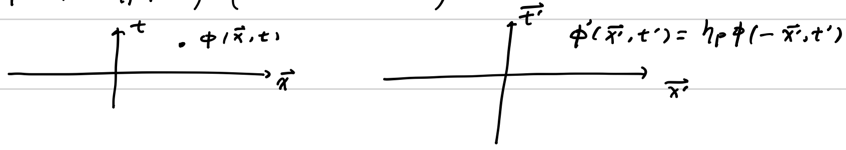
Charged scalar field. $\mathcal{P}$

- classical.

坐标 transformation:

$(\phi'(x') = \eta_p \phi(x))$; $(\phi'^*(x') = \eta_p^* \phi^*(x))$



$\phi'(\vec{x}', t') = \eta_p \phi(-\vec{x}', t')$

—— 此 transformation 不改变作用量 S:

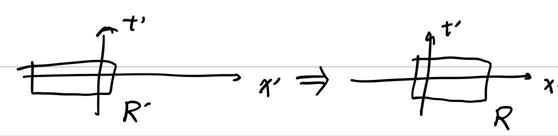
Non-charge Spin 0 field 的 lagrangian 是:

$$\mathcal{L}(\phi, \phi^*, \dot\phi, \dot\phi^*, \partial\phi, \partial\phi^*) = \frac{1}{2}\hbar^2 \frac{\partial \phi^*}{\partial x_\mu}\frac{\partial \phi}{\partial x^\mu} - \frac{1}{2}m^2c^2\phi\phi^* \; ; \; L = L[\phi, \dot\phi, \phi^*, \dot\phi^*]$$

则, S 作用量是: $S = \int_R d^4x \left(\frac{1}{2}\hbar^2 \frac{\partial \phi^*(x)}{\partial x_\mu}\frac{\partial \phi}{\partial x^\mu} - \frac{1}{2}m^2c^2\phi\phi^* \right)$

在经历坐标变换后: $S' = \int_{R'} d^4x' \left(\frac{1}{2}\hbar^2 \frac{\partial \phi'^*(x')}{\partial x'_\mu}\frac{\partial \phi'(x')}{\partial x'^\mu} - \frac{1}{2}m^2c^2\phi'(x')\phi'^*(x') \right)$

$$\begin{cases} \phi'(x') = \eta_p \phi(-\vec{x}', t) \\ \dfrac{\partial \phi'(x')}{\partial x'^i} = \eta_p \dfrac{\partial \phi(-\vec{x}', t)}{\partial x'^i} = -\eta_p \partial_i \phi(x) & (x = (-\vec{x}', t')) \\ \dfrac{\partial \phi'(x')}{\partial t'} = \eta_p \cdot \partial_0 \phi(x) & (x = (-\vec{x}', t')) \end{cases}$$



$$= \int_R \det\left(\left|\frac{\partial x'^\mu}{\partial x^\nu}\right|\right) \cdot d^4x \cdot \left(\frac{1}{2}\hbar^2 \partial^\mu\phi^* \partial_\mu\phi - \frac{1}{2}m^2c^2\phi^2 \right)|\eta_p\eta_p^*|$$

$S' = S \Rightarrow \eta_p = e^{i\delta}$

- 量子化:

原坐标系: $|\alpha\rangle$, $\phi(x)$, $\phi^\dagger(x)$

变化后: $|\alpha'\rangle = \mathcal{P}|\alpha\rangle$; $(\mathcal{P}^{-1} = \mathcal{P}^\dagger = \mathcal{P})$

关系:

$\langle\beta'|\phi(x')|\alpha'\rangle = \eta_p \langle\beta|\phi(x)|\alpha\rangle$ $\qquad$ $\langle\beta'|\phi^\dagger(x')|\alpha'\rangle = \eta_p^* \langle\beta|\phi^\dagger(x)|\alpha\rangle$

$\mathcal{P}^{-1}\phi(x')\mathcal{P} = \eta_p \phi(x)$

$\mathcal{P}\phi(x)\mathcal{P}^{-1} = \eta_p \phi(-\vec{x}, t)$ $\qquad$ $\mathcal{P}\phi^\dagger(\vec{x}, t)\mathcal{P}^{-1} = \eta_p^* \phi^\dagger(-\vec{x}, t)$

- Scalar & Pseudoscalar particle. 当 $\phi = \phi^\dagger$ 时, $\eta_p = \eta_p^* \Rightarrow \eta_p = \pm 1$ $\begin{cases} +1: \text{scalar}, \mathcal{P} \text{下不变} \\ -1: \text{pseudoscalar} \to \mathcal{P} \text{下改变}. \end{cases}$
- $\mathcal{P}$ operator 具体要求与形式.

$\phi(\vec{x}, t) = \int d^3p \left[a_p u_p(\vec{x}, t) + b_p^\dagger u_p^*(\vec{x}, t) \right]$ $\qquad$ $u_p(\vec{x}, t) = \dfrac{1}{\sqrt{2w_p(2\pi)^3}} \exp(i(w_p t - \vec{p}\cdot\vec{x}))$

$\left(w_p = \sqrt{m^2 + |\vec{p}|^2} \right)$

$u_p(-\vec{x}, t) = u_{-p}(\vec{x}, t)$

$$\int d^3p \cdot \left[\mathcal{P} a_p \mathcal{P}^{-1} u_p(\vec{x}, t) + \mathcal{P} b_p^\dagger \mathcal{P}^{-1} u_p^*(\vec{x}, t) \right]$$

$$= \eta_p \cdot \int d^3p \cdot \left[a_p u_p(-\vec{x}, t) + b_p^\dagger u_p^*(-x, t) \right]$$

$$\Downarrow$$

$$\begin{cases} \mathcal{P} a_p \mathcal{P}^{-1} = \eta_p \cdot a_{-p} \\ \mathcal{P} b_p^\dagger \mathcal{P}^{-1} = \eta_p \, b_{-p}^\dagger \end{cases} \Longrightarrow \begin{cases} \mathcal{P} a_p^\dagger \mathcal{P}^{-1} = \eta_p^* \, a_{-p}^\dagger \\ \mathcal{P} b_p \mathcal{P}^{-1} = \eta_p^* \, b_{-p} \end{cases}$$

$\downarrow$ real value η_p solutions

$$\mathcal{P} = \exp\left[i\frac{\pi}{2} \int d^3p \cdot \left((a_p^\dagger a_{-p} + b_p^\dagger b_{-p}) - \eta_p (a_p^\dagger a_p + b_p^\dagger b_p) \right) \right]$$

◦ $\mathcal{P}$ 的效果.

— 真空态. $|0\rangle$

用前面 $\mathcal{P}$ 的表达式.

$$\mathcal{P}|0\rangle = |0\rangle$$

— 单粒子态.

$$\mathcal{P}|p\rangle = \mathcal{P}a_p^\dagger|0\rangle = \mathcal{P}a_p^\dagger\mathcal{P}^{-1}\mathcal{P}|0\rangle = \eta_p|-\vec{p}\rangle$$

— Hamiltonian / Angular momentum / Momentum

$$\mathcal{P}H\mathcal{P}^{-1} = H \qquad \mathcal{P}\vec{P}\mathcal{P}^{-1} = -\vec{P} \qquad \mathcal{P}\vec{L}\mathcal{P} = \vec{L}$$

Time Reversal

- Classical:

$$\phi(x) \to \phi'(x')$$

$$\phi'(x') = \Lambda \phi(x) \qquad x' = (-t, \vec{x})$$

—— 对特定解的效果.

$$\phi(x) = e^{-iE_n t} \cdot u_n(\vec{x}) = u_n(x).$$

变换后:

$$\phi'(x') = \Lambda \phi(x) \Rightarrow \boxed{\phi'(-t, \vec{x}) = \Lambda e^{-iE_n t} u_n(\vec{x})}$$

道理上: $\phi'(-t, \vec{x}) = e^{iE_n t} \Lambda u_n(\vec{x})$ 很奇怪, 但先接受.

— 取, $\Lambda = \eta_T K$

$$\boxed{\begin{array}{l} KC = C^* K \\ \text{operator of complex conjugation.} \end{array}}$$

$|\eta_T|^2 = 1$, fase factor, $\underline{\Lambda^2 = 1}$ $\Rightarrow \eta_T K \eta_T K = \eta_T \eta_T^* K^2 = |\eta_T|^2$

↑ requirement.

$$\Lambda e^{iE_n t} = \eta_T K e^{iE_n t} = \eta_T e^{+iE_n t} K = e^{iE_n t} \Lambda.$$ (满足解的条件!)

—— 作用效果.

$$\phi'(-t, \vec{x}) = \Lambda \phi(t, \vec{x}) = \eta_T K \cdot \phi(t, \vec{x}) = \eta_T \phi^*(t, \vec{x})$$

可以直接理解为定义 $\phi'(x') = \eta_T \phi^*(x)$; $|\eta_T|^2 = 1$. (它可确实保证了 S invariant)

- Quantum.

—— Anti unitary Operator

1° Linear

$$V(C_1|\alpha_1\rangle + C_2|\alpha_2\rangle) = C_1^* V|\alpha_1\rangle + C_2^* V|\alpha_2\rangle$$

2° Hermitian conjugate satisfy (定义为)

$$\langle \alpha | V^\dagger | \beta \rangle = \langle \beta | V | \alpha \rangle$$

3° Anti-unitary

$$VV^\dagger = V^\dagger V = \mathbb{I}$$

4° preserve scalar norm while interchange "bra" "ket" vectors.

$|\alpha\rangle \to V|\alpha\rangle$; $|\beta\rangle \to V|\beta\rangle$ ordinary: $\langle \beta' | V | \alpha \rangle = \langle V^\dagger \beta' | \alpha \rangle = \langle \alpha | V^\dagger | \beta \rangle^*$

$$\langle \beta' | \alpha' \rangle = \langle \beta' | V | \alpha \rangle = \langle \alpha | V^\dagger | \beta' \rangle = \langle \alpha | V^\dagger V | \beta \rangle = \langle \alpha | \beta \rangle$$

5° $|\alpha\rangle = V|\alpha\rangle$ $|\beta\rangle = V|\beta\rangle$ / $\langle \beta' | A | \alpha' \rangle = \langle A^\dagger \beta' | \alpha' \rangle \overset{V^\dagger\text{定义.}}{=} \langle \alpha | V^\dagger | A^\dagger \beta' \rangle = \langle \alpha | V^\dagger A^\dagger V | \beta \rangle = \langle \alpha | (V^\dagger A V)^\dagger | \beta \rangle$

6° Can decompose into unitary op U & complex conjugate op K.

$$V = U \cdot K$$

—— Requirement of Time - Reverse operator

Classical: $\phi'(-t, \vec{x}) = \eta_T \phi^*(t, \vec{x})$

schrodinger: $\langle \alpha' | \phi(-t, \vec{x}) | \beta' \rangle = \langle \beta | \phi'(-t, \vec{x}) | \alpha \rangle = \eta_T \langle \beta | \phi^\dagger(t, \vec{x}) | \alpha \rangle$

↑ Change initial / final states.

Heisenberg : $\Downarrow$ From above.

$$\langle T\alpha | \phi(-t,\vec{x}) | T\beta\rangle = \eta_T \langle\beta| \phi^\dagger(t,\vec{x}) |\alpha\rangle$$

$\Big\}$ property 5° / T 是 Anti-unitary op.

$$\langle T\alpha | \phi(-t,\vec{x}) | T\beta\rangle = \langle\beta| (T^\dagger \phi(-t,\vec{x}) T)^\dagger |\alpha\rangle$$

$$\langle\beta| (T^\dagger \phi(-t,\vec{x}) T)^\dagger |\alpha\rangle = \eta_T \langle\beta| \phi^\dagger(t,\vec{x}) |\alpha\rangle$$

$$(T^\dagger \phi(-t,\vec{x}) T)^\dagger = \eta_T \phi^\dagger(t,\vec{x})$$

$$T^\dagger \phi(-t,\vec{x}) T = \eta_T^* \phi(t,\vec{x})$$ ⇐直接很难写出这个式子.

—— Requirement of Time-inverse operator

$$\int d^3p \cdot N_p \cdot (\hat{T} a_p \hat{T}^\dagger u_p^*(\vec{x},t) + \hat{T} b_p^\dagger T^{-1} u_p(\vec{x},t)) = \eta_T \int d^3p\, N_p \cdot (a_p u_p(\vec{x},-t) + b_p^\dagger u_p^*(\vec{x},-t))$$

$\{ u_p^*(\vec{x},t) = u_{-p}(x,-t)$

$$T a_p T^\dagger = \eta_T a_{-p} \qquad T b_p T^\dagger = \eta_T^* b_{-p}$$

$$T a_p^\dagger T^\dagger = \eta_T^* a_{-p}^\dagger \qquad T b_p^\dagger T^\dagger = \eta_T b_{-p}^\dagger$$

Charge Conjugation.

- Classical

$$\phi'(x) = \eta_c \phi^*(x) \qquad |\eta_c|^2 = 1$$

- Quantum ($C = C^\dagger = C^{-1}$) $C=C^{-1}$这个性质我不确定!

$$C^\dagger \phi(x) C = \eta_c \phi^\dagger(x), \qquad C^\dagger \phi^\dagger(x) C = \eta_c^* \phi(x)$$

$$C \phi(x) C^\dagger = \eta_c \phi^\dagger(x) \qquad C \phi^\dagger(x) C^\dagger = \eta_c^* \phi(x)$$

- Requirement of effect on generator/annihilator.

$$C a_p C^{-1} = \eta_c b_p \qquad C b_p C^{-1} = \eta_c^* a_p$$

$$C a_p^\dagger C^{-1} = \eta_c^* b_p^\dagger \qquad C b_p^\dagger C^{-1} = \eta_c a_p^\dagger$$

- Solution for $\boxed{\text{real value}}$ η_c.

$$C = \exp\left(-i\frac{\pi}{2}\int d^3p \cdot \left(b_p^\dagger a_p + a_p^\dagger b_p - \eta_c\left(a_p^\dagger a_p + b_p^\dagger b_p\right)\right)\right)$$

Dirac.

parity / space inversion

- Parity Trans for Dirac field.

$$\psi(x,t) \to \psi'(x') \qquad x' = (t, -\vec{x})$$

$$\psi'(x') = \gamma^0 \psi(t, -\vec{x})$$

$$\bar{\psi}'(x') = \psi'^{\dagger}(x')\gamma^0 = \psi^{\dagger}(t,-\vec{x})\gamma^{0\dagger}\gamma^0 = \bar{\psi}(t,-\vec{x})\gamma^0$$

- Quantum:

$$\langle \beta' | \psi(t,\vec{x}) | \alpha' \rangle = \langle \beta | \psi'(t,\vec{x}) | \alpha \rangle = \gamma^0 \langle \beta | \psi(t,-\vec{x}) | \alpha \rangle$$

$$P^{\dagger}\psi(t,\vec{x})P = \gamma^0\psi(t,-\vec{x}) \Rightarrow \text{Greiner 书中写为 } P\psi(t,\vec{x})P^{-1} = \gamma^0\psi(t,-\vec{x})$$

$$P^{\dagger}\bar{\psi}(t,\vec{x})P = \bar{\psi}(t,-\vec{x})\gamma^0 \qquad P\bar{\psi}(t,\vec{x})P^{-1} = \bar{\psi}(t,-\vec{x})\gamma^0$$

- Acting to creation/annihilation op $\quad \tilde{p} = (p^0, -\vec{p})$

$$\int d^3p \cdot N_p \cdot \sum_s \left(P b(p,s) P^{-1} u(p,s) e^{-ip\cdot x} + P d^{\dagger}(p,s) P^{-1} v(p,s) e^{ip\cdot x} \right)$$

$$= \int d^3p \, N_p \sum_s \left(b(p,s)\gamma^0 u(p,s) e^{-i\tilde{p}\cdot x} + d^{\dagger}(p,s)\gamma^0 v(p,s) e^{i\tilde{p}\cdot x} \right)$$

$$\gamma^0 u(p,s) = u(\tilde{p},s) \qquad \gamma^0 v(p,s) = -v(\tilde{p},s)$$

$$P b(p,s) P^{-1} = b(\tilde{p},s) \qquad P d(p,s) P^{-1} = -d(\tilde{p},s)$$

$$P b^{\dagger}(p,s) P^{-1} = b^{\dagger}(\tilde{p},s) \qquad P d^{\dagger}(p,s) P^{-1} = -d^{\dagger}(\tilde{p},s)$$

Particles and antiparticles have opposite intrinsic parity

- explicit expression for parity operator

$$P = \exp\left[i\frac{\pi}{2} \int d^3p \sum_s \left(b^{\dagger}(p,s) b(\tilde{p},s) + d^{\dagger}(p,s) d(\tilde{p},s) - b^{\dagger}(p,s) b(p,s) + d^{\dagger}(p,s) d(p,s) \right) \right]$$

- Transformation Law

— Momentum operator

$$P P^{\mu} P^{-1} = P_{\mu}$$

— Angular momentum Transforms as pseudo vector.

$$P \vec{J} P^{-1} = \vec{J}$$

Time Reversal.

$\gamma^5 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -A & B \\ -C & D \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -A & -B \\ C & D \end{pmatrix}$

- Classical,

$\psi'(-t, \vec{x}) = T\psi(\vec{x}, t) = T_0 K \psi(\vec{x}, t)$ (K : operator of complex conjugate)

$= T_0 \psi^*(\vec{x}, t)$ 它们都可视为定义.

T or T_0 satisfy

$T^{-1}\gamma^\mu T = \gamma_\mu$

$T_0^{-1}\gamma^\mu T_0 = \gamma_\mu^* = \gamma^{\mu T}$

$T_0 = i\gamma^1\gamma^3 = -i\gamma^5 C$ charge conjugation. $C = CK$

$T_0 = T_0^{-1} = T_0^\dagger = -T_0^* = -T_0^T$

—— check for T effect in free Dirac Equation

Dirac equation

$(i\gamma^0\partial_0 + i\gamma^k\partial_k - m)\psi(\vec{x}, t) = 0$

T from Left

$(T i T^{-1} T\gamma^0 T^{-1}\partial_0 + T i T^{-1} T\gamma^k T^{-1}\partial_k - m)\cdot T\psi(\vec{x}, t) = 0$

$T i T^{-1} = -i$

$T\gamma^0 T^{-1} = \gamma^0$

$T\gamma^k T^{-1} = \gamma_k$

$(-i\gamma^0\partial_0 - i\sum_k \gamma_k\partial_k - m)\cdot T\psi(\vec{x}, t) = 0$

↓ variable trans

$(i\gamma^0\partial_0 + i\sum_k \gamma^k\partial_k - m)\underbrace{T\psi(\vec{x}, -t)}_{\psi'(\vec{x}, t)} = 0$

- Quantum Anti-unitary

$|\alpha'\rangle = \hat{T}|\alpha\rangle$ $|\beta'\rangle = \hat{T}|\beta\rangle$ 又遇到了交换左右态的问题.

$\langle\alpha'|\psi(\vec{x}, t)|\beta'\rangle = \langle\beta|\psi'(\vec{x}, t)|\alpha\rangle = \langle\beta|T_0\psi^\dagger(\vec{x}, -t)|\alpha\rangle$

$\hat{T}$ is anti-unitary operator.

property 5° / T是 Anti-unitary op.

$\langle\hat{T}\alpha|\psi(t, \vec{x})|\hat{T}\beta\rangle = \langle\beta|(\hat{T}^\dagger\psi(t, \vec{x})\hat{T})^\dagger|\alpha\rangle$

$(\hat{T}^\dagger\psi(\vec{x}, t)\hat{T})^\dagger = T_0\psi^\dagger(\vec{x}, -t)$

$T_0^\dagger = T_0^{-1} = T_0$, $\hat{T}^{-1} = \hat{T}^\dagger$

$\hat{T}\psi(\vec{x}, t)\hat{T}^{-1} = T_0\psi(\vec{x}, -t)$

and $\hat{T}\bar{\psi}(\vec{x}, t)\hat{T}^{-1} = \bar{\psi}(\vec{x}, -t)T_0^T$

—— Characteristic property, spinors have to rotate twice to get reproduced.

$\hat{T}\hat{T}\psi(\vec{x}, t)\hat{T}^{-1}\hat{T}^{-1} = \hat{T}T_0\psi(\vec{x}, -t)\hat{T} = T_0^* T_0\psi(\vec{x}, t) = -\psi(\vec{x}, t)$

— Trans formation law for creation & annihilation operators.

$$\int d^3p\, N_p \cdot \sum_s \left(\hat{T}\, b(p,s)\, \hat{T}^{-1} u^*(p,s)\, e^{ip\cdot x} + \hat{T}\, d^\dagger(p,s)\, \hat{T}^{-1} v^*(p,s)\, e^{-ip\cdot x} \right)$$

$$= \int d^3p\, N_p \cdot \sum_s \left(b(p,s)\, T_0\, u(p,s)\, e^{-i\tilde{p}\cdot x} + d^\dagger(p,s)\, T_0\, v(p,s)\, e^{-i\tilde{p}\cdot x} \right)$$

$$\hat{p} = (p^0, -\vec{p})$$

— Trans Law for creation & annihilation op

$$\hat{T}\, b(p,s)\, \hat{T}^{-1} = i(-1)^{s-\frac{1}{2}}\, b(\hat{p},\tilde{s}) \qquad \hat{T}\, d(p,s)\, \hat{T}^{-1} = -i(-1)^{s-\frac{1}{2}}\, d(\tilde{p},\tilde{s})$$

$$\hat{T}\, b^\dagger(p,s)\, \hat{T}^{-1} = -i(-1)^{s-\frac{1}{2}}\, b^\dagger(\tilde{p},\tilde{s}) \qquad \hat{T}\, d^\dagger(p,s)\, \hat{T}^{-1} = i(-1)^{s-\frac{1}{2}}\, d^\dagger(\tilde{p},\tilde{s})$$

Charge Conjugation.

- Classical relativistic QM.

$$\psi'(x') = \psi_c(x) = C\cdot\gamma^0\psi^*(x) = C\,\bar{\psi}^T(x) \quad ; \text{同样的}\ \bar{\psi}'(x') = \psi'^\dagger(x')\gamma^0 = \psi^T(x)\gamma^0 C^T\gamma^0 = -\psi^T C^T$$

在 Dirac Eq 中 $\psi_c(x)$ 也 satisfy Dirac equation ; but with opposite charge!

C 满足性质 : $\begin{cases} C\gamma^\mu C^{-1} = -\gamma^{\mu T} \\ C^{-1} = C^\dagger = C^T = C \end{cases}$

C 的值 : $C = i\gamma^2\gamma^0$.

Note: $C\cdot C = i\gamma^2\gamma^0\cdot i\gamma^2\gamma^0 = -\gamma^2\gamma^0\gamma^2\gamma^0 = +\gamma^2\gamma^2\gamma^0\gamma^0 = (-1)(-1) = 1$.

- Quantum. Charge transformation 写为 $\hat{C}$

—— charge transformation 满足.

$$\hat{C}\,\psi(x)\,\hat{C}^{-1} = C\,\bar{\psi}^T(x)$$

还是那个问题, 为什么不是 $\hat{C}\,\psi(x)\cdot\hat{C}$?

$$\hat{C}\,\bar{\psi}(x)\,\hat{C}^{-1} = -\psi^T C^\dagger$$

—— 对第一式代入具体场 operator 模式展开.

$$\int d^3p\, N_p \sum_s \left(\hat{C}\, b(p,s)\,\hat{C}^{-1}\, u(p,s)\, e^{-ip\cdot x} + \hat{C}\, d^\dagger(p,s)\,\hat{C}^{-1}\, v(p,s)\, e^{ip\cdot x} \right)$$

$$= \int d^3p\, N_p \sum_s \left(b^\dagger(p,s)\, C\,\bar{u}^T(p,s)\, e^{+ip\cdot x} + d(p,s)\, C\,\bar{v}^T(p,s)\, e^{-ip\cdot x} \right)$$

spinor action under charge conjugation.

$$C\,\bar{u}^T(p,s) = v(p,s) \qquad C\,\bar{v}^T(p,s) = u(p,s)$$

$C\bar{u}^T(p,s)$ & $C\bar{v}^T(p,s)$ are eigenstate of spin-projection op

$$\Sigma(s) = \tfrac{1}{2}(1+\gamma^5 \cancel{s})$$

Q, S 的定义是什么.

$$\Sigma(s)\, C\,\bar{u}^T(p,s) = C\,\bar{u}^T(p,s) \qquad \Sigma(-s)\, C\,\bar{u}^T(p,s) = 0$$

same reasoning can be applied to $C\bar{v}^T(p,s)$

—— Transformation Law for creation & annihilation op.

$$\hat{C}\, b(p,s)\,\hat{C}^{-1} = d(p,s) \qquad \hat{C}\, d(p,s)\,\hat{C}^{-1} = b(p,s)$$

$$\hat{C}\, b^\dagger(p,s)\,\hat{C}^{-1} = d^\dagger(p,s) \qquad \hat{C}\, d^\dagger(p,s)\,\hat{C}^{-1} = b^\dagger(p,s).$$

—— Explicit construction of $\hat{C}$

$$\hat{C} = \exp\left[i\tfrac{\pi}{2}\cdot \int d^3p \sum_s \left(d^\dagger(p,s)\, b(p,s) + b^\dagger(p,s)\, d(p,s) - b^\dagger(p,s)\, b(p,s) - d^\dagger(p,s)\, d(p,s) \right) \right]$$

—— Energy / momentum op

$$\hat{C}\, P^\mu\, \hat{C} = P^\mu$$

Anti-symmetrized current op

$$j^\mu = \tfrac{1}{2}[\bar{\psi}, \gamma^\mu\psi]$$

$$\hat{C}\, j^\mu\, \hat{C}^{-1} = -j^\mu$$

- Note on

$$-i\eta_c \gamma^2 \psi^*(t,\vec{x}) = \eta_c (-i\gamma^2)\psi^*(t,\vec{x}) = \eta_c(-i\gamma^2)(\psi^\dagger)^T = \eta_c(-i\gamma^2)(\bar{\psi}\gamma^0)^T = \eta_c(-i)(\bar{\psi}\gamma^0\gamma^2)^T$$

$$C \cdot \psi C = -i\gamma^2\psi^* = -i(\bar{\psi}\gamma^0\gamma^2)^T$$

$$C\bar{\psi}C = C\psi^\dagger\gamma^0 C = (C\psi C)^\dagger\gamma^0 = (-i\gamma^2\psi^*)^\dagger\gamma^0 = (i)\ \psi^T\ (\gamma^2)^\dagger$$

$$= \boxed{-i\psi^T(\gamma^2)}$$

- QED 的相互作用项.

$$j^\mu = \frac{e}{2}[\bar{\Psi}(x), \gamma^\mu \psi(x)] \qquad H \sim j_\mu A^\mu$$

Equation of motion

$$\Box A^\mu = \frac{e}{2}[\bar{\Psi}(x), \gamma^\mu \psi(x)]$$

- Transformation of A^μ, 从 Equation of motion 出发.

$$\hat{P} A^\mu(\vec{x},t)\hat{P}^{-1} = A_\mu(-\vec{x},t)$$
$$\hat{C} A^\mu(\vec{x},t)\hat{C}^{-1} = -A^\mu(\vec{x},t)$$
$$\hat{T} A^\mu(\vec{x},t)\hat{T}^{-1} = A_\mu(\vec{x},-t)$$

- QED invarient under transformation.

$$\hat{U}\hat{H}U^{-1} = \hat{H}$$

- Parity

用 coulomb gauge. $\nabla\cdot A = 0$. (下面讨论也用这个 gauge)

$$\hat{A}(\vec{x},t) = \int d^3k\, N_k \cdot \sum_{\lambda=1}^{2} [\vec{\varepsilon}(\vec{k},\lambda)\, a_{k\lambda}\, e^{-ik\cdot x} + \varepsilon^*(\vec{k},\lambda)\cdot a^\dagger_{k\lambda}\, e^{ik\cdot x}]$$

parity effect.

$$P a_{k\lambda} P^{-1} = (-1)^\lambda \cdot a_{-k\lambda}$$

—— Space inversion change helicity.

spherical basis vectors.

$$\vec{\varepsilon}(\vec{k},\pm) = \frac{1}{\sqrt{2}}(\vec{\varepsilon}(\vec{k},1) \pm i\,\vec{\varepsilon}(\vec{k},2))$$
$$\hat{P} a_{k\sigma} \hat{P}^{-1} = -a_{-\vec{k},\sigma}$$

- Charge,

$$\hat{C} a_{k\lambda} \hat{C}^{-1} = -a_{k\lambda} \qquad \hat{C} a_{k\sigma} \hat{C}^{-1} = -a_{k\sigma}$$

—— Multi photon state is an eigenstate of charge conjugation op.

$$\hat{C}|n\gamma\rangle = \hat{C} a^\dagger_{k_1\lambda_1} \hat{C}^{-1} \cdots \hat{C} a^\dagger_{k_n\lambda_n} C^{-1}|0\rangle$$
$$= (-1)^n \cdot a^\dagger_{k_1\lambda_1} \cdots a^\dagger_{k_n\lambda_n}|0\rangle$$
$$= (-1)^n |n\gamma\rangle$$

—— Furry's theorem. QED does not allow transition between states of even & odd photons.

$$\langle n'\gamma|S|n\gamma\rangle = \langle n'\gamma|\hat{C}^{-1} S \hat{C}|n\gamma\rangle = (-1)^{n+n'}\langle n'\gamma|\hat{S}|n\gamma\rangle$$

Invariance of S matrix.

- Transformation law for free fields. ($\hat{u}^\dagger = \hat{u}^{-1}$, $\hat{S}^\dagger = \hat{S}^{-1}$)

$$\hat{u}\,\phi_{in/out}(x)\,\hat{u}^{-1} = \Lambda\,\phi_{in/out}(x') \Rightarrow \text{我认为应当写为 } \hat{u}^{-1}\,\phi_{in/out}(x')\,\hat{u} = \Lambda\phi(x)$$

S operator connects in & out fields.

$$\phi_{out}(x) = \hat{S}^{-1}\phi_{in}(x)\hat{S}.$$ Heisenberg Pic 下场演化.

代入:

$$\hat{u}^{-1}\phi_{out}(x')\hat{u} = \Lambda\phi_{out}(x)$$ 末态满足对称关系

$$= \Lambda\cdot\hat{S}^{-1}\phi_{in}(x)\hat{S}$$

$$\hat{u}^{-1}\phi_{out}(x')\hat{u} = \hat{u}^{-1}\cdot\hat{S}^{-1}\phi_{in}(x')\hat{S}\hat{u}$$ 初态满足对称关系

$$= \hat{u}^{-1}\hat{S}^{-1}\hat{u}\cdot\hat{u}^{-1}\phi_{in}(x')\hat{u}\;\hat{u}^{-1}\hat{S}\hat{u}$$

$$= (\hat{u}^{-1}\hat{S}\hat{u})^{-1}\Lambda\phi_{in}(x)(\hat{u}^{-1}\hat{S}\hat{u})$$

$$(\hat{u}^{-1}\hat{S}\hat{u})^{-1}\phi_{in}(x')(\hat{u}^{-1}\hat{S}\hat{u}) = \hat{S}^{-1}\phi_{in}(x)\hat{S}$$

$$\Downarrow$$

$$\hat{S} = \hat{u}^{-1}\hat{S}\hat{u}$$

$$\Downarrow$$

$$[\hat{S},\hat{u}] = 0$$

- S matrix vanish for states with different Symmetry.

$H(t\to-\infty)|\beta,in\rangle = E_\beta|\beta,in\rangle$

$H(t\to+\infty)\,S^\dagger|\beta,in\rangle = S^\dagger H(t\to-\infty)S\cdot S^\dagger|\beta,in\rangle = E_\beta\boxed{S^\dagger|\beta,in\rangle}$ (拓展的定义为: $S_{\beta\alpha} = \langle\beta;out|\alpha;in\rangle$)

$$S_{\beta\alpha} = \langle\beta;in|\hat{S}|\alpha;in\rangle = \langle\beta;in|\hat{u}^\dagger\hat{S}\hat{u}|\alpha;in\rangle = \langle\beta',in|\hat{S}|\alpha';in\rangle$$

$$= S_{\beta'\alpha'} = S_{U\beta,U\alpha}$$ $\hat{S} = \hat{u}^\dagger\hat{S}\hat{u}$

$$\hat{u}|\alpha,in\rangle = \lambda_\alpha|\alpha,in\rangle$$

$$\langle\beta;in|\hat{S}\hat{u}|\alpha;in\rangle = \lambda_\alpha S_{\beta\alpha}$$

$$= \langle\beta;in|\hat{u}\hat{S}|\alpha;in\rangle = \langle\beta;in|\hat{u}^\dagger\hat{S}|\alpha;in\rangle$$ $\hat{u}^\dagger = \hat{u}$

$$= \lambda_\beta S_{\beta\alpha}$$ $\hat{u}|\beta,in\rangle = \lambda_\beta|\beta,in\rangle$

$$\Downarrow$$

$$S_{\beta\alpha}(\lambda_\beta - \lambda_\alpha) = 0.$$

- Time Reverse.

—— 场 $\psi(x)$, classical Time-Reverse transformation. (只是为了回顾, $\psi(x)$ 不是 spin 场)

$$\psi'(x') = \Lambda\psi(x)$$ Λ might be anti-linear operator!

—— 场 $\psi(x)$, Quantum

主动观点; All 态矢量 $|\alpha\rangle \to \hat{T}|\alpha\rangle$ $\hat{T}$: Anti unitary op.

$$\langle\beta'|\psi(x')|\alpha'\rangle = \langle\alpha|\psi'(x')|\beta\rangle = \langle\alpha|\Lambda\psi(x)|\beta\rangle$$

$$= \Lambda\langle\alpha|\psi(x)|\beta\rangle$$

▫ 上述左侧,

Anti-unitary op. $\langle\beta|V|\alpha\rangle = \langle\alpha|V^\dagger|\beta\rangle$

$$\langle\beta'|\psi(x')|\alpha'\rangle = \langle\hat{T}\beta|\psi(x')|\hat{T}\alpha\rangle = \langle\psi^\dagger(x')\hat{T}\beta|\hat{T}\alpha\rangle = \langle\alpha|\hat{T}^\dagger\psi^\dagger(x')\hat{T}|\beta\rangle$$

In and out fields

- Asymptotic Region.
- In particulor existence of vacuum space $|0\rangle$, Normolized. $\langle 0|0\rangle = 1$
- Energy / angular momentum of vacuum space

$$\hat{P}_\mu |0\rangle = 0 \qquad \hat{M}_{\mu\nu}|0\rangle = |0\rangle$$

- Asymptotic field satisfy noninteracting field equation. / 什么是 in/out fields?

$$(\Box + m^2)\phi_{in}(x) = 0 \qquad (\Box + m^2)\phi_{out}(x) = 0.$$

← physical but not bare mass.

- $a_{p,out} = S^\dagger a_{p,in} S \qquad a^\dagger_{p,out} = S^\dagger a^\dagger_{p,in} S$

$$|P_1 \cdots P_n, in\rangle = a^\dagger_{P_1,in} \cdots a^\dagger_{P_n,in} |0\rangle$$
$$= S a^\dagger_{P_1,in} S^\dagger \cdots |0\rangle$$
$$= S|P_1 \cdots P_n, out\rangle$$

Stability & uniqueness of vacuum space $S^{-1}|0\rangle = S|0\rangle = |0\rangle$

$\Rightarrow$ $S_{fi} = \langle q_1 \cdots q_m; in | S | P_1 \cdots P_n, in\rangle$

$= \langle q_1 \cdots q_m, out | S | P_1 \cdots P_n; out\rangle$

- S commute with generator of symmetry Trans.

$$\langle q_1 \cdots q_m, in | \hat{Q}^\dagger \hat{S} Q | P_1 \cdots P_n, in\rangle = S_{fi}$$
$$\langle \beta | \hat{S} | \alpha\rangle = \langle \beta | \hat{Q}^\dagger \hat{S} Q | \alpha\rangle$$
$$\boxed{\hat{S} = \hat{Q}^\dagger \hat{S} Q}$$

← 从 invariance of S matrix 出发 更本质一点.

(p272)

- S op acts like unit op, if is restricted to subspace of single particle space

$$\hat{S}|P, in\rangle = |P, in\rangle \qquad |P; in\rangle = |P, out\rangle = |P\rangle$$

There is only one subspace of Single-particle states.

- single particle is necessarily free and doesn't experience an interaction.
- Always an unavoidable interaction with "cloud" of virtual field quanta, However / Taken into account $(\Box + m^2)\phi_{in/out}(x) = 0$ contains physical Mass.

- $(\Box + m^2)\phi(x) = j(x) \qquad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1; \quad j(x) = \dfrac{\partial \mathcal{L}_1}{\partial \phi(x)} + (m^2 - m_0^2)\phi^2(x)$

$$\phi(x) = \phi_{in}(x) - \int d^4x' \, \Delta_R(x - x') j(x') \qquad \text{Retarded propagator.}$$

Δ_R: $\Theta(x)\Delta(x)$

$$\phi(x) = \phi_{out}(x) - \int d^4x' \, \Delta_A(x - x') j(x') \qquad \text{Advanced propagator.}$$

Δ_A: $-\Theta(-x)\Delta(x)$

- $\lim_{x_0\to-\infty} \phi(x) = \sqrt{Z}\,\phi_{in}(x)$

$\lim_{x_0\to+\infty} \phi(x) = \sqrt{Z}\,\phi_{out}(x)$

$\hookrightarrow \langle 1|\phi(x)|0\rangle = \sqrt{Z}\,\langle 1|\phi_{in}(x)|0\rangle$

$Z\in[0,1)$. (Reduced to). | $\phi(x)$ can also create complicated many particle states.

$$i\delta^{(3)}(\vec{x}-\vec{y}) = \lim_{x_0\to-\infty} [\phi(x),\dot{\phi}(y)]\big|_{x_0=y_0} = Z\,[\phi_{in}(x),\dot{\phi}_{in}(y)]\big|_{x^0=y^0} = Z\cdot i\cdot\delta^{(3)}(x-y)$$

$\Uparrow$ Dilemma.

$\Downarrow$ Asymptotic relation interpreted using weak convergence.

$$\lim_{x_0\to-\infty} \langle b|\phi(x)|a\rangle = \sqrt{Z}\,\langle b|\phi_{in}(x)|a\rangle$$

$$\lim_{x_0\to+\infty} \langle b|\phi(x)|a\rangle = \sqrt{Z}\,\langle b|\phi_{out}(x)|a\rangle$$

—— project to spatially localized wave packet. /认为上面渐近条件不完善.

$$\phi^{a}(t) = i\int d^3x\, U_\alpha^*(\vec{x},t)\overleftrightarrow{\partial_0}\phi(\vec{x},t)$$

$$\phi^{\alpha}_{in/out}(t) = i\int d^3x\, U_\alpha^*(\vec{x},t)\overleftrightarrow{\partial_0}\phi_{in}(\vec{x},t)$$

Localized K-G solution. $(\Box+m^2)U_\alpha(\vec{x},t)=0$

▽ $\phi^{\alpha}_{in/out}$ 与 t 无关.

$$\partial_0\phi^{\alpha}_{in/out}(t) = i\int d^3x\,\partial_0\left(U_\alpha^*(\vec{x},t)\partial_0\phi_{in}(\vec{x},t) - \partial_0 U_\alpha^*(\vec{x},t)\,\phi_{in}(\vec{x},t)\right)$$

$$= i\int d^3x\cdot(U_\alpha^*\partial_0^2\phi_{in} - \partial_0^2 U_\alpha^*\,\phi_{in})$$

$$= i\int d^3x\cdot\left(U_\alpha^*\partial_0^2\phi_{in} - (\nabla^2-m^2)U_\alpha^*\,\phi_{in}\right) = i\int d^3x\cdot U_\alpha^*(\Box+m^2)\phi_{in} = 0$$

Integral by parts!

▽ 取 localized 满足渐近关系.

$$\lim_{x_0\to\mp\infty} \langle b|\phi^{a}(x_0)|a\rangle = \sqrt{Z}\langle b|\phi^{\alpha}_{in/out}|a\rangle$$

▽ 取 plane wave $U_p(x)$ / not localized wave op.

$$a_{\vec{p}}(x_0) = i\int d^3x\cdot U_{\vec{p}}^*(x)\cdot\overleftrightarrow{\partial_0}\phi(x)$$

$$\hat{a}_{\vec{p};in/out} = i\int d^3x\, U_{\vec{p}}^*(x)\overleftrightarrow{\partial_0}\phi_{in/out}(x)$$

$\Downarrow$

$$\lim_{x_0\to\mp\infty} \langle b|a_{\vec{p}}(x_0)|a\rangle = \sqrt{Z}\,\langle b|a_{\vec{p}}^{in/out}|a\rangle$$

- Yang – Feldman Equation derive.

$$\sqrt{Z}\,\langle b|a_{\vec{p},in}|a\rangle = \lim_{x^0\to-\infty} i\int d^3x'\,U_{\vec{p}}^*(x')\overleftrightarrow{\partial_0'}\langle b|\phi(x')|a\rangle$$

$$\int d^3x'\,F(x',-\infty) = \int d^3x'\,F(x',x^0) - \int d^3x'\int_{-\infty}^{x^0} dx^{0\prime}\partial_0' F(x',x^0)$$

$$\sqrt{Z}\,\langle b|a_{\vec{p};in}|a\rangle = i\int d^3x'\,U_{\vec{p}}^*(\vec{x}',x^0)\overleftrightarrow{\partial_0}\langle b|\phi(x')|a\rangle$$

$$-i\int d^3x' \int_{-\infty}^{x^0} dx^{0'} \cdot \partial_0' \left\{ u_{\vec{p}}^*(\vec{x}', x^{0'}) \cdot \overleftrightarrow{\partial_0} \langle b|\phi(\vec{x}', x^{0'})|a\rangle \right\}$$

$$= i\int d^3x' \, u_{\vec{p}}^*(\vec{x}', x^0) \overleftrightarrow{\partial_0} \langle b|\phi(x')|a\rangle$$
$$- i\int d^3x' \int_{-\infty}^{x^0} dx^{0'} \left\{ u_{\vec{p}}^*(\vec{x}', x^{0'}) \partial_0'^2 \langle b|\phi(\vec{x}', x^{0'})|a\rangle - \partial_0'^2 u_{\vec{p}}^*(\vec{x}', x^{0'}) \cdot \langle b|\phi(\vec{x}', x^{0'})|a\rangle \right\}$$

$$= i\int d^3x' \, u_{\vec{p}}^*(\vec{x}', x^0) \overleftrightarrow{\partial_0} \langle b|\phi(x')|a\rangle$$
$$- i\int d^3x' \int_{-\infty}^{x^0} dx^{0'} \left\{ u_{\vec{p}}^*(\vec{x}', x^{0'}) \partial_0'^2 \langle b|\phi(\vec{x}', x^{0'})|a\rangle - (\nabla'^2 - m^2) u_{\vec{p}}^*(\vec{x}', x^{0'}) \cdot \langle b|\phi(\vec{x}', x^{0'})|a\rangle \right\}$$

$$= i\int d^3x' \, u_{\vec{p}}^*(\vec{x}', x^0) \overleftrightarrow{\partial_0} \langle b|\phi(x')|a\rangle$$
$$- i\int d^3x' \int_{-\infty}^{x^0} dx^{0'} \left\{ u_{\vec{p}}^*(\vec{x}', x^{0'}) \partial_0'^2 \langle b|\phi(\vec{x}', x^{0'})|a\rangle - u_{\vec{p}}^*(\vec{x}', x^{0'}) (\nabla'^2 - m^2) \langle b|\phi(\vec{x}', x^{0'})|a\rangle \right\}$$

$$= i\int d^3x' \, u_{\vec{p}}^*(\vec{x}', x^0) \overleftrightarrow{\partial_0} \langle b|\phi(x')|a\rangle$$
$$- i\int d^3x' \int_{-\infty}^{x^0} dx^{0'} \left\{ u_{\vec{p}}^*(\vec{x}', x^{0'}) (\Box'^2 + m^2) \langle b|\phi(\vec{x}', x^{0'})|a\rangle \right\}$$

$$= \langle b|a_{\vec{p}}(x^0)|a\rangle - i\int d^3x' \int_{-\infty}^{x^0} dx^{0'} \cdot u_{\vec{p}}^*(\vec{x}', x^0) \langle b|j(x')|a\rangle$$

$$\sqrt{Z} \langle b|a_{\vec{p}; in}^\dagger|a\rangle = \langle b|a_{\vec{p}}^\dagger(x^0)|a\rangle + i\int d^3x' \int_{-\infty}^{x^0} dx^{0'} \cdot u_{\vec{p}}(\vec{x}', x^{0'}) \cdot \langle b|j(x')|a\rangle$$

$$\phi(x) = \int d^3p \cdot (a_p(x^0) u_{\vec{p}}(x) + a_p^\dagger(x^0) u_{\vec{p}}^*(x))$$

$$\sqrt{Z} \langle b|\phi_{in}(x)|a\rangle = \langle b|\phi(x)|a\rangle - i\int d^3x' \int_{-\infty}^{x^0} dx^{0'} \int d^3p \left(u_p(x) u_p^*(x') - u_p^*(x) u_p(x') \right) \cdot \langle b|j(x')|a\rangle$$

Pauli – Jordan function.

$$\int d^3p \cdot (u_{\vec{p}}(x) u_{\vec{p}}^*(x') - u_{\vec{p}}^*(x) u_p(x'))$$
$$= \int d^3p \frac{1}{2\omega_p (2\pi)^3} \left(e^{-ip\cdot(x-x')} - e^{ip\cdot(x-x')} \right)$$
$$= i\Delta(x - x')$$

$$\sqrt{Z} \langle b|\phi_{in}(x)|a\rangle = \langle b|\phi(x)|a\rangle + \int d^4x' \, \theta(x^0 - x'^0) \Delta(x - x') \cdot \langle b|j(x')|a\rangle$$

$$\langle b|\phi(x)|a\rangle = \sqrt{Z} \langle b|\phi_{in}(x)|a\rangle + \int d^4x' \cdot \Delta_R(x - x') \langle b|j(x')|a\rangle$$

Similarily

$$\langle b|\phi(x)|a\rangle = \sqrt{Z} \langle b|\phi_{out}(x)|a\rangle - \int d^4x' \cdot \Delta_A(x - x') \langle b|j(x')|a\rangle$$

Lehmann - Kallen Spectral representation

∘ Pauli - Jordan function

$$i\Delta'(x-y) \equiv \langle 0|[\phi(x),\phi(y)]|0\rangle$$
$$= \int_0^{+\infty} ds\ \rho(s)\,\Delta(x-y;s)$$

$$i\delta^{(3)}(\vec{x}-\vec{y}) = [\phi(x),\dot{\phi}(y)]_{x^0=y^0} = \partial_{y^0}\langle 0|[\phi(x),\phi(y)]|0\rangle$$
$$= i\,\partial_{y^0}\Delta'(x-y)|_{x^0=y^0}$$
$$= \int_0^{+\infty} ds\ \rho(s)\, i\,\partial_{y^0}\Delta(x-y;s)|_{x^0=y^0}$$
$$= \int_0^{+\infty} ds\ \rho(s)\ i\delta^{(3)}(\vec{x}-\vec{y})$$

$$\Rightarrow \boxed{\int ds\ \rho(s) = 1}$$

CP transformation of Weak Interaction term

parity transform: $\psi'(x') = \gamma^0 \psi(x)$ $x' = (x^0, -\vec{x})$

- C transformation:

$$C = i\gamma^2 K \sim \gamma^2 \quad P = \eta_P \gamma^0 \simeq \gamma^0 \qquad T = \eta_T \gamma^1\gamma^3 K \sim \gamma^1\gamma^3 K.$$

CP transform of field

$$\psi \to \gamma^2\gamma^0\psi^*$$

$$\bar\psi \to (\gamma^2\gamma^0\psi^*)^\dagger\gamma^0$$

$\sigma^1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\sigma^2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\sigma^3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\gamma^0 = \begin{pmatrix}0&I\\I&0\end{pmatrix}$ $\gamma^1 = \begin{pmatrix}0&\sigma^1\\-\sigma^1&0\end{pmatrix}$ $(\gamma^1)^T = (\gamma^1)^\dagger = -\gamma^1$ $(\gamma^2)^T = \gamma^2$ $(\gamma^2)^\dagger = -\gamma^2$

$\gamma^i = \begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}$ $\gamma^5 = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$ $(\gamma^3)^T = (\gamma^3)^\dagger = -\gamma^3$

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ $\{\gamma^\mu, \gamma^5\} = 0$

lagrangian:

$$\mathcal{L} \sim \bar\psi\gamma^\mu(1-\gamma^5)\psi$$

$P_L = \frac{1}{2}(1-\gamma^5) = \begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\begin{pmatrix}L\\R\end{pmatrix}$

Transform as

$$(\gamma^2\gamma^0\psi^*)^\dagger\gamma^0\gamma^\mu(1-\gamma^5)\gamma^2\gamma^0\psi^*$$
$$= \psi^T(\gamma^0)^\dagger(\gamma^2)^\dagger\gamma^0\gamma^\mu(1-\gamma^5)\gamma^2\gamma^0\psi^*$$
$$= -\psi^T\gamma^0\gamma^2\gamma^0\gamma^\mu(1-\gamma^5)\gamma^2\gamma^0\psi^*$$
$$= -(\psi^T\gamma^0\gamma^2\gamma^0\gamma^\mu(1-\gamma^5)\gamma^2\gamma^0\psi^*)^T$$
$$= -\psi^\dagger(\gamma^0)^T(\gamma^2)^T(1-\gamma^5)(\gamma^\mu)^T(\gamma^0)^T(\gamma^2)^T(\gamma^0)^T\psi$$

$$= -\psi^\dagger\gamma^0\gamma^2(1-\gamma^5)(\gamma^\mu)^T\gamma^0\gamma^2\gamma^0\psi$$

1° $\mu = 0$ $(\gamma^0)^T = \gamma^0$

$$-\psi^\dagger\gamma^0\gamma^2(1-\gamma^5)\gamma^0\gamma^0\gamma^2\gamma^0\psi$$
$$= -\psi^\dagger\gamma^0\gamma^2(1-\gamma^5)\gamma^2\gamma^0\psi$$
$$= -\psi^\dagger(1-\gamma^5)\gamma^0\gamma^2\gamma^2\gamma^0\psi$$
$$= \psi^\dagger(1-\gamma^5)\psi$$
$$= \psi^\dagger\gamma^0\gamma^0(1-\gamma^5)\psi$$
$$= \bar\psi\gamma^0(1-\gamma^5)\psi$$

2° $\mu = 1$ $(\gamma^1)^T = -\gamma^1$

$$+\psi^\dagger\gamma^0\gamma^2(1-\gamma^5)\gamma^1\gamma^0\gamma^2\gamma^0\psi$$
$$= \psi^\dagger(1-\gamma^5)\gamma^0\gamma^2\gamma^1\gamma^0\gamma^2\gamma^0\psi$$
$$= \psi^\dagger(1-\gamma^5)\gamma^1\gamma^0\gamma^2\gamma^0\gamma^2\gamma^0\psi$$
$$= -\psi^\dagger(1-\gamma^5)\gamma^1\gamma^0\gamma^2\gamma^2\gamma^0\gamma^0\psi$$
$$= \psi^\dagger(1-\gamma^5)\gamma^1\gamma^0\psi$$
$$= -\psi^\dagger\gamma^0\gamma^1(1-\gamma^5)\psi = -\bar\psi\gamma^1(1-\gamma^5)\psi$$

3° $\mu = 2$ $(\gamma^2)^T = \gamma^2$

$$-\psi^\dagger\gamma^0\gamma^2(1-\gamma^5)\gamma^2\gamma^0\gamma^2\gamma^0\psi$$
$$= -\psi^\dagger(1-\gamma^5)\gamma^0\gamma^2\gamma^2\gamma^0\gamma^2\gamma^0\psi$$
$$= \psi^\dagger(1-\gamma^5)\gamma^0\gamma^0\gamma^2\gamma^0\psi$$
$$= \psi^\dagger\gamma^2\gamma^0(1-\gamma^5)\psi$$
$$= -\psi^\dagger\gamma^0\gamma^2(1-\gamma^5)\psi$$
$$= -\bar\psi\gamma^2(1-\gamma^5)\psi$$

4° $\mu = 3$ $(\gamma^3)^T = -\gamma^3$

$$+\psi^\dagger\gamma^0\gamma^2(1-\gamma^5)\gamma^3\gamma^0\gamma^2\gamma^0\psi$$
$$= \psi^\dagger(1-\gamma^5)\gamma^0\gamma^2\gamma^3\gamma^0\gamma^2\gamma^0\psi$$
$$= \psi^\dagger(1-\gamma^5)\gamma^3\gamma^0\gamma^2\gamma^0\gamma^2\gamma^0\psi$$
$$= -\psi^\dagger(1-\gamma^5)\gamma^3\gamma^0\gamma^2\gamma^2\gamma^0\gamma^0\psi$$
$$= +\psi^\dagger\gamma^3\gamma^0(1-\gamma^5)\psi$$
$$= -\psi^\dagger\gamma^0\gamma^3(1-\gamma^5)\psi$$
$$= -\bar\psi\gamma^3(1-\gamma^5)\psi$$

$\bar\psi\gamma^\mu(1-\gamma^5)\psi$ Transform As $(j^0, -\vec{j})$ Under CP trans!

by The Way, Parity Transformation of This Term Acts as $P: \gamma^0$

$$\bar\psi\gamma^\mu(1-\gamma^5)\psi \to (\gamma^0\psi)^\dagger\gamma^0\gamma^\mu(1-\gamma^5)\gamma^0\psi$$
$$= \psi^\dagger(\gamma^0)^\dagger\gamma^0\gamma^\mu(1-\gamma^5)\gamma^0\psi$$
$$= \psi^\dagger\gamma^0\gamma^0\gamma^\mu\gamma^0(1+\gamma^5)\psi$$
$$= \psi^\dagger\gamma^\mu\gamma^0(1+\gamma^5)\psi$$
$$= \begin{cases}\psi^\dagger\gamma^0\gamma^\mu(1+\gamma^5)\psi & \mu = 0\\ -\psi^\dagger\gamma^0\gamma^i(1+\gamma^5)\psi & \mu = i \in \{1,2,3\}\end{cases}$$

(改变了手性, This Term is Not parity invariant)

by The Way:

T transformation of $\bar\psi\gamma^\mu(1-\gamma^5)\psi$

T transformation: $T=\gamma^1\gamma^3 K$

可以证明(证明方式类似)

$$j^\mu=\bar{\psi}\gamma^\mu(1-\gamma^5)\psi$$

transform under T as

$$j^\mu\to(-j^0,\vec{j})$$

当然, 在 CPT transformation 下.

$$j^\mu\to(-j^0,-\vec{j})$$

Notice, for Charge & parity transformation.

Field transform as.

$$\bar{\psi}'(x')\gamma^0(1-\gamma^5)\psi'(x')=\bar{\psi}(x)\gamma^0(1-\gamma^5)\psi(x)$$

$$\bar{\psi}'(x')\gamma^i(1-\gamma^5)\psi'(x')=-\bar{\psi}(x)\gamma^i(1-\gamma^5)\psi(x)$$

Hence,

$$\begin{aligned}\partial'_\mu\bar{\psi}'(x')\gamma^\mu(1-\gamma^5)\psi'(x')&=\partial_0\bar{\psi}'(x')\gamma^0(1-\gamma^5)\psi'(x')\\&\quad-\partial_i\bar{\psi}'(x')\gamma^i(1-\gamma^5)\psi'(x')\\&=\partial_\mu\bar{\psi}(x)\gamma^0(1-\gamma^5)\psi(x)\end{aligned}$$

Which means

$$\partial'_\mu\bar{\psi}'(x')\gamma^\mu(1-\gamma^5)\psi'(x')=\partial_\mu\bar{\psi}(x)\gamma^\mu(1-\gamma^5)\psi(x)$$

Charge conjugate for spinor

° Charge conjugation for creation / Annihilation operator.

$$\hat{C}\psi\hat{C} = \eta_c(-i)(\bar{\psi}\gamma^0\gamma^2)^T$$

场的 Mode Expansion.

$$\psi = \sum_{s=\pm 1/2}\int \tilde{dp}\,(b_s(p)u_s(p)e^{-ip\cdot x} + d_s^\dagger(p)v_s(p)e^{ip\cdot x})$$
$$= \sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left(b_s(p)u_s(p)e^{-ip\cdot x} + d_s^\dagger(p)v_s(p)e^{ip\cdot x}\right)$$
$$\bar{\psi} = \sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left(b_s^\dagger(p)\bar{u}_s(p)e^{-ip\cdot x} + d_s(p)\bar{v}_s(p)e^{ip\cdot x}\right)$$

$$\hat{C}\sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left(b_s(p)u_s(p)e^{-ip\cdot x} + d_s^\dagger(p)v_s(p)e^{ip\cdot x}\right)\hat{C}$$
$$=(\eta_c)(-i)\left(\sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left(b_s^\dagger(p)\bar{u}_s(p)e^{-ip\cdot x} + d_s(p)\bar{v}_s(p)e^{ip\cdot x}\right)\gamma^0\gamma^2\right)^T$$

$$=(-i)(\eta_c)\cdot(\gamma^2)^T(\gamma^0)^T\cdot\sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left(b_s^\dagger(p)\,\bar{u}_s^T(p)\,e^{-ip\cdot x} + d_s(p)\,\bar{v}_s^T(p)\,e^{ip\cdot x}\right)$$

$(\gamma^2)^T = \gamma^2$

$(\gamma^0)^T = \gamma^0$

$$=(-i)(\eta_c)\gamma^2\gamma^0\sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left(b_s^\dagger(p)\,\bar{u}_s^T(p)\,e^{-ip\cdot x} + d_s(p)\,\bar{v}_s^T(p)\,e^{ip\cdot x}\right)$$
$$=(-i)\eta_c\gamma^2\gamma^0\sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left(b_s^\dagger(p)(u_s^\dagger(p)\gamma^0)^T e^{-ip\cdot x} + d_s(p)(v_s^\dagger(p)\gamma^0)^T e^{ip\cdot x}\right)$$
$$=(-i)\eta_c\gamma^2\gamma^0\sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left(b_s^\dagger(p)(\gamma^0)^T u_s^*(p)e^{-ip\cdot x} + d_s(p)(\gamma^0)^T v_s^*(p)e^{ip\cdot x}\right)$$

$$=(-i)\eta_c\gamma^2\gamma^0\sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left(b_s^\dagger(p)\gamma^0 u_s^*(p)\,e^{-ip\cdot x} + d_s(p)\gamma^0\,v_s^*(p)e^{ip\cdot x}\right)$$
$$=(-i)\eta_c\gamma^2\sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left(b_s^\dagger(p)\,u_s^*(p)\,e^{-ip\cdot x} + d_s(p)\quad v_s^*(p)\,e^{ip\cdot x}\right)$$

—— Charge conjugation of Dirac spinor $\gamma^2 u_s^*(p)$

Notice: $u(p,s) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix}$ $\qquad \bar{u}_r(p)u_s(p) = 2m\delta_{rs}$

$\qquad u_r^\dagger(p)u_s(p) = 2E_p\delta_{rs}$

$v(p,s) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta^s\\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s\end{pmatrix}$ $\qquad \bar{v}_r(p)v_s(p) = -2m\,\delta_{rs}$

$\qquad v_r^\dagger(p)v_r(p) = 2E_p\delta_{rs}$

$\xi^1 \equiv \xi^\uparrow = \begin{pmatrix}\cos\theta/2\\ e^{i\varphi}\sin\theta/2\end{pmatrix}$ $\qquad \xi^2 \equiv \xi^\downarrow = \begin{pmatrix}-e^{-i\varphi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$ $\qquad \xi^s = \xi^s(\vec{p})$

$\eta^1 = \xi^{-1} = \xi^2 = \xi^\downarrow$ $\qquad \eta^2 = \xi^{-2} = -\xi^1 = -\xi^\uparrow$ $\qquad \eta^s = (-1)^{\frac{1}{2}s-\frac{1}{2}}\xi^{-s}$

$$\gamma^2 u^*(p,s) = \begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ -\sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix}^*$$

$$\gamma^2 u^*(p,+) = \begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\begin{pmatrix}\cos\theta/2\\ e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix}\\ +\sqrt{p\cdot\bar{\sigma}}\begin{pmatrix}\cos\theta/2\\ e^{i\varphi}\sin\theta/2\end{pmatrix}\end{pmatrix}^*$$

$$= \begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma^*}\begin{pmatrix}\cos\theta/2\\ e^{-i\varphi}\sin\frac{\theta}{2}\end{pmatrix}\\ \sqrt{p\cdot\bar{\sigma}^*}\begin{pmatrix}\cos\theta/2\\ e^{-i\varphi}\sin\frac{\theta}{2}\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}\sigma^2\sqrt{p\cdot\bar{\sigma}^*}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{-i\varphi}\sin\frac{\theta}{2}\end{pmatrix}\\ -\sigma^2\sqrt{p\cdot\sigma^*}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{-i\varphi}\sin\frac{\theta}{2}\end{pmatrix}\end{pmatrix}$$

Suppose $\sigma^2\sqrt{A} = \sqrt{\sigma^2 A\sigma^2}\,\sigma^2$ $\quad \{A\sqrt{B}A^{-1}\}^2 = ABA^{-1} \Rightarrow A\sqrt{B} = \sqrt{ABA^{-1}}\,A$

$$\sigma^2\sigma^\mu\sigma^2 = \sigma^2(\mathbb{I},\sigma^1,\sigma^2,\sigma^3)\sigma^2 = (\mathbb{I},-\sigma^1,\sigma^2,-\sigma^3)$$

$$\sigma^2\,\mathbb{I}\,\sigma^2 = \mathbb{I} \qquad \sigma^2\sigma^2\sigma^2 = \sigma^2$$

$$\sigma^2\sigma^1\sigma^2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}-i&0\\0&i\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}0&-1\\-1&0\end{pmatrix} = -\sigma^1$$

$$\sigma^2\sigma^3\sigma^2 = \begin{pmatrix}-i&0\\0&i\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}-i&0\\0&i\end{pmatrix} = \begin{pmatrix}-i&0\\0&-i\end{pmatrix}\begin{pmatrix}-i&0\\0&i\end{pmatrix} = \begin{pmatrix}-1&0\\0&1\end{pmatrix} = -\sigma^3$$

$$\sigma^2\,p\cdot\sigma^*\sigma^2 = \sigma^2\,p\cdot(\mathbb{I},\sigma^1,-\sigma^2,\sigma^3)\sigma^2 = p\cdot(\mathbb{I},-\sigma^1,-\sigma^2,-\sigma^3) = p\cdot\bar\sigma$$

$$\sigma^2\,p\cdot\bar\sigma^*\sigma^2 = \sigma^2\,p\cdot(\mathbb{I},-\sigma^1,\sigma^2,-\sigma^3)\sigma^2 = p\cdot(\mathbb{I},\sigma^1,\sigma^2,\sigma^3) = p\cdot\sigma$$

$$= \begin{pmatrix}\sqrt{p\cdot\sigma}\,\sigma^2\begin{pmatrix}\cos\theta/2\\ e^{-i\varphi}\sin\frac{\theta}{2}\end{pmatrix}\\ -\sqrt{p\cdot\bar\sigma}\,\sigma^2\begin{pmatrix}\cos\theta/2\\ e^{-i\varphi}\sin\theta/2\end{pmatrix}\end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix}0&i\\-i&0\end{pmatrix}$$

$$= i\begin{pmatrix}\sqrt{p\cdot\sigma}\begin{pmatrix}-e^{-i\varphi}\sin\theta/2\\ \cos\theta/2\end{pmatrix}\\ -\sqrt{p\cdot\bar\sigma}\begin{pmatrix}-e^{-i\varphi}\sin\theta/2\\ \cos\theta/2\end{pmatrix}\end{pmatrix}$$

$$= i\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^2\\ -\sqrt{p\cdot\bar\sigma}\,\xi^2\end{pmatrix} = i\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^{-1}\\ -\sqrt{p\cdot\bar\sigma}\,\xi^{-1}\end{pmatrix}$$

$$= i\begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta^1\\ -\sqrt{p\cdot\bar\sigma}\,\eta^1\end{pmatrix} = i\,v(p,+) \quad -(1)$$

From (1),

$$i\gamma^2 v(p,+) = \gamma^2\gamma^2 u^*(p,+) = -u^*(p,+)$$

$$+i\gamma^2 v^*(p,+) = -u(p,+) \Rightarrow \gamma^2 v^*(p,+) = +i\,u(p,+) \quad -(2)$$

$$\gamma^2 u^*(p,-) = \begin{pmatrix}0&\sigma^2\\-\sigma^2&0\end{pmatrix}\cdot\begin{pmatrix}\sqrt{p\cdot\sigma}\begin{pmatrix}-e^{-i\varphi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}\\ \sqrt{p\cdot\bar\sigma}\begin{pmatrix}-e^{-i\varphi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}\end{pmatrix}^*$$

$$= \begin{pmatrix}\sigma^2\sqrt{p\cdot\bar\sigma^*}\begin{pmatrix}-e^{+i\varphi}\sin\theta/2\\ \cos\theta/2\end{pmatrix}\\ -\sigma^2\sqrt{p\cdot\sigma^*}\begin{pmatrix}-e^{+i\varphi}\sin\theta/2\\ \cos\theta/2\end{pmatrix}\end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$= \begin{pmatrix}\sqrt{\sigma^2\,p\cdot\bar\sigma^*\sigma^2}\,\sigma^2\begin{pmatrix}-e^{+i\varphi}\sin\frac{\theta}{2}\\ \cos\theta/2\end{pmatrix}\\ -\sqrt{\sigma^2\,p\cdot\sigma^*\sigma^2}\,\sigma^2\begin{pmatrix}-e^{+i\varphi}\sin\frac{\theta}{2}\\ \cos\theta/2\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}\sqrt{p\cdot\sigma}\begin{pmatrix}-i\cos\theta/2\\ -i\,e^{+i\varphi}\sin\theta/2\end{pmatrix}\\ -\sqrt{p\cdot\bar\sigma}\begin{pmatrix}-i\cos\theta/2\\ -i\,e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix}\end{pmatrix}$$

$$= i\begin{pmatrix}\sqrt{p\cdot\sigma}\,(-\xi^1)\\ -\sqrt{p\cdot\bar\sigma}\,(-\xi^1)\end{pmatrix} = i\begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta^2\\ -\sqrt{p\cdot\bar\sigma}\,\eta^2\end{pmatrix} = i\,v(p,-)$$

$$\gamma^2 u^*(p,-) = i\,v(p,-)$$

$$\gamma^2\gamma^2 u^*(p,-) = i\gamma^2 v(p,-)$$

$$-u^*(p,-) = i\gamma^2 v(p,-)$$

$$-u(p,-) = i\gamma^2 v^*(p,-) \Rightarrow \gamma^2 v^*(p,-) = i\,u(p,-)$$

Conclusion (Indeed Changed charge of particle).

$$\gamma^2 u^*(p,\pm) = i\, v(p,\pm)$$

$$\gamma^2 v^*(p,\pm) = i\, u(p,\pm)$$

Used Wierd Equation

$$\sigma^2 \sqrt{A} = \sqrt{\sigma^2 A \sigma^2}\, \sigma^2$$

—— Calculate Transformation of creation/annihilation operator

$$\hat{C} \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b_s(p) u_s(p) e^{-ip\cdot x} + d_s^\dagger(p) v_s(p) e^{ip\cdot x} \right) \hat{C}$$

$$= (-i)\eta_c \gamma^2 \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b_s^\dagger(p) u_s^*(p) e^{-ipx} + d_s(p) v_s^*(p) e^{ipx} \right)$$

$$= \eta_c \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b_s^\dagger(p) v(p,s) e^{-ip\cdot x} + d_s(p) u(p,s) e^{ip\cdot x} \right)$$

$$\boxed{\begin{aligned} \hat{C} d_s^\dagger(p) \hat{C} &= \eta_c b_s^\dagger(p) \\ \hat{C} b_s(p) \hat{C} &= \eta_c d_s(p) \end{aligned}}$$

$\eta_c = 1$ 是合适的取法!

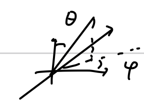
Time Reverse for Creation op & annihilation op.

∘ Time Reverse op.

$$\hat T \psi(t,\vec x)\hat T = \eta_T^* \gamma^1\gamma^3 \psi(-t,\vec x)$$

$$\hat T \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^3 2E_p}\left(b_s(p)u_s(p)e^{-ipx} + d_s^\dagger(p) v_s(p) e^{ipx}\right)\hat T$$

$$= \eta_T^*\gamma^1\gamma^3 \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^3 2E_p}\left(b_s(p)u_s(p)e^{iEt+i\vec p\cdot\vec x} + d_s^\dagger(p)v_s(p)e^{-iEt-i\vec p\cdot\vec x}\right)$$



$x \sim \sin\theta\cos\varphi$,
$y\sim\sin\theta\sin\varphi$

—— $\gamma^1\gamma^3$ Effects on dirac spinor u and v.

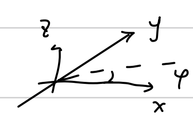
$$\gamma^1\gamma^3 = \begin{pmatrix}0&\sigma^1\\-\sigma^1&0\end{pmatrix}\begin{pmatrix}0&\sigma^3\\-\sigma^3&0\end{pmatrix} = \begin{pmatrix}-\sigma^1\sigma^3&0\\0&-\sigma^1\sigma^3\end{pmatrix} = \begin{pmatrix}-\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}&0\\0&-\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}-\begin{pmatrix}0&-1\\1&0\end{pmatrix}&0\\0&-\begin{pmatrix}0&-1\\1&0\end{pmatrix}\end{pmatrix}$$
$$= i\begin{pmatrix}\sigma^2&0\\0&\sigma^2\end{pmatrix}$$

$$\gamma^1\gamma^3 u(p,s) = i\begin{pmatrix}\sigma^2&0\\0&\sigma^2\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ +\sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}$$

$$= i\begin{pmatrix}\sigma^2\sqrt{p\cdot\sigma}\,\xi^s\\+\sigma^2\sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix} = i\begin{pmatrix}\sqrt{\sigma^2 p\cdot\sigma\sigma^2}\,\sigma^2\xi^s\\ -\sqrt{\sigma^2 p\cdot\bar\sigma\sigma^2}\,\sigma^2\xi^s\end{pmatrix}$$

$\{$ $\sigma^2 p\cdot\sigma = p'\cdot\bar\sigma$ $\quad p' = (p^0, p^1, -p^2, p^3)$

$$= i\begin{pmatrix}\sqrt{p'\cdot\bar\sigma}\,\sigma^2\xi^s\\ +\sqrt{p'\cdot\sigma}\,\sigma^2\xi^s\end{pmatrix}$$



$\vec p' = (p_x, -p_y, p_z)$

$$\left\{\begin{aligned}
&\xi^1 = \begin{pmatrix}\cos\theta/2\\ e^{i\varphi}\sin\theta/2\end{pmatrix} \qquad \xi^2 = \begin{pmatrix}-e^{-i\varphi}\sin\frac\theta2\\ \cos\frac\theta2\end{pmatrix}\\
&\sigma^2\xi^1 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\cos\frac\theta2\\e^{i\varphi}\sin\frac\theta2\end{pmatrix} = \begin{pmatrix}-ie^{i\varphi}\sin\frac\theta2\\ i\cos\frac\theta2\end{pmatrix} = i\begin{pmatrix}-e^{+i\varphi}\sin\frac\theta2\\ \cos\frac\theta2\end{pmatrix} = i\,\xi^2(p')\\
&\sigma^2\xi^2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}-e^{-i\varphi}\sin\frac\theta2\\ \cos\theta/2\end{pmatrix} = -i\begin{pmatrix}\cos\theta/2\\ e^{-i\varphi}\sin\theta/2\end{pmatrix} = -i\,\xi^1(p')\\
&\sigma^2\xi^1(\vec p) = i\,\xi^{-1}(\vec p') \qquad \sigma^2\xi^s(\vec p) = i(-1)^{\frac12 s-\frac12}\,\xi^{-s}(\vec p')\\
&\sigma^2\xi^{-1}(\vec p) = -i\,\xi^1(\vec p')
\end{aligned}\right.$$

$$= (-1)^{\frac12 s+\frac12}\begin{pmatrix}\sqrt{p'\cdot\bar\sigma}\,\xi^{-s}(p')\\ \sqrt{p'\cdot\sigma}\,\xi^{-s}(p')\end{pmatrix}$$

Notice. $\xi^s(\vec p) = -\xi^s(-\vec p)$

$p'' \equiv (p^0, -p^1, p^2, -p^3)$

$$= (-1)^{\frac12 s-\frac12}\, u(p'', -s)$$

$$\gamma^1\gamma^3 v(p,s) = i\begin{pmatrix}\sigma^2&0\\0&\sigma^2\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta^s\\ -\sqrt{p\cdot\bar\sigma}\,\eta^s\end{pmatrix}$$

$$= i\begin{pmatrix}\sqrt{\sigma^2p\cdot\sigma\sigma^2}\,\sigma^2\eta^s\\ -\sqrt{\sigma^2p\cdot\bar\sigma\sigma^2}\,\sigma^2\eta^s\end{pmatrix} = i\begin{pmatrix}\sqrt{p'\cdot\bar\sigma}\,\sigma^2\eta^s\\ -\sqrt{p'\cdot\sigma}\,\sigma^2\eta^s\end{pmatrix}$$

$$\left\{\begin{aligned}
&\eta^1 = \xi^{-1} \qquad \eta^2 = -\xi^1\\
&\eta^s = (-1)^{\frac12 s-\frac12}\,\xi^{-s} \qquad (s=\pm1)
\end{aligned}\right.$$

$$= i\begin{pmatrix}\sqrt{p'\cdot\bar\sigma}\,\sigma^2(-1)^{\frac12 s-\frac12}\xi^{-s}\\ -\sqrt{p'\cdot\sigma}\,\sigma^2(-1)^{\frac12 s-\frac12}\xi^{-s}\end{pmatrix}$$

$$= i\begin{pmatrix}\sqrt{p'\cdot\bar\sigma}\,(-1)^{\frac12 s-\frac12}\sigma^2\xi^{-s}\\ -\sqrt{p'\cdot\sigma}\,(-1)^{\frac12 s-\frac12}\sigma^2\xi^{-s}\end{pmatrix}$$

$$\left\{ \sigma^2 \xi^s(\vec{p}) = i(-1)^{\frac{1}{2}s-\frac{1}{2}} \xi^{-s}(\vec{p}') \right.$$

$$= i \begin{pmatrix} \sqrt{p'\cdot\bar{\sigma}} \; (-1)^{\frac{1}{2}s-\frac{1}{2}}(i)(-1)^{-\frac{1}{2}s-\frac{1}{2}} \xi^s(p') \\ -\sqrt{p'\cdot\sigma} \; (-1)^{\frac{1}{2}s-\frac{1}{2}}(i)(-1)^{-\frac{1}{2}s-\frac{1}{2}} \xi^s(p') \end{pmatrix}$$

$$= (i)(i)\cdot(-1)^{\frac{1}{2}s-\frac{1}{2}} (-1)^{-\frac{1}{2}s-\frac{1}{2}} \begin{pmatrix} \sqrt{p'\cdot\bar{\sigma}} \; \xi^s(p') \\ -\sqrt{p'\cdot\sigma} \; \xi^s(p') \end{pmatrix}$$

$$\left\{ \eta^s = (-1)^{\frac{1}{2}s-\frac{1}{2}} \xi^{-s} \right.$$ Notice. $p'' \equiv (p^0, -p^1, p^2, -p^3)$ $\quad \xi^s(\vec{p}'') = -\xi^s(\vec{p}')$

$p'\cdot\sigma = p''\cdot\bar{\sigma}$ $\quad p'\cdot\bar{\sigma} = p''\cdot\sigma$

$$= (-1)^{\frac{1}{2}s+\frac{1}{2}} \begin{pmatrix} \sqrt{p'\cdot\bar{\sigma}} \; \eta^{-s}(p') \\ -\sqrt{p'\cdot\sigma} \; \eta^{-s}(p') \end{pmatrix}$$

$$= (-1)^{\frac{1}{2}s-\frac{1}{2}} v(p'', -s)$$

Transformation of creation & Annihilation operator.

$$\hat{T} \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b_s(p) u_s(p) e^{-ip\cdot x} + d_s^\dagger(p) v_s(p) e^{ip\cdot x} \right) \hat{T}$$

$$= \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 \, 2E_p} \left(T b_s(p) T \cdot u_s^*(p) e^{ip\cdot x} + T d_s^\dagger(p) T \, v_s^*(p) e^{-ip\cdot x} \right)$$

$$\left\{ u_s(p) = \begin{bmatrix} \sqrt{p\cdot\sigma} \begin{vmatrix} \cos\theta/2 \\ e^{+i\varphi}\sin\frac{\theta}{2} \end{vmatrix} \\ \sqrt{p\cdot\bar{\sigma}} \begin{vmatrix} \cos\theta/2 \\ e^{+i\varphi}\sin\frac{\theta}{2} \end{vmatrix} \end{bmatrix} \right.$$

$u_s^*(p) = u_s(p')$ $\quad p' = (p^0, p^1, -p^2, p^3)$

$\vec{p} \to \vec{p}' \Leftrightarrow \varphi \to -\varphi$

$v^*(p,s) = v(p', s)$

$p' = (p^0, p^1, -p^2, p^3)$ $\quad p'' = (p^0, -p^1, p^2, -p^3)$

$$= \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 \, 2E_p} \left(T b_s(\vec{p}) T \, u_s(p') e^{ip\cdot x} + T d_s^\dagger(\vec{p}) T \, v_s(p') e^{-ip\cdot x} \right)$$

$$= \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 \, 2E_p} \left(T b(-\vec{p}, -s) T \, u(p'', -s) e^{iEt + i\vec{p}\cdot\vec{x}} + T d^\dagger(-\vec{p}, -s) \, v(\vec{p}'', -s) e^{-iEt - i\vec{p}\cdot\vec{x}} \right)$$

On The other hand, Shows that,

$$= \eta_T^* \gamma^1 \gamma^3 \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b_s(p) u_s(p) e^{iEt + i\vec{p}\cdot\vec{x}} + d_s^\dagger(p) v_s(p) e^{-iEt - i\vec{p}\cdot\vec{x}} \right)$$

$$= \eta_T^* (-1)^{\frac{1}{2}s-\frac{1}{2}} \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b_s(p) \cdot u(p'', -s) e^{iEt + i\vec{p}\cdot\vec{x}} + d_s^\dagger(p) v(p'', -s) e^{-iEt - i\vec{p}\cdot\vec{x}} \right)$$

$$\boxed{\begin{aligned} T b(\vec{p}, s) T &= (\eta_T)^* (-1)^{\frac{1}{2}s-\frac{1}{2}} b(-\vec{p}, -s) \\ T d^\dagger(\vec{p}, s) T &= (\eta_T)^* (-1)^{\frac{1}{2}s-\frac{1}{2}} d^\dagger(-\vec{p}, -s) \end{aligned}}$$

LSZ Reduction formalism.

标量场 LSZ Reduction.(是 non-charged scalar field)

回顾量子化过程

$$\hat{H} = \int d^3x \, \frac{1}{2} \left(\hat{\pi}(\vec{x},t)^2 + (\nabla\hat{\phi}(\vec{x},t))^2 + m^2 \hat{\phi}(\vec{x},t)^2 \right)$$

Equal Time Commutation Relation

$$[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{x}',t)] = i\delta^{(3)}(\vec{x}-\vec{x}')$$

$$[\hat{\phi}(\vec{x},t), \hat{\phi}(\vec{x}',t)] = [\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{x}',t)] = 0$$

量子化后的场及生成/湮灭算符的 commutation Relation.

正能量指 $-i\omega t$

$$\hat{\phi}(x,t) = \int d^3p \left(\hat{a}_p u_p(x,t) + \hat{a}_p^\dagger u_p^*(x,t) \right) = \hat{\phi}^{(+)}(x,t) + \hat{\phi}^{(-)}(x,t)$$

$$\hat{\pi}(x,t) = \frac{\partial}{\partial t}\hat{\phi}(x,t) = \int d^3p \, (-i\omega_p t)\left(\hat{a}_p u_p(x,t) - \hat{a}_p^\dagger u_p^*(x,t) \right) = \hat{\pi}^{(+)}(x,t) - \hat{\pi}^{(-)}(x,t)$$

定义 u 为:
$$u_p(x,t) = N_p e^{-ip\cdot x} = \frac{1}{\sqrt{2\omega_p(2\pi)^3}} e^{-i(\omega_p t - p\cdot x)}$$

$$(\partial_0^2 - \nabla^2 + m^2)\, u_p(x,t) = 0$$

生成/湮灭 op 对易关系
$$[\hat{a}_p, \hat{a}_{p'}^\dagger] = \delta^{(3)}(\vec{p}-\vec{p}') \qquad [\hat{a}_p, \hat{a}_{p'}] = [\hat{a}_p^\dagger, \hat{a}_{p'}^\dagger] = 0$$

$a_p, a_p^\dagger$ 从场函数中反解.
$$\hat{a}_p = i\int d^3x \, u_p^*(x,t) \overleftrightarrow{\partial_0} \hat{\phi}(x,t)$$
$$\hat{a}_p^\dagger = -i\int d^3x \, u_p(x,t) \overleftrightarrow{\partial_0} \hat{\phi}(x,t)$$

Heisenberg picture 有 interacting

$$H = H_0 + H_1 \qquad H_0 = \int d^3x \, \frac{1}{2} \left(\hat{\pi}(\vec{x},t)^2 + (\nabla\hat{\phi}(\vec{x},t))^2 + m^2 \hat{\phi}(\vec{x},t)^2 \right)$$

H_0: 自由场

H_1: interacting, 不含时.

场用生成/湮灭 op 展开为

$$\phi(\vec{x},t) = \int d^3p \left(\hat{a}_p(t) u_p(x,t) + \hat{a}_p^\dagger(t) u_p^*(x,t) \right)$$

$$u_{\vec{p}}(\vec{x},t) = \frac{1}{\sqrt{2\omega_p(2\pi)^3}} e^{-i(\omega_p t - p\cdot x)}$$

$$\int d^3x \, u_{\vec{p}}(\vec{x},t) \cdot u_{\vec{p}'}(\vec{x},t) = \frac{1}{2\omega_p} \cdot e^{-i2\omega_p t} \delta^{(3)}(\vec{p}+\vec{p}')$$

$$\int d^3x \, u_{\vec{p}}(\vec{x},t) u_{\vec{p}'}^*(\vec{x},t) = \frac{1}{2\omega_p} \cdot \delta^{(3)}(\vec{p}-\vec{p}')$$

—— 反解生成/湮灭算符.

$$a_p(t) = i\int d^3x \, u_p^*(\vec{x},t) \overleftrightarrow{\partial_0} \phi(\vec{x},t)$$

$$= i\iint d^3x \, d^3p' \Big\{ u_{\vec{p}}^*(\vec{x},t) \left(-i\omega_{p'} a_{\vec{p}'}(t) u_{\vec{p}'}(\vec{x},t) + i\omega_{p'} a_{\vec{p}'}^\dagger(t) u_{\vec{p}'}^*(\vec{x},t) \right)$$
$$- i\omega_{\vec{p}} \left(a_{\vec{p}'}(t) u_{\vec{p}'}(\vec{x},t) + a_{\vec{p}'}^\dagger(t) u_{\vec{p}'}^*(\vec{x},t) \right) \Big\}$$

$$= i\int d^3p' \Big\{ -i\omega_{p'} a_{\vec{p}'}(t) \frac{1}{2\omega_p} \delta^{(3)}(\vec{p}-\vec{p}') + i\omega_{p'} a_{p'}^\dagger(t) \frac{1}{2\omega_p} \delta^{(3)}(\vec{p}+\vec{p}') e^{i2\omega_p t}$$
$$- i\omega_p a_{p'}(t) \frac{1}{2\omega_p} \delta^{(3)}(\vec{p}-\vec{p}') - i\omega_p a_{p'}^\dagger(t) \frac{1}{2\omega_p} \delta^{(3)}(p+p') e^{i2\omega_p t} \Big\}$$

$$= a_p(t)$$

$$a_p^\dagger(t) = -i\int d^3x \, u_p(x,t) \overleftrightarrow{\partial_0} \phi(\vec{x},t)$$

—— 生成/湮灭 operator 初末态的差：

$$a_p(+\infty) - a_p(-\infty) = \Delta\left[i\int d^3x\, u_p^*(x,t)\overleftrightarrow{\partial_0}\hat{\phi}(x,t)\right]\Big|_{-\infty}^{+\infty}$$

$$= i\int d^3x\,\Delta\left[u_p^*(\vec{x},t)\partial_0\phi(\vec{x},t) - \phi(\vec{x},t)\partial_0 u_p^*(\vec{x},t)\right]\Big|_{-\infty}^{+\infty}$$

$$= i\int d^4x\,\partial_0\left[u_p^*(\vec{x},t)\partial_0\phi(\vec{x},t) - \phi(\vec{x},t)\partial_0 u_p^*(\vec{x},t)\right]$$

$$= i\int d^4x\left\{u_p^*(\vec{x},t)\partial_0^2\phi(\vec{x},t) - \phi(\vec{x},t)\partial_0^2 u_p^*(\vec{x},t)\right\}$$

$$= i\int d^4x\, u_p^*(\vec{x},t)\,(\omega_p^2 + \partial_0^2)\,\phi(\vec{x},t)$$

$$= i\int d^4x\, u_p^*(\vec{x},t)\left(\partial_0^2 + |\vec{k}|^2 + m^2\right)\phi(\vec{x},t)$$

Integrate by parts

$$\int dx\, f(x)\frac{d^2}{dx^2}g(x)$$

$$= \Delta\Big(\underbrace{f(x)\frac{dg(x)}{dx}}_{=0}\Big) - \int\frac{df(x)}{dx}\cdot\frac{dg(x)}{dx}\,dx$$

$$= -\Delta\Big(\underbrace{\frac{df(x)}{dx}\cdot g(x)}_{=0}\Big) + \int\frac{d^2f(x)}{dx^2}\,g(x)\cdot dx$$

$$= i\int d^4x\, u_p^*(\vec{x},t)\left(\partial_0^2 - \overleftarrow{\nabla}^2 + m^2\right)\phi(\vec{x},t)$$

$$= +i\int d^4x\, u_p^*(\vec{x},t)\,(\partial^2 + m^2)\,\phi(\vec{x},t)$$

$$a_p^\dagger(+\infty) - a_p^\dagger(-\infty) = -i\int d^4x\, u_p(\vec{x},t)(\partial^2+m^2)\phi(\vec{x},t)$$

—— 初/末态 的生成/湮灭算符 用来生成粒子.

{初/末态 a；$a^\dagger$ 与时间无关/ 有对易关系 $[a, a^\dagger] = \delta^{(3)}(\sim)$/ 可用于将 H 表示为 $a^\dagger a$ 形式!

可用于产生/湮灭粒子.

$$|i\rangle = a_{p_1}^\dagger(-\infty)\, a_{p_2}(-\infty)\cdots|0\rangle$$

不是自由场的真空态矢量. 是 Heisenberg Pic 下能量最小态。Gell-Mann-Low Theorem 中的 $|\Omega\rangle$

$$|f\rangle = a_{g_1}^\dagger(+\infty)\, a_{g_2}^\dagger(+\infty)\cdots|0\rangle$$

Heisenberg picture $\langle f|i\rangle$ 为 从 $|i\rangle \to |f\rangle$ 态的概率

—— 计算 $\langle f|i\rangle$ (Suppose 初末态的 $|0\rangle$ 是一样的)

$$\langle f|i\rangle = \langle 0|T a_{g_m}(+\infty)\cdots a_{g_1}(+\infty)\, a_{p_1}^\dagger(-\infty)\cdots a_{p_n}^\dagger(-\infty)|0\rangle$$

$$= \langle 0|T a_{g_m}(+\infty)\cdots a_{g_1}(+\infty)\left(a_{p_1}^\dagger(+\infty) + i\int d^4x\, u_{p_1}(\vec{x},t)(\partial^2+m^2)\phi(\vec{x},t)\right)\cdots a_{p_n}^\dagger(-\infty)|0\rangle$$

$$= \langle 0|T a_{g_m}(+\infty)\cdots a_{g_1}(+\infty)\, a_{p_1}^\dagger(+\infty)\cdots a_{p_n}^\dagger(-\infty)|0\rangle + i\int d^4x\, u_{p_1}(\vec{x},t)(\partial^2+m^2)$$
$$\langle 0|T a_{g_m}(+\infty)\cdots a_{g_1}(+\infty)\cdot\phi(\vec{x},t)\, a_{p_2}^\dagger(-\infty)\cdots a_{p_n}^\dagger(-\infty)|0\rangle.$$

⇓ 忽略第一项 (P_1 不参与相互作用)

$$= i\int d^4x_1\, u_{p_1}(x_1)(\partial_1^2 + m^2)\,\langle 0|T a_{g_m}(+\infty)\cdots a_{g_1}(+\infty)\,\phi(x_1)\cdot a_{p_2}^\dagger(-\infty)\cdots a_{p_n}^\dagger(-\infty)|0\rangle$$

$$= (i)^{n+m}\int d^4x_1\, u_{p_1}(x_1)(\partial_1^2+m^2)$$
$$\vdots$$
$$\int d^4x_n\, u_{p_n}(x_n)(\partial_n^2+m^2)$$
$$\int d^4y_1\, u_{g_1}^*(y_1)(\partial_{y_1}^2+m^2)$$
$$\vdots$$
$$\int d^4y_m\, u_{g_m}^*(y_m)(\partial_{y_m}^2+m^2)\cdot\langle 0|T\left[\phi(y_m)\phi(y_{m-1})\cdots\phi(y_1)\phi(x_1)\cdots\phi(x_n)\right]|0\rangle$$

∘ LSZ Reduction formula for spin-$\frac{1}{2}$ fields.

Free asymptotic in & out field introduced satisfy weak limit condition.

$\lim_{x_0\to-\infty} \langle b|\psi(x)|a\rangle = \sqrt{Z_2}\langle b|\psi_{in}|a\rangle$ $\lim_{x_0\to+\infty}\langle b|\psi(x)|a\rangle = \sqrt{Z_2}\langle b|\psi_{out}|a\rangle$ Z_2 is customary name of fermion renormalization const.

Field quantized. $\psi_{in}(x) = \int d^3p \cdot \sum_s \cdot (b_{in}(p,s)\, U_{ps}(x) + d^\dagger_{in}(p,s)\, V_{ps}(x))$

$$U_{ps}(x) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{w_p}}\; u(p,s)\, e^{-ip\cdot x}$$

$$V_{ps}(x) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{w_p}}\cdot v(p,s)\, e^{+ipx}$$

solution satisfy dirac equation

$$(i\not\partial - m)\, U_{ps}(x) = (\not p - m)\, U_{ps}(x) = 0$$

$$(i\not\partial - m)\, V_{ps}(x) = (-\not p - m)\, V_{ps}(x) = 0$$

Creation / annihilation op from asymptotic field.

$$b^\dagger_{in}(p,s) = \int d^3x \cdot \psi^\dagger_{in}(x)\, U_{ps}(x)$$

$$d^\dagger_{in}(p,s) = \int d^3x\; \psi_{out}(x)\, V^\dagger_{ps}(x)$$

$$S_{fi} = \langle f, out | i, in\rangle$$

$$= \langle g_1 r_1, \cdots, g_m r_m; \bar g_1 \bar r_1 \cdots \bar g_{\bar m} \bar r_{\bar m}; out | p_1 s_1 \cdots p_n s_n; \bar p_1 \bar s_1 \cdots \bar p_{\bar n} \bar s_{\bar n}; in\rangle$$

$$= \lim_{x_0\to-\infty} \int d^3x \cdot \langle f, out|\; b^\dagger_{in}(p_1, s_1)\, |i - (p_1 s_1); in\rangle$$

$$= \frac{1}{\sqrt{Z_2}} \cdot \lim_{x_0\to-\infty} \int d^3x \cdot \langle f; out|\, \psi^\dagger(x)\, |i - (p_1 s_1); in\rangle\, U_{p_1 s_1}(x)$$

$$= \frac{1}{\sqrt{Z_2}} \lim_{x_0\to+\infty} \int d^3x\, \langle f; out|\, \psi^\dagger(x)\, | i - (p_1 s_1); in\rangle\, U_{p_1 s_1}(x)$$

$$- \frac{1}{\sqrt{Z_2}}\left(\lim_{x_0\to+\infty} - \lim_{x_0\to-\infty}\right)\cdot \int d^3x \cdot \langle f; out|\, \psi^\dagger(x)\, |i - (p_1 s_1); in\rangle\, U_{p_1 s_1}(x)$$

$$= \langle f; out|\, b^\dagger_{out}(p_1, s_1)\, |i - (p_1 s_1); in\rangle$$

$$- \frac{1}{\sqrt{Z_2}} \int d^4x \cdot \partial_0 [\psi^\dagger(x)\, U_{p_1 s_1}(x)]$$

Vanish under assumption all initial & final momenta are different.

$$= -\frac{1}{\sqrt{Z_2}} \cdot \int d^4x \cdot \psi^\dagger(x) (\overleftarrow{\partial_0} + \overrightarrow{\partial_0})\, U_{p_1 s_1}(x)$$

Dirac Equation. Real Mass

$(i\not\partial - m) U_{ps}(x) = 0 \Rightarrow (i\gamma^\mu \partial_\mu - m)\, U_{ps}(x) = 0$

$i\gamma^0\partial_0 U_{ps}(x) + i\gamma^i \partial_i U_{ps}(x) - m\, U_{ps}(x) = 0$

Times γ^0 from left, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$i\partial_0 U_{ps}(x) + i\gamma^0\gamma^k\partial_k U_{ps}(x) - \gamma^0 m\, U_{ps}(x) = 0$

$\partial_0 U_{ps}(x) = i\gamma^0 (i\gamma^k\partial_k - m)\, U_{ps}(x)$

$$= -\frac{1}{\sqrt{Z_2}} \int d^4x\; \psi^\dagger(x) \left(\overleftarrow{\partial_0} + i\gamma^0 (i\gamma^k \partial_k - m)\right)\cdot U_{p_1 s_1}(x)$$

$$= -\frac{i}{\sqrt{Z_2}} \cdot \int d^4x\; \psi^\dagger(x)\, \gamma^0 \left(-i\gamma^0 \overleftarrow{\partial_0} - i\gamma^k \overleftarrow{\partial_k} - m\right) U_{p_1 s_1}(x)$$

$$= -\frac{i}{\sqrt{Z_2}} \int d^4x \; \bar{\psi}(x) \overleftarrow{(-i\not{\partial}_x - m)} \, u_{p_1 s_1}(x)$$

$$S_{fi} = \frac{-i}{\sqrt{Z_2}} \int d^4x \; \langle f, out | \bar{\psi}(x) | i - (p_1 s_1); in \rangle \overleftarrow{(-i\not{\partial}_x - m)} \, u_{p_1 s_1}(x)$$

Similarily, other first stage reduction.

Antiparticle, initial state

$$S_{fi} = \frac{i}{\sqrt{Z_2}} \int d^4x \; \bar{v}_{\bar{p}_1 \bar{s}_1} \overrightarrow{(i\not{\partial}_x - m)} \langle f, out | \psi(x) | i - (\bar{p}_1 \bar{s}_1); in \rangle$$

particle $(q_1 r_1)$, final state.

$$S_{fi} = -\frac{i}{\sqrt{Z_2}} \int d^4x \; \bar{u}_{q_1 r_1} \overrightarrow{(i\not{\partial}_x - m)} \langle f - (q_1 r_1); out | \psi(x) | i; in \rangle$$

Antiparticle $(\bar{q}_1 \bar{r}_1)$ final state

$$S_{fi} = \frac{i}{\sqrt{Z_2}} \int d^4x \; \langle f - (\bar{q}_1 \bar{r}_1); out | \bar{\psi}(x) | i, in \rangle \overleftarrow{(-i\not{\partial}_x - m)} \, v_{\bar{q}_1 \bar{r}_1}(x)$$

LSZ Reduction formalism for spin- $\frac{1}{2}$ particles.

$$\begin{aligned}
S_{fi} = &\left(\frac{-i}{\sqrt{Z_2}}\right)^{n+m} \left(\frac{i}{\sqrt{Z_2}}\right)^{\bar{n}+\bar{m}} \cdot \int d^4x_1 \cdots d^4x_n \, d^4\bar{x}_1 \cdots d^4\bar{x}_{\bar{n}} \, d^4y_1 \cdots d^4y_m \, d^4\bar{y}_1 \cdots d^4\bar{y}_{\bar{m}} \\
&\bar{u}_{q_m r_m}(y_m) \overrightarrow{(i\not{\partial}_{y_m} - m)} \cdots \bar{u}_{q_1 r_1}(y_1) \overrightarrow{(i\not{\partial}_{y_1} - m)} \\
&\bar{v}_{\bar{p}_{\bar{n}} \bar{s}_{\bar{n}}}(\bar{x}_{\bar{n}}) \overrightarrow{(i\not{\partial}_{\bar{x}_{\bar{n}}} - m)} \cdots \bar{v}_{\bar{p}_1 \bar{s}_1}(\bar{x}_1) \overrightarrow{(i\not{\partial}_{\bar{x}_1} - m)} \\
&\langle 0 | T \; \bar{\psi}(\bar{y}_{\bar{m}}) \cdots \bar{\psi}(\bar{y}_1) \psi(y_m) \cdots \psi(y_1) \bar{\psi}(x_1) \cdots \bar{\psi}(x_n) \psi(\bar{x}_1) \cdots \psi(\bar{x}_{\bar{n}}) | 0 \rangle \\
&\overleftarrow{(-i\not{\partial}_{x_1} - m)} u_{p_1 s_1}(x_1) \cdots \overleftarrow{(-i\not{\partial}_{x_n} - m)} u_{p_n s_n}(x_n) \\
&\overleftarrow{(-i\not{\partial}_{\bar{y}_1} - m)} v_{\bar{q}_1 \bar{r}_1}(\bar{y}_1) \cdots \overleftarrow{(-i\not{\partial}_{\bar{y}_{\bar{m}}} - m)} v_{\bar{q}_{\bar{m}} \bar{r}_{\bar{m}}}(\bar{y}_{\bar{m}})
\end{aligned}$$

Path Integral for fermion Fields

- Grassman variables. (Grassman 数)

Grassman variable 满足反对易性. ⟶ 数学中叫作 exterior algebra!

$$\{\Theta_i, \Theta_j\} = 0 \Longrightarrow \Theta_i^2 = 0$$

—— Any finite dimensional Grassmann Algebra can be expanded into finite sum.

coefficients are ordinary variable.

$$g(\Theta) = \underset{\dim=1}{g^{(0)}} + \sum_i \underset{\dim=n}{g_i^{(1)}} \Theta_i + \sum_{i_1<i_2} \underset{\dim=C_n^2}{g_{i_1 i_2}^{(2)}} \Theta_{i_1}\Theta_{i_2} + \cdots + g^{(n)}\Theta_1\Theta_2\cdots\Theta_n$$

The dimension of Grassmann Algebra :

$$D = \sum_{p=0}^{n}\binom{n}{p} = \sum_{p=0}^{n}\frac{n!}{p!(n-p)!} = \sum_{p=0}^{n}\frac{n!}{p!(n-p)!}\cdot(1)^p\cdot(1)^{n-p} = 2^n$$

—— Example, Grassmann algebra of order $n=2$ with generators Θ_1 & Θ_2 have basis $\{1, \Theta_1, \Theta_2, \Theta_1\Theta_2\}$

- Rules of differentiation

$$\frac{d}{d\Theta_i}1 = 0 \qquad \frac{d}{d\Theta_i}\Theta_j = \delta_{ij} \qquad \frac{d}{d\Theta_i}\Theta_1\Theta_2 = \delta_{i1}\Theta_2 - \delta_{i2}\Theta_1$$

$$\frac{d}{d\Theta_j}\Theta_{i_1}\cdots\Theta_{i_m} = \delta_{j i_1}\Theta_{i_2}\cdots\Theta_{i_m} + (-1)\cdot\delta_{j i_2}\Theta_{i_1}\Theta_{i_3}\cdots\Theta_{i_m} + \cdots + (-1)^{m-1}\delta_{j i_m}\Theta_{i_1}\cdots\Theta_{i_{m-1}}$$

$$\left\{\frac{d}{d\Theta_i}, \Theta_j\right\} = \delta_{ij} \qquad \left\{\frac{d}{d\Theta_i}, \frac{d}{d\Theta_j}\right\} = 0$$

- Rules of Integral.

$$\int d\Theta\, 1 = 0 \quad \Longleftarrow \quad \int_{-\infty}^{+\infty} dx \frac{d}{dx}f(x) = f(+\infty) - f(-\infty) = 0 \quad (\text{function drop to zero at infinity})$$

$$\int d\Theta\,\Theta = 1$$

⟵ Normalization condition amounts to defining a scale for Grassmann variable.

- Change variable in Integral.

—— one dimensional $g(\Theta) = a^{(0)} + a^{(1)}\Theta$

$$\Theta' = \underset{\text{grassmann}}{\eta} + \overset{\text{ordinary}}{a}\Theta$$

Θ' 相当于原变量, Θ 相当于变换后的变量.

$$\int d\Theta'\, g(\Theta') = \left(\int d\Theta'\cdot 1\right)a^{(0)} + a^{(1)}\left(\int d\Theta'\,\Theta'\right) = a^{(1)}$$

$$\int d\Theta\, g(\Theta'(\Theta)) = \int d\Theta\left(a^{(0)} + a^{(1)}\eta + a^{(1)}a\Theta\right)$$
$$= a\,a^{(1)}$$

$$\int d\Theta'\, g(\Theta') = \int d\Theta\cdot\left(\frac{d\Theta'}{d\Theta}\right)^{-1}\cdot g(\Theta'(\Theta))$$

—— Change of n-dimensional variable.

$$\left\{\ \Theta_i = \sum_j a_{ij}\Theta'_j + \eta_j\right.$$

ordinary integral:

$$\int d\Theta'_n \cdots d\Theta'_1 \; g(\Theta') \xrightarrow{\text{想用此积分表示}} \int d\Theta_n \cdots d\Theta_1 \cdot g(\Theta'(\Theta))$$

Consider Transformation of integral variable: ← i 是行 index，j 是列 index.

$$\int d\Theta'_n \cdots d\Theta'_1 \; g(\Theta') = \int d\Theta_n \cdots d\Theta_1 \cdot \left[\det\left(\frac{\partial \Theta'_i}{\partial \Theta_j}\right)\right]^{-1} \cdot g(\Theta'(\Theta))$$

proof of above equation (by induction)

Dimension $=1$

$$g(\Theta) = g^{(0)} + g^{(1)}\Theta \qquad \Theta' = a\Theta + b$$

$$\int d\Theta' \; g(\Theta') = g^{(1)}$$

$$\int d\Theta \; g(\Theta'(\Theta)) = \int d\Theta \left(g^{(0)} + g^{(1)}(a\Theta + b)\right) = a\, g^{(1)}$$

$$\int d\Theta' g(\Theta') = \int d\Theta \cdot \left(\frac{d\Theta'}{d\Theta}\right)^{-1} \cdot g(\Theta'(\Theta))$$

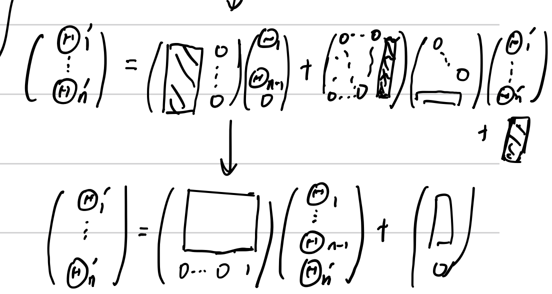
Dimension $=(n-1)\times(n-1)$

suppose that

$$\int d\Theta'_{n-1} \cdots d\Theta'_1 \; g(\Theta') = \int d\Theta_{n-1} \cdots d\Theta_1 \left(\det\left(\frac{\partial \Theta'_i}{\partial \Theta_j}\right)_{n-1\times n-1}\right)^{-1} g(\Theta'(\Theta))$$

$$\Theta'_i = a_{ij}\Theta_j + c_i \quad (n \text{ dimension})$$

$$\Theta_n = (\cdots\cdots)\begin{pmatrix}\Theta'_1\\ \vdots \\ \Theta'_n\end{pmatrix} + c_n$$



Dimension $= n\times n$ ⌐ 将它理解为最后被积分.

$$\int d\Theta'_n \int d\Theta'_{n-1} \cdots d\Theta'_1 \; g(\Theta') = \int d\Theta'_n \int d\Theta_{n-1} \cdots d\Theta_1 \cdot \left(\det\left(\frac{\partial \Theta'_i}{\partial \Theta_j}\right)_{n-1\times n-1}\right)^{-1} g(\Theta'(\Theta))$$

└ 具体来说，变量是 $(\Theta_1 \cdots \Theta_{n-1}, \Theta'_n)$

$$\Downarrow \; \Theta'_n = (\sim)\cdot\Theta_n + (\sim)\Theta_1 \cdots + (\sim)$$

$$= \int d\Theta_n \cdots d\Theta_1 \cdot \left(\frac{\partial \Theta'_n}{\partial \Theta_n}\bigg|_{\Theta_n}\right)^{-1} \cdot \left(\det\left(\frac{\partial \Theta'_i}{\partial \Theta_j}\right)_{n-1\times n-1}\right)^{-1} g(\Theta'(\Theta))$$

└ 将它理解为第一个被积分的. └ 自变量：$(\Theta_1 \cdots \Theta_{n-1}, \Theta_n)$

变量变化：$(\Theta_1 \cdots \Theta_{n-1}\, \Theta'_n) \longrightarrow (\Theta_1 \cdots \Theta_n)$

记为 Θ'_n 记为 Θ_n（每一项都可写为 $(\cdots \Theta'_n)$ 的变量）

上式中的 Jacobi determinant: $\Theta_n = \Theta_n(\Theta_1 \cdots \Theta'_n)$ $\Theta_1 = \Theta_1 \cdots \Theta_{n-1} = \Theta_{n-1}$

$$\frac{\partial \Theta'_i}{\partial \Theta_j}\bigg|_{\Theta'_n} = \frac{\partial \Theta'_i}{\partial \Theta_j}\bigg|_{\Theta_n} + \frac{\partial \Theta'_i}{\partial \Theta_n}\bigg|_{\Theta_n} \cdot \frac{\partial \Theta_n}{\partial \Theta_j}\bigg|_{\Theta'_n} \quad (i \text{ or } j \leq n-1)$$

$$\frac{\partial \Theta'_n}{\partial \Theta_j}\bigg|_{\Theta'_n} = 0 \qquad (j \leq n-1)$$

$$\frac{\partial \Theta'_n}{\partial \Theta_j}\bigg|_{\Theta_n} + \frac{\partial \Theta'_n}{\partial \Theta_n}\bigg|_{\Theta_n} \frac{\partial \Theta_n}{\partial \Theta_j}\bigg|_{\Theta'_n} = 0$$

$$\frac{\partial \Theta_n}{\partial \Theta_j}\bigg|_{\Theta'_n} = -\frac{\partial \Theta'_n}{\partial \Theta_j}\bigg|_{\Theta_n}\left(\frac{\partial \Theta'_n}{\partial \Theta_n}\bigg|_{\Theta_n}\right)^{-1}$$

$$\frac{\partial \Theta'_i}{\partial \Theta_j}\bigg|_{\Theta'_n} = \frac{\partial \Theta'_i}{\partial \Theta_j}\bigg|_{\Theta_n} - \frac{\partial \Theta'_i}{\partial \Theta_n}\bigg|_{\Theta_n} \frac{\partial \Theta'_n}{\partial \Theta_j}\bigg|_{\Theta_n}\left(\frac{\partial \Theta'_n}{\partial \Theta_n}\bigg|_{\Theta_n}\right)^{-1}$$

$$\det(a_{ij})_{n\times n} = a_{nn}\cdot\det\left(a_{ij} - a_{in}a_{nj}a_{nn}^{-1}\right)_{n-1\times n-1}$$

$$a_{ij} = \left(\frac{\partial\Theta_i'}{\partial\Theta_j}\right)\Big|_{\Theta_n}$$

$$\left(\frac{\partial\Theta_n'}{\partial\Theta_n}\right)\Big|_{\Theta_n}\cdot\det\left[\frac{\partial\Theta_i'}{\partial\Theta_j}\Big|_{\Theta_n} - \frac{\partial\Theta_i'}{\partial\Theta_n}\Big|_{\Theta_n}\frac{\partial\Theta_n'}{\partial\Theta_j}\Big|_{\Theta_n}\left(\frac{\partial\Theta_n'}{\partial\Theta_n}\Big|_{\Theta_n}\right)^{-1}\right]$$

$$= \det(a_{ij})_{n\times n}$$

$$= \int d\Theta_n\cdots d\Theta_1\cdot\det\left(\frac{\partial\Theta_i'}{\partial\Theta_j}\right)^{-1}\Big|_{\Theta_n,\,n\times n}\cdot g(\Theta'(\Theta))$$

和普通的variable变换相反！

- Grassmann变量的一个典型积分.

$$\int d\Theta_n\cdots d\Theta_1\quad \exp\left(-\tfrac{1}{2}\Theta^T A\Theta\right) = (\det A)^{1/2}(-1)^{n/2} \Rightarrow \int d\Theta_1\cdots d\Theta_n\exp\left(-\tfrac{1}{2}\Theta^T A\Theta\right) = \sqrt{\det A}$$

with A a real antisymmetric matrix of even dimensions.

用 real & antisymmetric matrix 构建 hermitean matrix：

$$(iA)^\dagger = -iA^\dagger = iA$$

Hermit matrix decomposition (Ad is diagonal − real matrix, with diagonal as iA's eigen value)

$$A_d = U\,iA\,U^\dagger$$

Ad 中的eigenvalue正负成对出现：

$$\det(iA-\lambda\mathbb{I}) = \det(iA-\lambda\mathbb{I})^T = \det(-iA-\lambda\mathbb{I}) = \det(iA+\lambda\mathbb{I}) = 0$$

Ad 的形式是：

$$A_d = \begin{pmatrix}\lambda_1 & & & & & \\ & -\lambda_1 & & & & \\ & & \lambda_3 & & & \\ & & & -\lambda_3 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n/2} \\ & & & & & & -\lambda_{n/2}\end{pmatrix} \Big\} n$$

构建 R matrix $\quad R_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1\\ 1 & i\end{pmatrix}\quad R_2^t = R_2^t$

$$R_2\begin{pmatrix}1&0\\0&-1\end{pmatrix}R_2^\dagger = -i\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$R = \begin{pmatrix}R_2 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_2\end{pmatrix}\Big\} n \quad 共 n/2 个 R_2\text{; } R \text{ is unitary}$$

$$\begin{bmatrix}R_2 & & \\ & \ddots & \\ & & R_2\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\lambda_1 & \\ & -\lambda_1\end{bmatrix} & \\ & \ddots\end{bmatrix}\begin{bmatrix}R_2^\dagger & & \\ & \ddots & \\ & & R_2^\dagger\end{bmatrix} = i\begin{bmatrix}\begin{bmatrix}0 & \lambda_1\\ -\lambda_1 & 0\end{bmatrix} & \\ & \ddots\end{bmatrix}$$

$$R\,A_d\,R^\dagger = iA'$$

$$R A_d R^\dagger = RU\,iA\,U^\dagger R^\dagger = i\,RUAU^\dagger R^\dagger = iA'$$

定义 $\begin{cases} A' = RUAU^\dagger R^\dagger \\ \Theta' = RU\Theta \longrightarrow \det\left(\frac{\partial\Theta_i'}{\partial\Theta_j}\right) = 1\end{cases}$

$$\int d\Theta_n'\cdots d\Theta_1'\ \exp\left(-\tfrac{1}{2}\Theta'A'\Theta'\right) = \int d\Theta_n'\cdots d\Theta_1'\cdot\exp\left(-\tfrac{1}{2}(\lambda_1\Theta_1'\Theta_2' - \lambda_1\Theta_2'\Theta_1'\cdots)\right)$$