+k)\sqrt{\frac{m}{(2\pi)^3E_{p_1}}}\, u(p_1,s_1)$$
$$- \sqrt{\frac{m}{(2\pi)^3E_{p_2'}}}\,\bar{u}(p_2',s_2')(-ie)\gamma_\mu (2\pi)^4\delta^{(4)}(p_2'-p_1-k)\sqrt{\frac{m}{(2\pi)^3E_{p_1}}}\, \bar{u}(p_1,s_1)\cdot iD_F^{\mu\nu}(k)$$
$$\sqrt{\frac{m}{(2\pi)^3E_{p_1'}}}\,\bar{u}(p_1',s_1')(-ie)\gamma_\nu (2\pi)^4\delta^{(4)}(p_1'-p_2+k)\sqrt{\frac{m}{(2\pi)^3E_{p_2}}}\, u(p_2,s_2)\Big\}$$

= ( − )

验证!

• Feynman relues of QED in momentum space.

— Each internal line is assigned a momentum variable p_i or k_i

— vertex $-ie(\gamma_\mu)_{\alpha\beta}$

— Internal photon line $i D_F^{\mu\nu}(k_i)$

— Internal fermion line $N_p u(p,s)$ incoming electron

$N_p \bar{v}(p,s)$ incoming positron.

$N_p \bar{u}(p,s)$ outgoing electron

$N_p v(p,s)$ outgoing positron.

— External photon line $N_k \, \varepsilon^{\mu}(k, \lambda)$ incoming photon

$N_k \, \varepsilon^{\mu *}(k, \lambda)$ outgoing photon. $N_p = \sqrt{\frac{m}{(2\pi)^3 E_p}}$ $N_k = \sqrt{\frac{1}{(2\pi)^3 2\omega_k}}$

— All momenta of internal lines integrate over.

$$\int \frac{d^4 p}{(2\pi)^4}$$

— Each closed fermion loop leads to factor -1.

— Each vertex is associated with factor $(2\pi)^4 \delta^{(4)}(p' - p \pm k)$

QED — Feynman Diagram具体例子.

Compton 散射 ⟶ 电子-光子散射!

Electron!

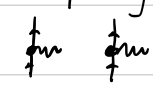
Initial : $b^{+}_{p,s}\, a^{+}_{k,\lambda}\, |0\rangle = |i\rangle$

Final : $\langle 0|\, a_{k',\lambda'}\, b_{p',s'} = \langle f|$

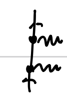
S matrix element $\langle f|\, S^{(2)}\, |i\rangle = S_{fi}$

Topological figure :

$$S^{(2)} = \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \;:\bar\psi(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar\psi(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (a)$$



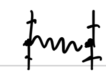
$$+ :\bar\psi(x_1)\gamma_\mu\underbrace{\psi(x_1)A^\mu(x_1)\;\bar\psi(x_2)}\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (b)$$



$$+ :\underbrace{\bar\psi(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar\psi(x_2)\gamma_\nu\psi(x_2)}A^\nu(x_2): \quad (c)$$

按定义, (b)=(c)

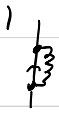
$$+ :\bar\psi(x_1)\gamma_\mu\psi(x_1)\underbrace{A^\mu(x_1)\;\bar\psi(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2)}: \quad (d)$$



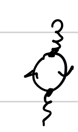
$$+ :\bar\psi(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar\psi(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (e)$$

$$+ :\bar\psi(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar\psi(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (f)$$

(e)=(f)



$$+ :\bar\psi(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar\psi(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (g)$$



$$+ :\bar\psi(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\;\bar\psi(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (h)$$

(b) 和 (c) 有什么关系, 且 (b)=(c)

$$:\bar\psi(x_1)\gamma_\mu\underbrace{\psi(x_1)A^\mu(x_1)\bar\psi(x_2)}\gamma_\nu\psi(x_2)A^\nu(x_2):$$

$$:\bar\psi_a(x_1)(\gamma_\mu)_{ab}\underbrace{\psi_b(x_1)A^\mu(x_1)\bar\psi_c(x_2)}(\gamma_\nu)_{cd}\psi_d(x_2)A^\nu(x_2):$$

normal order 中的交换引起(±)

$$= :\bar\psi_a(x_1)(\gamma_\mu)_{ab}A^\mu(x_1)\underbrace{(\gamma_\nu)_{cd}\psi_d(x_2)A^\nu(x_2)}:\ \underbrace{\psi_b(x_1)\bar\psi_c(x_2)}$$

Contraction 中的交换引起(±)

$$= :(\gamma_\nu)_{cd}\psi_d(x_2)A^\nu(x_2)\;\bar\psi_a(x_1)(\gamma_\mu)_{ab}A^\mu(x_1):\ \underbrace{\bar\psi_c(x_2)\psi_b(x_1)}$$

$$= :\bar\psi(x_2)\gamma_\nu\underbrace{\psi(x_2)A^\nu(x_2)\bar\psi(x_1)(\gamma_\mu)\psi(x_1)}A^\mu(x_1):$$

$$= 2\cdot\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\,\langle 0|\, b_{p',s'}\, a_{k',\lambda'} :\bar\psi(x_1)\gamma_\mu\underbrace{\psi(x_1)A^\mu(x_1)\bar\psi(x_2)}\gamma_\nu\psi(x_2)A^\nu(x_2): b^{+}_{p,s}\, a^{+}_{k,\lambda}|0\rangle$$

$$= 2\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\; iS_{F\,bc}(x_1-x_2)\,\langle 0|\, b_{p',s'}\, a_{k',\lambda'} :\bar\psi_a(x_1)(\gamma_\mu)_{ab}A^\mu(x_1)(\gamma_\nu)_{cd}\psi_d(x_2)A^\nu(x_2): b^{+}_{p,s}\, a^{+}_{k,\lambda}|0\rangle$$

$$= 2\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\; iS_{F\,bc}(x_1-x_2)\,\langle 0|\, b_{p',s'}\, a_{k',\lambda'} :\bar\psi^{(-)}_a(x_1)(\gamma_\mu)_{ab}(\gamma_\nu)_{cd}\psi^{(+)}_d(x_2): \;:A^\mu(x_1)A^\nu(x_2):\; b^{+}_{p,s}\, a^{+}_{k,\lambda}|0\rangle$$

$$= 2\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \cdot iS_{F\,bc}(x_1-x_2) \langle 0|\, b_{p',s'}\, a_{k',\lambda'} : \bar\psi_a^{(-)}(x_1)(\gamma_\mu)_{ab}(\gamma_\nu)_{cd}\,\psi_d^{(+)}(x_2) :$$
$$: A^{\mu(-)}(x_1)A^{\nu(+)}(x_2) + A^{\mu(+)}(x_1)A^{\nu(-)}(x_2) : \quad b^+_{p,s}\, a^+_{k,\lambda}|0\rangle$$

$$= 2\cdot\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \cdot$$
$$\sum_{\sigma_1\sigma_2}\sum_{\lambda_1\lambda_2}\int\frac{d^3q_1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{q_1}}}\cdot\int\frac{d^3q_2}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{q_2}}}\cdot\int\frac{d^3k_1}{\sqrt{(2\pi)^3 2W_{k_1}}}\int\frac{d^3k_2}{\sqrt{(2\pi)^3 2W_{k_2}}}$$

$$\bar u_a(q_1,\sigma_1)\, e^{iq_1\cdot x_1}(\gamma_\mu)_{ab}\cdot iS_{Fbc}(x_1-x_2)\,(\gamma_\nu)_{cd}\, u_d(q_2,\sigma_2)\, e^{-iq_2\cdot x_2}$$
$$\Big\{ \varepsilon^{\mu *}(k_1,\lambda_1)e^{ik_1\cdot x_1}\cdot\varepsilon^\nu(k_2,\lambda_2)e^{-ik_2\cdot x_2}\underbrace{\langle 0|\, b_{p's'}\, a_{k'\lambda'} : b^+_{q_1\sigma_1}\, b_{q_2\sigma_2} : \; : a^+_{k_1\lambda_1}\, a_{k_2\lambda_2} : b^+_{p,s}\, a^+_{k,\lambda}|0\rangle}_{1^\circ}$$
$$+\ \varepsilon^\mu(k_1,\lambda_1)\, e^{-ik_1\cdot x_1}\,\varepsilon^{*\nu}(k_2,\lambda_2)\, e^{ik_2\cdot x_2}\underbrace{\langle 0|\, b_{p's'}\, a_{k'\lambda'} : b^+_{q_1\sigma_1}\, b_{q_2\sigma_2} : \; : a_{k_1\lambda_1}\, a^+_{k_2\lambda_2} : b^+_{p,s}\, a^+_{k\lambda}|0\rangle}_{2^\circ}\Big\}$$

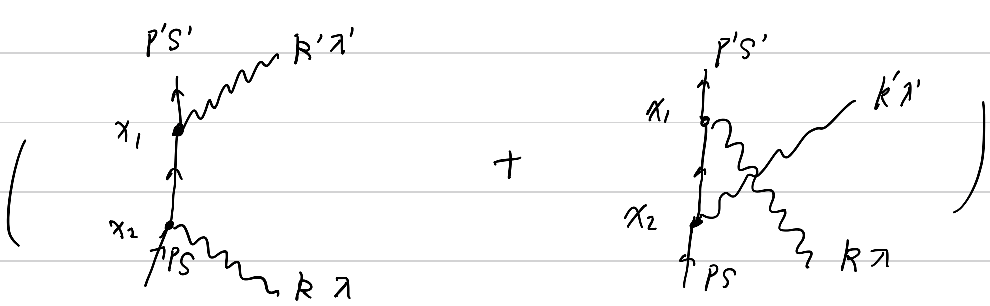
$$\{ b_{p,s},\, b^+_{p's'}\} = \delta^{(3)}(p-p')\,\delta_{ss'}$$
$$[a_{k,\lambda},\, a^+_{k',\lambda'}] = -g_{\lambda\lambda'}\,\delta^{(3)}(k-k')$$

1° $\quad \delta^{(3)}(p'-q_1)\,\delta_{s'\sigma_1}\cdot\delta^{(3)}(q_2-p)\,\delta_{\sigma_2,s}\quad g_{\lambda'\lambda_1}\,\delta^{(3)}(k_1-k')\quad g_{\lambda\lambda_2}\,\delta^{(3)}(k-k_2)$

2° $\quad \delta^{(3)}(q_1-p')\,\delta_{\sigma_1 s'}\quad\delta^{(3)}(q_2-p)\,\delta_{\sigma_2 s}\quad g_{\lambda'\lambda_2}\,\delta^{(3)}(k'-k_2)\quad g_{\lambda\lambda_1}\,\delta^{(3)}(k_1-k)$

$$= 2\cdot\frac{(-ie)^2}{2!}\int d^4x_1\cdot d^4x_2\quad\frac{1}{(2\pi)^6}\sqrt{\frac{m}{E_p}}\sqrt{\frac{m}{E_{p'}}}\cdot\frac{1}{\sqrt{2W_k}}\quad\frac{1}{\sqrt{2W_{k'}}}$$

$$\bar u_a(p',s')\, e^{ip'\cdot x_1}(\gamma_\mu)_{ab}\cdot iS_{Fbc}(x_1-x_2)\,(\gamma_\nu)_{cd}\, u_d(p,s)\, e^{-ip\cdot x_2}$$
$$\Big\{\varepsilon^{\mu *}(k',\lambda')\, e^{ik'\cdot x_1}\quad\varepsilon^\nu(k,\lambda)\, e^{-ik\cdot x_2}\qquad(\lambda \text{ or } \lambda'\in\{1,2\})$$
$$+\ \varepsilon^\mu(k,\lambda)\, e^{-ik\cdot x_1}\quad\varepsilon^\nu(k',\lambda')\, e^{ik'\cdot x_2}\Big\}$$



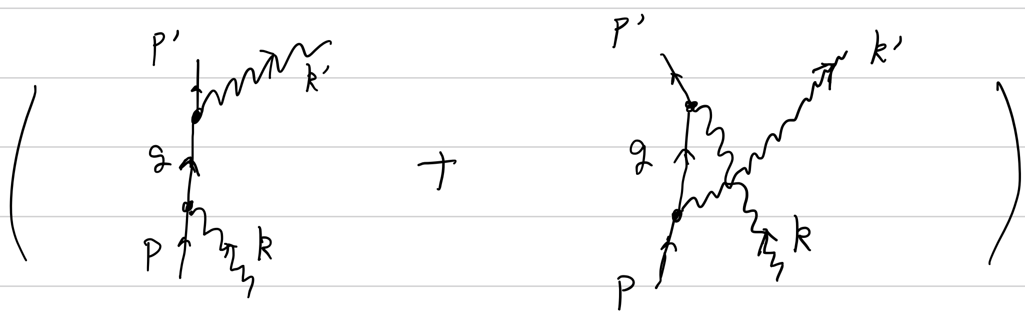
$$iS_F(x_1-x_2) = i\int\frac{d^4q}{(2\pi)^4}\cdot S_F(q)\, e^{-iq\cdot(x_1-x_2)}$$

(下面引出动量 Feynman 图!)

$$= 2\cdot\frac{(-ie)^2}{2!}\int d^4x_1\cdot d^4x_2\quad\frac{1}{(2\pi)^6}\sqrt{\frac{m}{E_p}}\sqrt{\frac{m}{E_{p'}}}\cdot\frac{1}{\sqrt{2W_k}}\quad\frac{1}{\sqrt{2W_{k'}}}$$

$$\bar u(p',s')\, e^{ip'\cdot x_1}(\gamma_\mu)\int\frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot(x_1-x_2)}\, iS_F(q)\quad(\gamma_\nu)\quad u(p,s)\, e^{-ip\cdot x_2}$$
$$\Big\{\varepsilon^{\mu *}(k',\lambda')\, e^{ik'\cdot x_1}\quad\varepsilon^\nu(k,\lambda)\, e^{-ik\cdot x_2}\qquad(\lambda \text{ or } \lambda'\in\{1,2\})$$
$$+\ \varepsilon^\mu(k,\lambda)\, e^{-ik\cdot x_1}\quad\varepsilon^\nu(k',\lambda')\, e^{ik'\cdot x_2}\Big\}$$

$$
\begin{aligned}
= \int \frac{d^4 q}{(2\pi)^4} \cdot \Bigg(& \sqrt{\frac{m}{(2\pi)^3 E_{p'}}}\ \bar{u}(p', s') \cdot (-ie)\gamma_\mu \cdot (2\pi)^4 \cdot \delta^{(4)}(p' + k' - q) \cdot i S_F(q) \cdot \\
& (-ie)\gamma_\nu \cdot (2\pi)^4 \delta^{(4)}(q - k - p) \cdot \sqrt{\frac{m}{(2\pi)^3 E_p}} \cdot u(p, s) \\
& \sqrt{\frac{1}{(2\pi)^3 2\omega_{k'}}}\ \varepsilon^{\mu *}(k', \lambda') \ \sqrt{\frac{1}{(2\pi)^3 2\omega_k}} \cdot \varepsilon^{\nu}(k, \lambda) \\
& + \sqrt{\frac{m}{(2\pi)^3 E_{p'}}}\ \bar{u}(p', s') \cdot (-ie)\gamma_\mu \cdot (2\pi)^4 \cdot \delta^{(4)}(p' - k - q) \cdot i S_F(q) \cdot \\
& (-ie)\gamma_\nu \cdot (2\pi)^4 \delta^{(4)}(q + k' - p) \cdot \sqrt{\frac{m}{(2\pi)^3 E_p}} \cdot u(p, s) \\
& \sqrt{\frac{1}{(2\pi)^3 2\omega_k}}\ \varepsilon^{\mu}(k, \lambda) \ \sqrt{\frac{1}{(2\pi)^3 2\omega_{k'}}} \cdot \varepsilon^{\nu}(k', \lambda') \Bigg)
\end{aligned}
$$



Compton scattering —— positron.

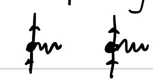
Initial state : $|i\rangle = a^{\dagger}_{k\lambda}\, d^{\dagger}_{ps}\, |0\rangle$ (λ and λ' = 1 or 2)

Final state : $\langle f| = \langle 0|\, d_{p's'}\, a_{k'\lambda'}$

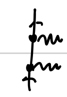
S operator 的矩阵元 : $S_{fi} = \langle f|S|i\rangle$

Topological figure :

$$S^{(2)} = \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \; :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\,\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (a)$$



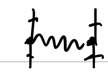
$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\,\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (b)$$



$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\,\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (c)$$

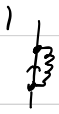
按定义, (b)=(c)

$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\,\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (d)$$



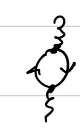
$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\,\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (e)$$

(e)=(f)



$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\,\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (f)$$

$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\,\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (g)$$



$$+ :\bar{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\,\bar{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2): \quad (h)$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_s \left(b_{ps}\, u(p,s)\, e^{-ip\cdot x} + d^{\dagger}_{ps}\, v(p,s)\, e^{ip\cdot x}\right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_s \left(d_{ps}\, \bar{v}(p,s)\, e^{-ip\cdot x} + b^{\dagger}_{p,s}\, \bar{u}(p,s)\, e^{ip\cdot x}\right)$$

$$A_\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3\cdot 2\omega_k}}\sum_\lambda \left(a_{k,\lambda}\, \varepsilon^\mu(k,\lambda)\, e^{-ik\cdot x} + a^{\dagger}_{k,\lambda}\, \varepsilon^{*\mu}(k,\lambda)\, e^{ik\cdot x}\right)$$

对 S_{fi} 有贡献的 S 部分是: (b); (c) ((b)=(c) 的原因在 Compton — electron 部分说了!)

$$S_{fi} = 2\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\, \langle 0|\, d_{p's'}\, a_{k'\lambda'}$$

$$:\bar{\psi}(x_1)\gamma_\mu\underbrace{\psi(x_1)\bar{\psi}(x_2)}\gamma_\nu\psi(x_2): \; :A^\mu(x_1)A^\nu(x_2):\; a^{\dagger}_{k\lambda}\, d^{\dagger}_{ps}\, |0\rangle$$

$$= 2\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\, \langle 0|\, d_{p's'}\, a_{k'\lambda'}$$

$$:\bar{\psi}^{(+)}(x_1)\gamma_\mu\underbrace{\psi(x_1)\bar{\psi}(x_2)}\gamma_\nu\psi^{(-)}(x_2): \; :A^{\mu(+)}(x_1)A^{\nu(-)}(x_2) + A^{\mu(-)}(x_1)A^{\nu(+)}(x_2):\; a^{\dagger}_{k\lambda}\, d^{\dagger}_{ps}\, |0\rangle$$

$$= 2\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2$$

$$\int d^4q_1\, d^4q_2\cdot d^4k_1\, d^4k_2\cdot \frac{1}{(2\pi)^6}\sqrt{\frac{m}{E_{q_1}}}\sqrt{\frac{m}{E_{q_2}}}\sqrt{\frac{1}{2\omega_k}}\sqrt{\frac{1}{2\omega_{k'}}}$$

$$\bar{v}(q_1,\sigma_1)\, e^{-iq_1\cdot x_1}\cdot\gamma_\mu\cdot iS_F(x_1-x_2)\cdot\gamma_\nu\cdot v(q_2,\sigma_2)\, e^{iq_2\cdot x_2}$$

$$\Big(\varepsilon^\mu(k_1,\lambda_1)\, e^{-ik_1\cdot x_1}\,\varepsilon^{\nu *}(k_2,\lambda_2)\, e^{ik_2\cdot x_2}$$

$$1^\circ \langle 0|\, d_{p's'}\, a_{k'\lambda'} :d_{q_1\sigma_1}\, d^{\dagger}_{q_2\sigma_2}: \; :a_{k_1\lambda_1}\, a^{\dagger}_{k_2\lambda_2}:\; a^{\dagger}_{k\lambda}\, d^{\dagger}_{ps}\, |0\rangle$$

$$+\, \varepsilon^{\mu *}(k_1,\lambda_1)\, e^{ik_1\cdot x_1}\,\varepsilon^{\nu}(k_2,\lambda_2)\, e^{-ik_2\cdot x_2}$$

$$2^\circ \langle 0|\, d_{p's'}\, a_{k'\lambda'} :d_{q_1\sigma_1}\, d^{\dagger}_{q_2\sigma_2}: \; :a^{\dagger}_{k_1\lambda_1}\, a_{k_2\lambda_2}:\; a^{\dagger}_{k\lambda}\, d^{\dagger}_{ps}\, |0\rangle \Big)$$

1° $\langle 0| d_{p's'}\, a_{k'\lambda'} :d_{q_1\sigma_1}\, d^\dagger_{q_2\sigma_2}: \, :a_{k_1\lambda_1}\, a^\dagger_{k_2\lambda_2}: \, a^\dagger_{k\lambda}\, d^\dagger_{ps} |0\rangle$

$= -\langle 0| d_{p's'}\, a_{k'\lambda'}\, d^\dagger_{q_2\sigma_2}\, d_{q_1\sigma_1}\, a^\dagger_{k_2\lambda_2}\, a_{k_1\lambda_1}\, a^\dagger_{k\lambda}\, d^\dagger_{ps}|0\rangle$

$$\begin{cases} \{ d_{ps},\, d^\dagger_{p's'} \} = \delta_{ss'}\, \delta^{(3)}(p-p') \\ [a_{p\lambda},\, a^\dagger_{p'\lambda'}] = -g_{\lambda\lambda'}\, \delta^{(3)}(k-k') \end{cases}$$

$= -\delta^{(3)}(q_2-p')\,\delta_{\sigma_2 s_2}\,\delta^{(3)}(q_1-p)\,\delta_{\sigma_1 s}\; g_{\lambda'\lambda_2}\,\delta^{(3)}(k_2-k')\; g_{\lambda_1\lambda}\,\delta^{(3)}(k-k_1)$

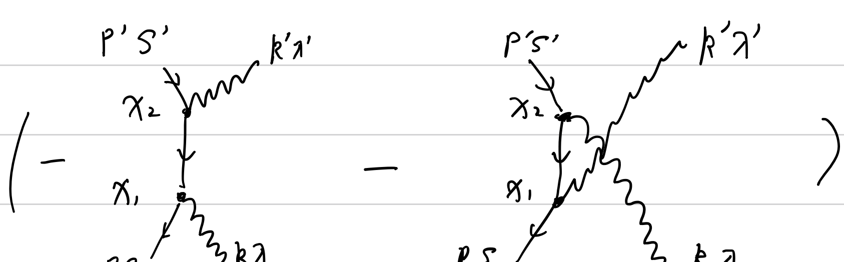
2° $\langle 0| d_{p's'}\, a_{k'\lambda'} :d_{q_1\sigma_1}\, d^\dagger_{q_2\sigma_2}: \, :a^\dagger_{k_1\lambda_1}\, a_{k_2\lambda_2}: \, a^\dagger_{k\lambda}\, d^\dagger_{ps} |0\rangle$

$= -\langle 0| d_{p's'}\, a_{k'\lambda'}\, d^\dagger_{q_2\sigma_2}\, d_{q_1\sigma_1}\, a^\dagger_{k_1\lambda_1}\, a_{k_2\lambda_2}\, a^\dagger_{k\lambda}\, d^\dagger_{ps}|0\rangle$

$= -\delta^{(3)}(q_2-p')\,\delta_{\sigma_2 s'}\,\delta^{(3)}(q_1-p)\,\delta_{\sigma_1 s}\; g_{\lambda'\lambda_1}\,\delta^{(3)}(k_1-k')\; g_{\lambda_2\lambda}\,\delta^{(3)}(k-k_2)$

$$\begin{aligned}
= 2\,\frac{(-ie)^2}{2!} \int d^4x_1\, d^4x_2 \\
\int d^4q_1\, d^4q_2\, d^4k_1\, d^4k_2\, \frac{1}{(2\pi)^6} \sqrt{\frac{m}{E_{q_1}}} \sqrt{\frac{m}{E_{q_2}}} \sqrt{\frac{1}{2\omega_k}} \sqrt{\frac{1}{2\omega_{k'}}} \\
\bar{v}(q_1,\sigma_1)\, e^{-iq_1\cdot x_1}\cdot \gamma_\mu \cdot iS_F(x_1-x_2)\cdot \gamma_\nu \cdot v(q_2,\sigma_2)\, e^{iq_2\cdot x_2} \\
\Big(-\varepsilon^\mu(k_1,\lambda_1)\, e^{-ik_1\cdot x_1}\, \varepsilon^{\nu *}(k_2,\lambda_2)\, e^{ik_2\cdot x_2} \\
\delta^{(3)}(q_2-p')\,\delta_{\sigma_2 s_2}\,\delta^{(3)}(q_1-p)\,\delta_{\sigma_1 s}\; g_{\lambda'\lambda_2}\,\delta^{(3)}(k_2-k')\; g_{\lambda_1\lambda}\,\delta^{(3)}(k-k_1) \\
-\varepsilon^{\mu *}(k_1,\lambda_1)\, e^{ik_1\cdot x_1}\, \varepsilon^{\nu}(k_2,\lambda_2)\, e^{-ik_2\cdot x_2} \\
\delta^{(3)}(q_2-p')\,\delta_{\sigma_2 s'}\,\delta^{(3)}(q_1-p)\,\delta_{\sigma_1 s}\; g_{\lambda'\lambda_1}\,\delta^{(3)}(k_1-k')\; g_{\lambda_2\lambda}\,\delta^{(3)}(k-k_2) \Big)
\end{aligned}$$

$$\begin{aligned}
= (-ie)^2 \int d^4x_1\, d^4x_2\, \frac{1}{(2\pi)^6} \sqrt{\frac{m}{E_p}} \sqrt{\frac{m}{E_{p'}}} \sqrt{\frac{m}{E_k}} \sqrt{\frac{m}{E_{k'}}} \\
\bar{v}(p,s)\, e^{-ip\cdot x_1}\, \gamma_\mu\, iS_F(x_1-x_2)\, \gamma_\nu\, v(p',s')\, e^{ip'\cdot x_2} \\
\Big(-\varepsilon^\mu(k,\lambda)\, e^{-ik\cdot x_1}\, \varepsilon^{\nu *}(k',\lambda')\, e^{ik'\cdot x_2} \\
-\varepsilon^{\mu *}(k',\lambda')\, e^{ik'\cdot x_1}\, \varepsilon^{\nu}(k,\lambda)\, e^{-ik\cdot x_2} \Big)
\end{aligned}$$



Change to momentum Feynman Diagram

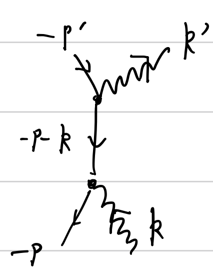
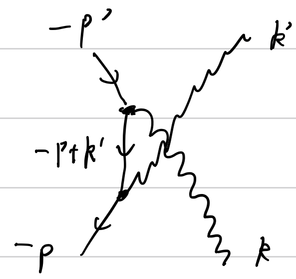
$$iS_F(x_1-x_2) = \int \frac{d^4q}{(2\pi)^4}\, iS_F(q)\, e^{-iq\cdot(x_1-x_2)}$$

$$\begin{aligned}
= (-ie)^2 \int d^4x_1\, d^4x_2\, \frac{1}{(2\pi)^6} \sqrt{\frac{m}{E_p}} \sqrt{\frac{m}{E_{p'}}} \sqrt{\frac{m}{E_k}} \sqrt{\frac{m}{E_{k'}}} \\
\bar{v}(p,s)\, e^{-ip\cdot x_1}\, \gamma_\mu \int \frac{d^4q}{(2\pi)^4}\, iS_F(q)\, e^{-iq(x_1-x_2)}\, \gamma_\nu\, v(p',s')\, e^{ip'\cdot x_2} \\
\Big(-\varepsilon^\mu(k,\lambda)\, e^{-ik\cdot x_1}\, \varepsilon^{\nu *}(k',\lambda')\, e^{ik'\cdot x_2} \\
-\varepsilon^{\mu *}(k',\lambda')\, e^{ik'\cdot x_1}\, \varepsilon^{\nu}(k,\lambda)\, e^{-ik\cdot x_2} \Big)
\end{aligned}$$

$$\begin{aligned}
= \int \frac{d^4q}{(2\pi)^4} \Big[-\sqrt{\frac{m}{(2\pi)^3 E_p}} \cdot \bar{v}(p,s) \cdot (-ie)\gamma_\mu \cdot (2\pi)^4 \cdot \delta^{(4)}(-p-q-k) \cdot iS_F(q) \cdot (-ie)\gamma_\nu \cdot (2\pi)^4 \delta^{(4)}(q+p'+k') \cdot \\
\sqrt{\frac{m}{(2\pi)^3 E_{p'}}}\, v(p',s') \sqrt{\frac{1}{(2\pi)^3 2E_k}}\, \varepsilon^\mu(k,\lambda) \sqrt{\frac{1}{(2\pi)^3 2E_{k'}}}\, \varepsilon^{\nu *}(k',\lambda')
\end{aligned}$$

$$- \sqrt{\frac{m}{(2\pi)^3 E_p}}\, \bar{v}(p,s) \; (-ie)\gamma_\mu \; (2\pi)^4 \delta^{(4)}(-p-q+k') \; iS_F(q) \; (-ie)\gamma_\nu \; (2\pi)^4 \delta^{(4)}(q+p'-k)$$

$$\sqrt{\frac{m}{(2\pi)^3 E_{p'}}}\, v(p',s') \; \sqrt{\frac{1}{(2\pi)^3 2E_{k'}}} \; \varepsilon^{\mu *}(k',\lambda') \; \sqrt{\frac{1}{(2\pi)^3 2E_k}} \; \varepsilon^{\nu}(k,\lambda) \Big]$$

$= \Big(-$  $-$  $\Big)$

(为了让节点动量 conserve，才把 positron 标负，动量标值是指箭头方向的动量!)

$$\begin{cases} v(p,-s)\exp(ip^\mu x_\mu) = \sqrt{\frac{E}{m}}\, \Psi_{-\vec{p},-,\uparrow} \\ \bar{v}(p,s)\exp(ip^\mu x_\mu) = \sqrt{\frac{E}{m}}\, \Psi_{-\vec{p},-,\downarrow} \end{cases}$$

Anti-particles behave like negative-energy particles that move backward in Time!

photon — photon scattering.

Delbruck scattering.

○ 是 Greiner 中的 problem 8.5 (计算 4 阶 Feynman 图, 证明其中的 $\frac{1}{n!}$ term 会消除!)

4 阶 S operator 的矩阵元可写为:

$$S_{fi} = \frac{(-ie)^4}{4!}\int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4 \cdot$$

$$\langle f|T[:\bar\psi(x_1)\not A(x_1)\psi(x_1)::\bar\psi(x_2)\not A(x_2)\psi(x_2)::\bar\psi(x_3)\not A(x_3)\psi(x_3)::\bar\psi(x_4)\not A(x_4)\psi(x_4):]|i\rangle$$

初末态为:

$$|i\rangle = a^{+}_{k_2\lambda_2}\, a^{+}_{k_1\lambda_1}|0\rangle \quad \langle f| = \langle 0|\, a_{k_1'\lambda_1'}\, a_{k_2'\lambda_2'} \qquad \lambda_1;\lambda_2;\lambda_1';\lambda_2' \in \{1 \text{ or } 2\}$$

对 S_{fi} 有贡献 → ψ 要相互 contract (共 6 种 contract 方式)!

(1234) + (1243) + (1324) + (1342) + (1423) + (1432)

↓ 代表 1 的 ψ 和 2 的 $\bar\psi$ contract…

而因为 x_i 可以随意调换顺序, 则此 6 个相等! ⟶ 提供一个 factor 6.

$$(1234) = :\bar\psi(x_1)\not A(x_1)\psi(x_1)\,\bar\psi(x_2)\not A(x_2)\psi(x_2)\,\bar\psi(x_3)\not A(x_3)\psi(x_3)\,\bar\psi(x_4)\not A(x_4)\psi(x_4):$$

在 Dirac field contract 过后, 要记算:

$$S_{fi} = 6\cdot\frac{(-ie)^4}{4!}\cdot\int d^4x_1\cdot d^4x_2\cdot d^4x_3\cdot d^4x_4.$$

$$\langle 0|\, a_{k_2'\lambda_2'}\, a_{k_1'\lambda_1'} :\bar\psi(x_1)\not A(x_1)\psi(x_1)\,\bar\psi(x_2)\not A(x_2)\psi(x_2)\,\bar\psi(x_3)\not A(x_3)\psi(x_3)\,\bar\psi(x_4)\not A(x_4)\psi(x_4): a^{+}_{k_2\lambda_2}a^{+}_{k_1\lambda_1}|0\rangle$$

4 个 A 要有 2 个取 a^+, 两个 a. 共有 $C_4^2 = 6$ 种 configuration!

计算其中一种 configuration:

$$S_{fi} = 6\cdot\frac{(-ie)^4}{4!}\cdot\int d^4x_1\cdot d^4x_2\cdot d^4x_3\cdot d^4x_4.$$

$$\langle 0|\, a_{k_2'\lambda_2'}\, a_{k_1'\lambda_1'} :\bar\psi(x_1)\not A^{(-)}(x_1)\psi(x_1)\,\bar\psi(x_2)\not A^{(-)}(x_2)\psi(x_2)\,\bar\psi(x_3)\not A^{(+)}(x_3)\psi(x_3)\,\bar\psi(x_4)\not A^{(+)}(x_4)\psi(x_4): a^{+}_{k_2\lambda_2}a^{+}_{k_1\lambda_1}|0\rangle$$

$$= 6\cdot\frac{(-ie)^4}{4!}\cdot\int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\cdot\int d^4q_1\cdot d^4q_2\cdot d^4q_3\cdot d^4q_4\cdot\sum_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}$$

$$\mathrm{tr}\,\{\gamma_{\mu_1}\varepsilon^{\mu_1 *}(q_1,\epsilon_1)\, iS_F(x_1-x_2)\cdot\gamma_{\mu_2}\,\varepsilon^{\mu_2 *}(q_2,\epsilon_2)\cdot iS_F(x_2-x_3)\cdot\gamma_{\mu_3}\cdot\varepsilon^{\mu_3}(q_3,\epsilon_3)\cdot iS_F(x_3-x_4)\cdot$$

$$\gamma_{\mu_4}\varepsilon^{\mu_4}(q_4,\epsilon_4)[-iS_F(x_4-x_1)]\}$$

$$\langle 0|\, a_{k_1'\lambda_1'}\, a_{k_2'\lambda_2'} : a^{+}_{q_1\epsilon_1}\, a^{+}_{q_2\epsilon_2}\, a_{q_3\epsilon_3}\cdot a_{q_4\epsilon_4}: a^{+}_{k_2\lambda_2}\, a^{+}_{k_1\lambda_1}|0\rangle$$

$$\langle 0|\, a_{k_1'\lambda_1'}\, a_{k_2'\lambda_2'} : a^{+}_{q_1\epsilon_1}\, a^{+}_{q_2\epsilon_2}\, a_{q_3\epsilon_3}\cdot a_{q_4\epsilon_4}: a^{+}_{k_2\lambda_2}\, a^{+}_{k_1\lambda_1}|0\rangle$$

$$= \langle 0|\, a_{k_1'\lambda_1'}\, a_{k_2'\lambda_2'}\, a^{+}_{q_1\epsilon_1}\, a^{+}_{q_2\epsilon_2}\, a_{q_3\epsilon_3}\, a_{q_4\epsilon_4}\, a^{+}_{k_2\lambda_2}\, a^{+}_{k_1\lambda_1}|0\rangle$$

$$= \langle 0|\, a_{k_1'\lambda_1'}\left(a^{+}_{q_1\epsilon_1}a_{k_2'\lambda_2'} - g_{\lambda_2'\epsilon_1}\,\delta^{(3)}(q_1-k_2')\right)\cdot a^{+}_{q_2\epsilon_2}$$

$$a_{q_3\epsilon_3}\left(a^{+}_{k_2\lambda_2}a_{q_4\epsilon_4} - g_{\lambda_2\epsilon_4}\,\delta^{(3)}(q_4-k_2)\right)a^{+}_{k_1\lambda_1}|0\rangle$$

$$= \left(g_{\lambda_1'\epsilon_1}\,\delta^{(3)}(k_1'-q_1)\cdot g_{\lambda_2'\epsilon_2}\cdot\delta^{(3)}(k_2'-q_2) + g_{\lambda_2'\epsilon_1}\,\delta^{(3)}(q_1-k_2')\, g_{\lambda_1'\epsilon_2}\,\delta^{(3)}(q_2-k_1')\right)$$

$$\times\left(g_{\lambda_2\epsilon_3}\,\delta^{(3)}(k_2-q_3)\, g_{\lambda_1\epsilon_4}\,\delta^{(3)}(k_1-q_4) + g_{\lambda_2\epsilon_4}\,\delta^{(3)}(q_4-k_2)\, g_{\lambda_1\epsilon_3}\,\delta^{(3)}(k_1-q_3)\right)$$

(产生 or 湮灭的光子的 λ = 1 or 2)

$$= 6\cdot\frac{(-ie)^4}{4!}\cdot\int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\cdot$$

$$\Big[\ \mathrm{tr}\,\{\gamma_{\mu_1}\varepsilon^{\mu_1 *}(k_1',\lambda_1')\,iS_F(x_1-x_2)\cdot\gamma_{\mu_2}\,\varepsilon^{\mu_2 *}(k_2',\lambda_2')\cdot iS_F(x_2-x_3)\cdot\gamma_{\mu_3}\cdot\varepsilon^{\mu_3}(k_2,\lambda_2)\cdot iS_F(x_3-x_4)\cdot$$
$$\gamma_{\mu_4}\varepsilon^{\mu_4}(k_1,\lambda_1)[-iS_F(x_4-x_1)]\}\ \exp(-ik_1'\cdot x_1+ik_2'\cdot x_2-ik_2\cdot x_3-ik_1\cdot x_4)$$

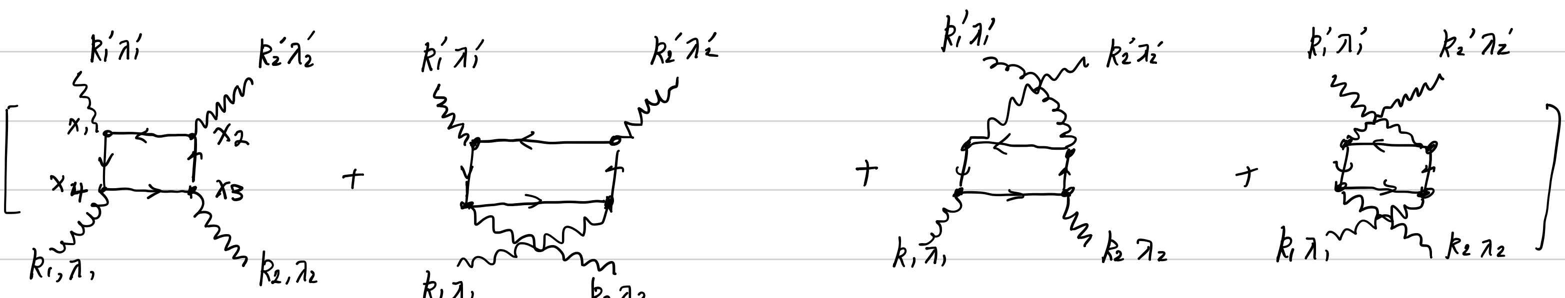
$$+\ \mathrm{tr}\,\{\gamma_{\mu_1}\varepsilon^{\mu_1 *}(k_1',\lambda_1')\,iS_F(x_1-x_2)\,\gamma_{\mu_2}\varepsilon^{\mu_2 *}(k_2',\lambda_2')\,iS_F(x_2-x_3)\,\gamma_{\mu_3}\varepsilon^{\mu_3}(k_1,\lambda_1)\,iS_F(x_3-x_4)\cdot$$
$$\gamma_{\mu_4}\varepsilon^{\mu_4}(k_2,\lambda_2)(-iS_F(x_4-x_1))\}\ \exp(ik_1'\cdot x_1+ik_2'\cdot x_2-ik_1\cdot x_3-ik_2\cdot x_4)$$

$$+\ \mathrm{tr}\,\{\gamma_{\mu_1}\varepsilon^{\mu_1 *}(k_2',\lambda_2')\,iS_F(x_1-x_2)\cdot\gamma_{\mu_2}\,\varepsilon^{\mu_2 *}(k_1',\lambda_1')\cdot iS_F(x_2-x_3)\cdot\gamma_{\mu_3}\cdot\varepsilon^{\mu_3}(k_2,\lambda_2)\cdot iS_F(x_3-x_4)\cdot$$
$$\gamma_{\mu_4}\varepsilon^{\mu_4}(k_1,\lambda_1)[-iS_F(x_4-x_1)]\}\ \exp(ik_2'\cdot x_1+ik_1'\cdot x_2-ik_2\cdot x_3-ik_1\cdot x_4)$$

$$+\ \mathrm{tr}\,\{\gamma_{\mu_1}\varepsilon^{\mu_1 *}(k_2',\lambda_2')\,iS_F(x_1-x_2)\,\gamma_{\mu_2}\varepsilon^{\mu_2 *}(k_1',\lambda_1')\,iS_F(x_2-x_3)\,\gamma_{\mu_3}\varepsilon^{\mu_3}(k_1,\lambda_1)\,iS_F(x_3-x_4)\cdot$$
$$\gamma_{\mu_4}\varepsilon^{\mu_4}(k_2,\lambda_2)(-iS_F(x_4-x_1))\}\ \exp(ik_2'\cdot x_1+ik_1'\cdot x_2-ik_1\cdot x_3-ik_2\cdot x_4)\ \Big]$$

图表示：

$\frac{6}{4!}\cdot\Big[$



(每4个一组，共6组)

对于不同的取 $a^\dagger$ 与 a 的办法，($C_4^2=6$；一共有24个图形，由于轮换对称，(1234)=(2341)…) ⇓

图表示：

$\frac{6}{4!}\cdot 4\cdot\Big[$

$6\cdot 4=4!$