$$= \int d\Theta_n' \cdots d\Theta_1' \exp(-\lambda_1 \Theta_1' \Theta_2' - \lambda_3 \Theta_3' \Theta_4' \cdots - \lambda_{n-1} \Theta_{n-1}' \Theta_n')$$

$$= \frac{1}{(n/2)!} \cdot \int d\Theta_n' \cdots d\Theta_1' (-1)^{n/2} \cdot (\lambda_1 \Theta_1' \Theta_2' \cdots + \lambda_{n-1} \Theta_{n-1}' \Theta_n')^{n/2}$$

$$= \int d\Theta_n' \cdots d\Theta_1' (-1)^{n/2} (\lambda_1 \cdots \lambda_{n-1}) \cdot \Theta_1' \cdots \Theta_n'$$

$$= (-1)^{n/2} \cdot \sqrt{\det A}$$

$\det(A) = \det(U^{-1}) \det(i A_d) \det(U) = i^n \det(A_d)$

$= (\lambda_1)(-\lambda_1) \cdots (-1)^{n/2}$

$= (\lambda_1 \cdots \lambda_{n-1})^2$

- Grassmann的常见积分 2：

$$\int d\Theta_1 \cdots d\Theta_n \exp(-\tfrac{1}{2} \Theta^T A \Theta + \rho^T \Theta) = (\det A)^{1/2} \exp(-\tfrac{1}{2} \rho^T A^{-1} \rho)$$

证明方式：定义 $\Theta' = \Theta + A^{-1}\rho$

- Complex Grassmann variable.

$$(\Theta_i)^* = \Theta_i^*$$

$$(\Theta_{i1} \cdots \Theta_{im})^* = \Theta_{im}^* \cdots \Theta_{i1}^*$$

$$(\Theta_i^*)^* = \Theta_i$$

$$(\lambda \Theta_i)^* = \lambda^* \Theta_i^*$$

在 Integration & differentiation 中，Θ_i and Θ_i^* are treated as independent variable.

—— complex grassmann variable integration.

$$\int d\Theta_1^* \cdots d\Theta_n^* d\Theta_1 \cdots d\Theta_n \exp(-\Theta^\dagger A \Theta) = \det A$$

grassmann variable.

$$\int d\Theta_1^* \cdots d\Theta_n^* d\Theta_1 \cdots d\Theta_n \exp(-\Theta^\dagger A \Theta + \Theta^\dagger \rho + \rho^\dagger \Theta) = \det A \cdot \exp(-\rho^\dagger A^{-1} \rho)$$

Easily proved for anti-hermitean matrix $A^\dagger = -A$ but validity is more general.

- Green's Function or n-point function.

$$G^{(2n)}(y_1,\cdots y_n;x_1\cdots x_n)=\langle 0|T(\psi(y_n)\cdots\psi(y_1)\bar\psi(x_1)\cdots\bar\psi(x_n))|0\rangle$$

- Generating function

$$W_0[\eta,\bar\eta]=\mathcal{N}\int\mathcal{D}\bar\psi\int\mathcal{D}\psi\,\exp\Big[\frac{i}{\hbar}\int d^4x\,\big(\bar\psi(x)(i\hbar\gamma^\mu\partial_\mu-m)\psi(x)+\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x)\big)\Big]$$

↑ Grassmann function (all are)

$$G^{(2n)}(y_1\cdots y_n;x_1\cdots x_n)=\left(\frac{\hbar}{i}\right)^{2n}\frac{\delta^{2n}W_0[\eta,\bar\eta]}{\delta\eta(x_n)\cdots\delta\eta(x_1)\delta\bar\eta(y_1)\cdots\delta\bar\eta(y_n)}$$

- Generating function for vacuum field.

Using integral formula:

$$\int d\Theta_1^*\cdots d\Theta_n^*\,d\Theta_1\cdots d\Theta_n\,\exp(-\Theta^\dagger A\Theta+\Theta^\dagger\rho+\rho^\dagger\Theta)=\det A\cdot\exp(-\rho^\dagger A^{-1}\rho)$$

With

$$\rho(x)=\frac{i}{\hbar}\eta(x)\qquad\bar\rho(x)=\frac{i}{\hbar}\bar\eta(x)$$

$$A(x',x)=-\frac{i}{\hbar}(-i\hbar\gamma\cdot\partial-m)\,\delta^{(4)}(x'-x)$$ ← 偏微分作用在 δ func 上!

$$W_0[\eta,\bar\eta]=\mathcal{N}\det A\,\exp\Big(-\frac{1}{\hbar^2}\int d^4x'\,d^4x\,\bar\eta(x')A^{-1}(x',x)\,\eta(x)\Big)$$

$$A(x',x)=-\frac{i}{\hbar}(-i\hbar\gamma\cdot\partial-m)\int\frac{d^4p}{(2\pi\hbar)^4}\exp(-ip\cdot(x'-x)/\hbar)$$

$$=-\frac{i}{\hbar}\int\frac{d^4p}{(2\pi\hbar)^4}\exp\Big(-\frac{i}{\hbar}p\cdot(x'-x)\Big)(\gamma\cdot p-m)$$

$$A^{-1}(x',x)=-\frac{i}{\hbar}\int\frac{d^4p}{(2\pi\hbar)^4}\exp\Big(-\frac{i}{\hbar}p\cdot(x'-x)\Big)\frac{1}{\gamma\cdot p-m}$$

$$=iS_F(x'-x)$$

$$S_F(x'-x)=(i\hbar\gamma\cdot\partial_{x'}+m)\,\Delta_F(x'-x)$$

$$W_0[\eta,\bar\eta]=\exp\Big[-\frac{i}{\hbar}\int d^4x'\,d^4x\,\bar\eta(x')S_F(x'-x)\eta(x)\Big]$$

$$=\exp\Big(-\frac{i}{\hbar}(\bar\eta,S_F\eta)\Big)\qquad\text{(简写)}$$

Normalization condition $W_0[\eta,\bar\eta]=1$

- Two-point function

$$G_0^{(2)}(y;x)=\left(\frac{\hbar}{i}\right)^2\frac{\delta^2W_0[\eta,\bar\eta]}{\delta\eta(x)\delta\bar\eta(y)}=\left(\frac{\hbar}{i}\right)^2\cdot\Big(-\frac{i}{\hbar}\Big)\cdot\frac{\delta}{\delta\eta(x)}\cdot\int d^4x'\,S_F(y-x')\eta(x')\exp\Big(-\frac{i}{\hbar}(\bar\eta,S_F\eta)\Big)\Big|_{\eta=0}$$

$$=i\hbar S_F(y-x)$$

表示为：

$$y\longleftarrow x$$

- Four-point function $G_0^{(4)}(y_1,y_2;x_1,x_2)=\begin{array}{l}y_1\longleftarrow x_1\\ y_2\longleftarrow x_2\end{array}-\begin{array}{l}y_1\quad x_1\\ y_2\quad x_2\end{array}$

Yukawa potential.

- Generating function

$$W[\eta, \bar{\eta}, J] = \mathcal{N} \exp\left(\int d^4x \ \frac{i}{\hbar} \mathcal{L}_{int}\left(\frac{\hbar}{i}\frac{\delta}{\delta\eta(x)}, \frac{\hbar}{i}\frac{\delta}{\delta\bar{\eta}(x)}, \frac{\hbar}{i}\frac{\delta}{\delta J(x)}\right)\right) W_0[\eta, \bar{\eta}, J]$$

$$= \mathcal{N} \int \mathcal{D}\phi \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left[\int d^4x \frac{i}{\hbar}\left(\mathcal{L}_0(\phi) + \mathcal{L}_0(\bar{\psi}, \psi) + J\phi + \bar{\psi}\eta + \bar{\eta}\psi + \mathcal{L}_{int}(\bar{\psi}, \psi, \phi)\right)\right]$$

$$\mathcal{L}_{int} = g\bar{\psi}(x)\psi(x)\phi(x)$$

Normalize $W[\eta, \bar{\eta}, J]\big|_{\eta, \bar{\eta}, J=0} = 1$

Explictly:

$$W_0[\eta, \bar{\eta}, J] = \exp\left[-\frac{i}{\hbar}\int d^4x d^4y \ \bar{\eta}(x) S_F(x-y) \eta(y)\right]$$

$$\times \exp\left[-\frac{i}{2\hbar} \int d^4x d^4y \ J(x) \Delta_F(x-y) J(y)\right]$$

Lorentz Transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} \cdot x^{\nu} + a^{\nu}$$

$$|\Psi'\rangle = U(\Lambda, a) \cdot |\Psi\rangle$$

$(\Lambda_1, a_1) \cdot (\Lambda_2, a_2)$

$\Downarrow$

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)$$

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a)$$

$$U^{-1}(\Lambda, a) = \Lambda(\Lambda^{-1}, -\Lambda^{-1} a) \qquad (\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$$

$$U(1+w, \varepsilon) = 1 - \frac{i}{2} J^{\mu\nu} w_{\mu\nu} - i P^{\mu} \varepsilon_{\mu}$$

$$U^{-1}(\Lambda, a) \cdot U(1+w, \varepsilon) \cdot U(\Lambda, a) = U(\Lambda^{-1}, -\Lambda^{-1} a)\, U(1+w, \varepsilon) \cdot U(\Lambda, a)$$

展开 $U(1+w, \varepsilon)$ 　　　　再乘，再对 w, ε 小量展开.

$$U^{-1}(\Lambda, a) \cdot J^{\mu\nu} \cdot U(\Lambda, a) = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} \cdot J^{\rho\sigma} + \Lambda^{\mu}{}_{\rho} \cdot a^{\nu} \cdot P^{\rho} - \Lambda^{\nu}{}_{\rho} \cdot a^{\mu} \cdot P^{\rho}$$

$$U^{-1}(\Lambda, a)\, P^{\mu}\, U(\Lambda, a) = \Lambda^{\mu}{}_{\nu} P^{\nu}$$

Lorentz Algebra

$$U^{-1}(\Lambda, a)\, J^{\mu\nu}\, U(\Lambda, a) = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\rho} J^{\rho\sigma} + \Lambda^{\mu}{}_{\rho} a^{\nu} P^{\rho} - \Lambda^{\nu}{}_{\rho} \cdot a^{\mu} P^{\rho}.$$

$\downarrow$

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} \cdot J^{\mu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma).$$

$$[J^{\mu\nu}, P^{\rho}] = i(g^{\nu\rho} \cdot P^{\mu} - g^{\mu\rho} P^{\nu})$$

$$J^{i} = \frac{1}{2} \varepsilon^{ijk} \cdot J^{jk}. \qquad K^{i} = J^{0i}$$

$$\theta^{i} = -\frac{1}{2} \varepsilon^{ijk} \cdot w_{jk} \qquad \xi^{i} = -w_{0i}$$

$$\vec{J} = (J^{23}, J^{31}, J^{12}). \qquad \vec{K} = (J^{01}, J^{02}, J^{03}).$$

$$\theta = (-w \sim \;) \qquad \xi = (\sim)$$

$$U(1+w) = 1 + i\vec{\theta} \cdot \vec{J} + i\vec{\xi} \cdot \vec{K}.$$

$$[J^{i}, J^{j}] = i\varepsilon^{ijk}. J^{k}.$$

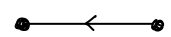
$$[J^{k}, K^{j}] = i\varepsilon^{ijk}. K^{k}.$$

$$[K^{i}, K^{j}] = -i\varepsilon^{ijk}. J^{k}$$

$\mathbb{C}$ SO(3)

- $\int d^4x\, d^4y\; i\bar\eta(x) \frac{1}{i} S(x-y) \cdot i\eta(y)$

points toward the bolb.



points away from the blob.

$x_2 \longrightarrow x_1 \;=\; \frac{1}{i} S(x_1 - x_2).$

- Generating Function

$$\mathcal{L}_1 = g \cdot \varphi \bar\Psi \Psi$$

$$Z[\bar\eta, \eta, J] \propto \exp\left[ig \int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(i \frac{\delta}{\delta \eta_\alpha(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta \bar\eta_\alpha(x)}\right)\right] \cdot Z_0[\bar\eta, \eta, J]$$

$$Z_0[\bar\eta, \eta, J] = \exp\left(i \int d^4x\, d^4y\; \bar\eta(x) S(x-y) \eta(y)\right)\left(\frac{i}{2} \cdot \int d^4x\, d^4y\; J(x) \Delta(x-y) J(y)\right)$$

$$Z_0[\bar\eta, \eta, J] = \exp[\underline{i W(\bar\eta, \eta, J)}]$$

Connected diagram / $iW[\bar\eta, \eta, J]$)

- Process needs to consider.

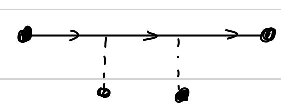
$e^-\varphi \to e^-\varphi$, $e^+\varphi \to e^+\varphi$ $\longrightarrow$ $\langle 0 | T\, \Psi \bar\Psi\, \varphi\varphi | 0\rangle_c$

$e^+e^- \to e^+e^-$, $e^-e^- \to e^+e^+$ $\longrightarrow$ $\langle 0 | T\, \Psi\bar\Psi\Psi\bar\Psi | 0\rangle_c$

- $\langle 0 | T\, \Psi\bar\Psi \varphi\varphi | 0 \rangle_c$.

$iW[\eta, \bar\eta, J]$ contributes to generating function of connected diagram.

$iW[\eta, \bar\eta, J] =$

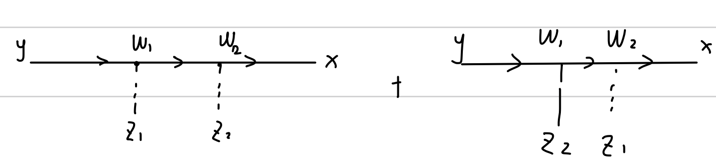


$\longleftarrow$ Symmetry factor.

忌是另为 $\Psi \Psi$ 形.

Contributes to $\langle 0 | T\, \Psi_\alpha(x) \bar\Psi_\beta(y) \varphi(z_1) \varphi(z_2) | 0 \rangle_c$

$$= \frac{1}{i}\frac{\delta}{\delta \bar\eta_\alpha(x)} \cdot i \frac{\delta}{\delta \eta_\beta(y)}\; \frac{1}{i}\frac{\delta}{\delta J(z_1)}\; \frac{1}{i}\frac{\delta}{\delta J(z_2)} \cdot \left(iW[\eta, \bar\eta, J]\right)$$

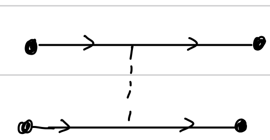


$$= \left(\frac{1}{i}\right)^5 \cdot (ig)^2 \cdot \int d^4w_1\, d^4w_2 \cdot [S(x-w_2) S(w_2-w_1) S(w_1-y)]_{\alpha\beta}\; \Delta(z_1 - w_1) \cdot \Delta(z_2 - w_2)$$

$$+ (z_1 \leftrightarrow z_2) + O(g^4).$$

- $\langle 0 | T\, \Psi\bar\Psi\Psi\bar\Psi | 0 \rangle_c$.

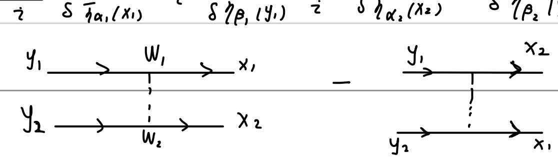
$iW[\eta, \bar\eta, J] =$



$\langle 0 | T\, \Psi_{\alpha_1}(x_1)\, \bar\Psi_{\beta_1}(y_1)\, \Psi_{\alpha_2}(x_2)\, \bar\Psi_{\beta_2}(y_2) | 0 \rangle_c$

$$= \frac{1}{i}\frac{\delta}{\delta \bar\eta_{\alpha_1}(x_1)}\; i\frac{\delta}{\delta \eta_{\beta_1}(y_1)}\; \frac{1}{i}\frac{\delta}{\delta \bar\eta_{\alpha_2}(x_2)}\; i\frac{\delta}{\delta \eta_{\beta_2}(y_2)}\; iW(\bar\eta, \eta, J)\Big|_{\bar\eta = \eta = J = 0}.$$

=



- Sign Rule.

  1° Draw each Fermion lines horizontal ; arrows left → right.

  2° Label left with same order for each diagram. ($y_1, y_2, \vdots$)

  3° Note label on right fermion lines. $\begin{cases} \text{Even permutation} \Rightarrow +1 \\ \text{Odd permutation} \Rightarrow -1 \end{cases}$

- Scattering Amplitude.

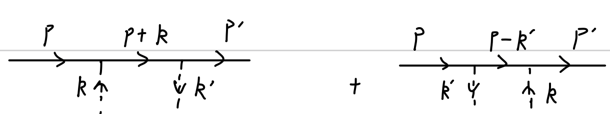
  (4.5.12 → 4.5.15) $\Big\}$ $b_s^\dagger(\vec{p})_{in} \Rightarrow i\int d^4y\ \bar{\Psi}(y) \cdots$

- $\langle f|i\rangle = \langle 0|T\ a(k')_{out}\ b_{s'}(p')_{out}\ b_s^\dagger(p)_{out}\ a^\dagger(k)_{in}|0\rangle$

$$iT_{e^-\varphi\to e^-\varphi} = \frac{1}{i}(ig)^2\,\bar{u}_{s'}(p')\cdot\left[\frac{\not{p}+\not{k}+m}{-s+m^2} + \frac{\not{p}-\not{k}'+m}{-u+m^2}\right]u_s(p)$$

$u_s(\vec{p})$: Fermi − external line points toward vertex

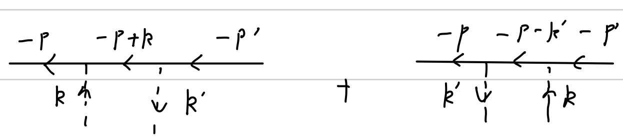
$\bar{u}_s(\vec{p})$: Fermi − external line points away vertex.



$s = +(p+k)^2 \qquad u = +(p-k')^2$

- $\langle f|i\rangle = \langle 0|T\ a(k')_{out}\ d_{s'}(p')_{out}\ d_s^\dagger(p)_{in}\ a^\dagger(k)_{in}|0\rangle$

  $\bar{\Psi}$



Label the $\boxed{\text{external}}$ fermion line with minus their four momentums

Same phenomenon occurs for complex scalar fields.

$-\bar{v}_s(p)$ away from vertex

$-v_s(p)$ point toward vertex.

$$iT_{e^+\varphi\to e^+\varphi} = \frac{1}{i}\cdot(ig)^2\cdot\bar{v}_s(p)\cdot\left(\frac{-\not{p}+\not{k}'+m}{-u+m^2} + \frac{-\not{p}-\not{k}+m}{-s+m^2}\right)\cdot v_{s'}(p')$$

$s = -(p+k)^2 \qquad u = -(p-k')^2$

$\left(\frac{|p|}{m}, \frac{p^0 \vec{p}}{m|p|}\right)$ $\quad \frac{|\vec{p}|^2}{m^2} - \frac{(p^0)^2}{m^2|\vec{p}|^2}\cdot|\vec{p}|^2$

$\frac{|p|^2}{m^2}\left[1 - |p|^2\right.$ $\quad \frac{|p|^2}{m^2}\left(1 - \frac{(p^0)^2|\vec{p}|^2}{|\vec{p}|^4}\right]$

$\left(\frac{|\vec{p}|}{m}, \frac{p^0}{m|\vec{p}|}\cdot\vec{p}\right)$

内积/与自己):

$$\frac{|\vec{p}|^2}{m^2} - \frac{(p^0)^2}{m^2|\vec{p}|^2}|\vec{p}|^2 = \frac{|\vec{p}|^2}{m^2} - \frac{(p^0)^2}{m^2} = -\frac{(p^0)^2 - |\vec{p}|^2}{m^2} = \boxed{(-1)}$$

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(\varepsilon^\mu(p,\lambda)e^{-ip\cdot x}a_{p,\lambda} + \varepsilon^{\mu *}(p,\lambda)a^\dagger_{p,\lambda}e^{ip\cdot x}\right)$$

$$(\hat{p}\cdot\vec{J})^\mu{}_\nu\,\varepsilon^\nu(p,\lambda) = \lambda\varepsilon^\mu(p,\lambda)$$

Lorentz Trans Generator & . Field , commutation.

$$[a_{p,\lambda}, \hat{p}\cdot\vec{J}] = \lambda a_{p,\lambda}.$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

Euler Lagrange Equation.

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\cdot(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

$$= -\frac{1}{2}\left[\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu\right] - \frac{1}{2\xi}(\partial_\mu A^\mu)(\partial_\mu A^\mu).$$

Euler - Lagrange Equation

$$\frac{\partial\mathcal{L}}{\partial(\phi)} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

$$\frac{\partial\left((\partial_\rho A_\sigma)(\partial^\rho A^\sigma)\right)}{\partial(\partial_\mu A^\nu)} = \frac{\partial\left((\partial_\mu A^\nu)(\partial^\mu A_\nu)\right)}{\partial(\partial_\mu A^\nu)} = \frac{\partial\left(\sum_{\mu\nu} g^{\mu\mu}\cdot g_{\nu\nu}\cdot(\partial_\mu A^\nu)(\partial_\mu A^\nu)\right)}{\partial(\partial_\mu A^\nu)}$$

$$= 2\cdot(\partial^\mu A_\nu)$$

$$\frac{\partial\left((\partial_\mu A_\nu)(\partial^\nu A^\mu)\right)}{\partial(\partial_\mu A^\nu)} = \frac{\partial\left[(\partial_\mu A^\nu)(\partial_\nu A^\mu)\right]}{\partial(\partial_\mu A^\nu)} = \partial_\nu A^\mu$$

$$\frac{\partial\mathcal{L}}{\partial(A^\nu)} = 0$$

不求和

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)} = \left(-\frac{1}{2}\right)\cdot 2\cdot\partial^\mu A_\nu + \frac{1}{2}\partial_\nu A^\mu - \frac{1}{\xi}\cdot(\partial^\nu A_\nu)\,\delta_{\mu,\nu}$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) = -\partial_\mu\partial^\mu A_\nu + \frac{1}{2}\partial_\nu\cdot\partial^\mu A^\mu - \frac{1}{\xi}\cdot[\partial_\nu\partial^\nu]A_\nu = 0 \quad ?$$

$$\begin{cases}(b_{p,0} - b_{p,3})|\Phi\rangle = 0. \\ \langle\Phi|(b^\dagger_{p,0} - b^\dagger_{p,3}) = 0\end{cases} \Rightarrow \langle\Phi|(b^\dagger_{p,0} - b^\dagger_{p,3})(b_{p,0} - b_{p,3})|\Phi\rangle$$

$$= \langle\Phi|b^\dagger_{p,0}b_{p,0} + b$$

Non abelian Gauge Theory.

Gauge Transformation

Shut up and calculate!

$$\phi(x) \longrightarrow \exp(-ig\Gamma(x))\phi(x) \qquad \phi(y) \to \exp(-ig\Gamma(y))\phi(y).$$

$$W(x,y) \longrightarrow \exp(-ig\Gamma(x))\,W(x,y)\,\exp(ig\Gamma(y))$$

$W(x,y)\phi(y)$ Transform as $\phi(x)$

联络

Suppose.

$$W(x, x+\delta x) = 1 - igA_\mu(x)\cdot\delta x^\mu$$

A transform as

$$W(x, x+\delta x) = 1 - igA_\mu(x)\delta x^\mu$$

$$\longrightarrow \exp(-ig\Gamma(x))W(x,x+\delta x)\exp(+ig\Gamma(x+\delta x))$$

$$(1-ig\Gamma(x))\cdot(1-igA_\mu(x)\delta x^\mu)\cdot(1+ig\Gamma(x+\delta x))$$

$$= 1 - ig\Gamma(x) + ig\Gamma(x+\delta x) - igA_\mu(x)\delta x^\mu$$

$$= 1 + ig\partial_\mu\Gamma(x)\delta x^\mu - igA_\mu(x)\delta x^\mu$$

于是 $A_\mu(x) \longrightarrow A_\mu(x) - \partial_\mu\Gamma(x)$

Yang-Mills Theory. $\boxed{SU(N)}$

$$W_{\alpha\beta}(x,y) \to U(x)W_{\alpha\beta}(x,y)U^{-1}(y) = \exp(-ig\Gamma^a(x)T^a_R)W(x,y)\exp(+ig\Gamma^a(y)T^a_R)$$

suppose

$$W = 1 - ig A_{\mu,\alpha\beta}\cdot\delta x^\mu$$

suppose: $W \in SU(N)$ wierd! $U(x), U(y) \in SU(N)$ 理解. $W \in SU(N)$, 怪!

$$A_\mu = A^a_\mu T^a_R = A^a_\mu(x)T^a_R$$

SU(N)'s generator

$U = (1+iT)$; $U\cdot U^\dagger = 1 \Rightarrow (1+iT)(1-iT^\dagger) = 1 + i(T - T^\dagger) = 0$

$\boxed{T = T^\dagger.}$ $N\times N$ $\quad \det(U) = 0 \Rightarrow Tr(T) = 0$

$$(N-1)+\cdots+1 = \frac{N-1+1}{2}(N-1) = \frac{N(N-1)}{2} \Rightarrow 2\times\frac{N(N-1)}{2} = N^2 - N$$

$\rightarrow (N-1)$ (Real, Traceless)

$$N^2 - N + N - 1 = N^2 - 1$$

Transformation of A

Matrix!

$$W'(x, x+\delta x) = 1 - igA'_\mu(x)\delta x^\mu$$

$$= \exp(-ig\Gamma(x))\cdot(1 - igA_\mu(x)\delta x^\mu)\exp(ig\Gamma(x+\delta x))$$

$$= 1 + ig\Gamma(x+\delta x) - ig\Gamma(x) - ig\exp(-ig\Gamma(x))A_\mu(x)\exp(ig\Gamma(x+\delta x))\delta x^\mu$$

$$= i + ig\partial_\mu\Gamma(x)\delta x^\mu - ig\exp(-ig\Gamma(x))A_\mu(x)\exp(ig\Gamma(x+\delta x))\delta x^\mu$$

$$A'_\mu(x) = \exp(-ig\Gamma(x))A_\mu(x)\exp(ig\Gamma(x)) - \partial_\mu\Gamma(x)$$

$$A'_\mu(x) = U(x) A_\mu(x) U(x) - \frac{i}{g}(\partial_\mu U(x)) U^{-1}(x).$$

$$A_\mu(x) \longrightarrow (1 - ig\Gamma^b(x) T^b_R)[A^a_\mu(x) T^a_R](1 + ig\Gamma^c(x) T^c_R) - \partial_\mu \Gamma^a(x) T^a_R$$

$$= A^a_\mu(x) T^a_R - ig\Gamma^b(x) A^a_\mu(x) T^b_R T^a_R + igA^a_\mu \Gamma^b(x) T^a_R T^b_R - \partial_\mu \Gamma^a(x) T^a_R$$

$$= A^a_\mu(x) T^a_R - ig A^a_\mu(x) \Gamma^b(x) [T^R_b, T^R_a] - \partial_\mu \Gamma^a(x) T^a_R$$

$$= A^a_\mu(x) T^a_R - ig A^a_\mu(x) \Gamma^b(x) (+i f_{bac}) T^R_c - \partial_\mu \Gamma^a(x) T^a_R$$

$$A^a_\mu(x) \to A^a_\mu(x) - \partial_\mu \Gamma^a(x) + g A^b_\mu(x) \Gamma^c(x) f_{cba}$$

$$= A^a_\mu(x) - [\partial_\mu \delta^{ac} + g f^{abc} A^b_\mu(x)] \Gamma^c(x)$$

$$(T^a_A)^{bc} = -i f^{abc}$$

$$= A^a_\mu(x) - [\partial_\mu \delta^{ac} - ig A^b_\mu(x) (T^b_A)^{ac}] \Gamma^c(x).$$

$$D^{ab}_\mu \Gamma^b(x). \qquad D_\mu = \mathbb{I}\cdot\partial_\mu - ig A^c_\mu \cdot T^c_A$$

Define Gauge Field strength.

$$F_{\mu\nu,\alpha\beta} = \frac{i}{g}[D_\mu, D_\nu]_{\alpha\beta} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} \cdot A^b_\mu A^c_\nu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].$$

$$F_{\mu\nu} \longrightarrow U F_{\mu\nu} U^{-1}.$$

$\mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) \Leftarrow$ gauge invariant.

$$D_\mu(x) \to V(x) D_\mu(x) U^{-1}(x).$$

$$\mathcal{L}_{YM} = \bar{\psi}_\alpha (i\not{D}_{\alpha\beta} - m\delta_{\alpha\beta})\psi_\beta - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

QCD & SU(3)

$$\mathcal{L}_{QCD} = \sum_{\text{Flavours}} \bar{q}_i (i\not{D}_{ij} - m_q \delta_{ij}) q_j - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}.$$

i, j : SU(3) index $\Rightarrow$ $\boxed{\text{color index}}$

Quantization of Yang Mills Theory.

$$Z[J, \eta, \bar{\eta}] = \int \mathcal{D}A\, \mathcal{D}\bar{\Psi}\, \mathcal{D}\Psi\, \exp\left(iS_{YM} + i\int d^4x (\bar{\eta}\Psi + \eta\bar{\Psi} + J^{a\mu}(x) A^a_\mu(x))\right)$$

$$A^a_\mu(x) \longrightarrow A^a_\mu(x) - D^{ab}_\mu(x) \Gamma^b(x).$$

$$G^a(x) \equiv \partial^\mu A^a_\mu(x) - w^a(x)$$

$$1 = \int \mathcal{D}\Gamma\ \delta[G(A_\Gamma)]\ \det\left(\frac{\delta G(A_\Gamma)}{\delta \Gamma}\right)$$

$$A_\Gamma = A^a_\mu - D^{ab}_\mu(x)\Gamma^b(x)$$

$$G^a(A_\Gamma) = \partial^\mu(A^a_{\Gamma\mu}) - w^a(x) = \partial^\mu(A^a_\mu - D^{ab}_\mu(x)\Gamma^b(x)) - w^a(x) = \partial^\mu A^a_\mu - \partial^\mu D^{ab}_\mu(x)\Gamma^b(x) - w^a(x).$$

$$\frac{\delta G^a(A_\Gamma)(x)}{\delta \Gamma^b(y)} = \frac{\delta}{\delta \Gamma^b(y)} \cdot \left(-\partial^\mu D^{ab}_\mu(x)\Gamma^b(x)\right) = -\partial^\mu D^{ab}_\mu(x)\,\delta^{(4)}(x-y)$$

$$\int \mathcal{D}A \int \mathcal{D}\Gamma\ \delta[G(A_\Gamma)] \cdot \det\left(\frac{\delta G(A_\Gamma)}{\delta \Gamma}\right) \exp\left(i\int d^4x\,(-\tfrac{1}{4})F^a_{\mu\nu}F^{a\,\mu\nu}\right)$$

$$G(A_\Gamma) = \partial^\mu A^a_{\mu\Gamma}(x) - w^a(x)$$

$$= \int \mathcal{D}A_\Gamma \int \mathcal{D}\Gamma\ \delta[G(A_\Gamma)]\ \det\left(\frac{\delta G(A_\Gamma)}{\delta \Gamma}\right) \exp(i\ -)$$

对 $\int \mathcal{D}\Gamma$ 的积分吸收.

$$N(\xi)\int \mathcal{D}w\ \exp\Big(-i\int d^4x\ \underbrace{w^2(x)}_{w^a(x)w^a(x)}/2\xi\Big) \cdot \int \mathcal{D}A \cdot \delta\left(\partial^\mu A^a_\mu(x) - w^a(x)\right) \det\left(\frac{\delta G(A)}{\delta \Gamma}\right)$$

$$\sim \int \mathcal{D}w \int \mathcal{D}A \cdot \det\left(\frac{\delta G[A]}{\delta \Gamma}\right) \exp\left(i\int d^4x \left[(-\tfrac{1}{4})\,\mathrm{Tr}(\sim) - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right]\right)$$

$$\det\left(\frac{\delta G[A_\Gamma]}{\delta \Gamma}\right) = \underline{\left(-\partial^\mu D^{ab}_\mu(x)\,\delta^4(x-y)\right)}$$

$$\det(M) = \int d^n\psi\, d^n\phi\ \exp\Big(\sum_{i,j}\phi_i^\dagger M_{ij}\psi_j\Big) = \det(M).$$

$$\det\left(-\partial^\mu D^{ab}_\mu(x)\,\delta^4(x-y)\right) = \int dc \cdot dc' \cdot \exp\left(-\int d^4x\,d^4y\ c'^a(x)\,\delta^4(x-y)\,\partial^\mu_x D^{ab}_\mu(x)\,c^b(y)\right)$$

$$\det(-\partial^\mu_x D^{ab}_{x,\mu}) \propto \int \mathcal{D}c\,\mathcal{D}\bar c\ \exp\left(-i\int d^4x\ \partial^\mu \bar c^a(x) D^{ab}_\mu c^b(x)\right).$$

Symmetries In Quantum Field Theory.

- Classical Field Theory and Noether's Theorem.

Change of Lagragian:

$$\delta \mathcal{L}(x) \equiv \frac{\partial \mathcal{L}}{\partial \varphi_a}(x)\,\delta\varphi_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x)\,\delta(\partial_\mu \varphi_a(x)) \qquad —(1)$$

Change of Action

$$S = S[\varphi_a] \qquad \mathcal{L} = \mathcal{L}(\varphi_a, \partial_\mu \varphi_a).$$

$$\frac{\delta S}{\delta \varphi_a(x)} = \int d^4y \,\frac{\delta \mathcal{L}(y)}{\delta \varphi_a(x)}\,\delta\varphi_a(x)$$

$$= \int d^4y \left(\frac{\partial \mathcal{L}}{\partial \varphi_b}(y)\,\frac{\delta \varphi_b(y)}{\delta \varphi_a(x)} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_b)}(y)\,\frac{\delta\, \partial_\mu \varphi_b(y)}{\delta \varphi_a(x)} \right)$$

$$= \int d^4y \left(\frac{\partial \mathcal{L}}{\partial \varphi_b}(y)\,\delta_{ab}\,\delta^{(4)}(y-x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_b)}(y)\,\delta_{ab}\,\partial_\mu \delta^{(4)}(y-x) \right)$$

$$= \int d^4y \left(\frac{\partial \mathcal{L}}{\partial \varphi_a}(y)\,\delta^{(4)}(y-x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(y)\,\partial_\mu \delta^{(4)}(y-x) \right)$$

$\downarrow$ Integral by parts.

$$= \frac{\partial \mathcal{L}}{\partial \varphi_a}(x) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x) \right) \qquad —(2)$$

Combine (1) & (2)

$$\delta \mathcal{L}(x) = \left(\frac{\delta S}{\delta \varphi_a(x)} + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x) \right) \right) \delta\varphi_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x)\,\partial_\mu \delta\varphi_a(x)$$

$$= \frac{\delta S}{\delta \varphi_a(x)}\,\delta\varphi_a(x) + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x) \times \delta\varphi_a(x) \right)$$

$$\Downarrow \quad j^\mu(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x) \times \delta\varphi_a(x)$$

$$\partial_\mu\left(j^\mu(x)\right) = \delta\mathcal{L}(x) - \frac{\delta S}{\delta \varphi_a(x)}\,\delta\varphi_a(x).$$

—— $\delta \mathcal{L} = 0$, $\delta S = 0$, conserved current.

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x) \times \delta\varphi_a(x)$$

$$\partial_\mu(j^\mu(x)) = 0$$

—— $\delta\mathcal{L}(x) = \partial^\mu\left(K_\mu(x)\right)$ $\qquad \delta S = 0$.

- —— 1° $\varphi_a(x) \longrightarrow \varphi_a(x+a) = \varphi_a(x) + a^\nu \partial_\nu \varphi_a(x)$. $(\delta S = 0)$

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x+a) = \mathcal{L}(x) + a^\nu \partial_\nu \mathcal{L}(x)$$

$$\delta\mathcal{L} = a^\nu \partial_\nu\left(\mathcal{L}(x)\right)$$

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x) \times \delta\varphi_a(x)$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x) \times a^\nu \partial_\nu\left(\varphi_a(x)\right)$$

$$\Downarrow \quad \left\{ \partial_\mu\left(j^\mu(x)\right) = \delta\mathcal{L}(x) - \frac{\delta S}{\delta\varphi_a(x)}\delta\varphi_a(x). \right.$$

$$\partial_\mu\left\{ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x)\times a^\nu\partial_\nu\left(\varphi_a(x)\right)\right\} = a^\mu\partial_\mu\left(\mathcal{L}(x)\right)$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x)\times a^\nu\partial_\nu\left(\varphi_a(x)\right) - a^\mu\mathcal{L}(x)\right\} = 0.$$

$$a_\nu\partial_\mu\left\{ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x)\times\partial^\nu\left(\varphi_a(x)\right) - g^{\mu\nu}\mathcal{L}(x)\right\} = 0$$

$$T^{\mu\nu}(x) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x)\times\partial^\nu\left(\varphi_a(x)\right) - g^{\mu\nu}\mathcal{L}(x)$$

$$a_\nu\partial_\mu\left(T^{\mu\nu}(x)\right) = 0.$$

0 —— 2° $\quad \varphi_a(x) \longrightarrow \varphi_a(x+\delta\omega\cdot x) \qquad (\delta S = 0)$ (后面又算了一遍, 比这里详细)

$\qquad\qquad \mathcal{L}(x) \longrightarrow \mathcal{L}(x+\delta\omega\cdot x)$

$$\left|\begin{aligned} \delta\mathcal{L}(x) &= \delta\omega^\mu{}_\nu\, x^\nu\, \partial_\mu\left(\mathcal{L}(x)\right) \\ &= \partial_\mu\left(\mathcal{L}(x)\right)\delta\omega^\mu{}_\nu\, x^\nu \\ \delta\varphi_a(x) &= \partial_\mu\left(\varphi_a(x)\right)\times(\delta\omega)^\mu{}_\nu\, x^\nu \end{aligned}\right.$$

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x)\times\delta\varphi_a(x)$$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x)\times\partial_\mu\left(\varphi_a(x)\right)\times(\delta\omega)^\mu{}_\nu\, x^\nu$$

$$\Downarrow \leftarrow \left\{ \partial_\mu\left(j^\mu(x)\right) = \delta\mathcal{L}(x) - \frac{\delta S}{\delta\varphi_a(x)}\delta\varphi_a(x). \right.$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x)\times\partial_\mu\left(\varphi_a(x)\right)\times(\delta\omega)^\mu{}_\nu\, x^\nu\right) = \partial_\mu\left(\mathcal{L}(x)\right)\delta\omega^\mu{}_\nu\, x^\nu$$

$$\downarrow \leftarrow \left\{\begin{aligned} (1+\omega)^T\eta(1+\omega) &= \eta \\ \omega^T\eta + \eta\omega &= 0 \\ \eta\omega^T\eta &= -\omega \\ \eta_{ii}\,\omega_{ji}\,\eta_{jj} &= -\omega_{ij} \Rightarrow \omega_{ii} = 0. \end{aligned}\right.$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x)\times\partial^\rho\left(\varphi_a(x)\right)\times(\delta\omega)_{\rho\nu}\, x^\nu - \mathcal{L}(x)\, g^{\mu\rho}\,\delta\omega_{\rho\nu}\, x^\nu\right) = 0$$

$$\partial_\mu\left(x^\nu T^{\mu\rho}\right) = 0 \implies \partial_\mu\left(x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}\right) = 0$$

Note As $\quad \partial_\mu M^{\mu\nu\rho} = 0.$

- Notes on Why $\mathcal{L}(x)$ Transforms as $\mathcal{L}(x) \to \mathcal{L}(x+\omega x)$

Lagrangian written as

$$\mathcal{L}(x) = \partial^\mu \bar{\psi}(x)\, \partial_\mu \psi(x)$$

Transformation of fields

$$\psi_a'(x') = \exp\left(\tfrac{1}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\right)\psi_b(x)$$

$$\psi_a'(x) = \exp\left(\tfrac{1}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\right)\psi_b(x-\omega x)$$

$$\bar{\psi}_a'(x) = \bar{\psi}_b(x-\omega x)\exp\left(-\tfrac{1}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\right)$$

$$\partial_\mu \psi_a'(x) = \exp\left(\tfrac{1}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\right)\partial_\mu \psi_b(x-\omega x)$$

$$\left\{\begin{aligned} &\psi_b(x+\delta x - \omega(x+\delta x)) = \psi_b(x^\mu - \omega^\mu{}_\nu x^\nu) + \partial_\mu \psi_b \cdot (\delta x^\mu - \omega^\mu{}_\nu \delta x^\nu) \\ &\partial_\mu \psi_b(x-\omega x) = (\partial_\mu \psi_b)(x-\omega x) - \omega^\alpha{}_\mu (\partial_\alpha \psi_b)(x-\omega x) \end{aligned}\right.$$

$$\partial^\mu \bar{\psi}_a' \partial_\mu \psi_a'(x) = \left\{(\partial^\mu \bar{\psi}_b)(x-\omega x) - \omega^{\alpha\mu}(\partial_\alpha \bar{\psi}_b)(x-\omega x)\right\}$$

$$\exp\left(-\tfrac{1}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\right)\exp\left(\tfrac{1}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ac}\right) \qquad \exp(-\square)\times\exp(\square) = 1$$

$$\left\{(\partial_\mu \psi_c)(x-\omega x) - \omega^\alpha{}_\mu (\partial_\alpha \psi_c)(x-\omega x)\right\}$$

$$= \left\{(\partial^\mu \bar{\psi}_a)(x-\omega x) - \omega^{\alpha\mu}(\partial_\alpha \bar{\psi}_a)(x-\omega x)\right\}$$

$$\times\left\{(\partial_\mu \psi_a)(x-\omega x) - \omega^\alpha{}_\mu (\partial_\alpha \psi_a)(x-\omega x)\right\}$$

$$= (\partial^\mu \bar{\psi}_a)(\partial_\mu \psi_a)(x-\omega x) - \omega_{\alpha\mu}(\partial^\mu \bar{\psi}_a)(\partial^\alpha \psi_a)\circ(x-\omega x)$$

$$- \omega_{\alpha\mu}(\partial^\alpha \bar{\psi}_a)(\partial^\mu \psi_a)\circ(x-\omega x)$$

$$\left|\ \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0\right.$$

$$= (\partial^\mu \bar{\psi}_a)(\partial_\mu \psi_a)(x-\omega x)$$

In all $\mathcal{L}'(x) = \mathcal{L}(x-\omega x)$

- Lorentz transformation & angular momentum.

Conclusion derived before

$$\delta\mathcal{L} = \frac{\delta S}{\delta \varphi_a(x)}\delta\varphi_a(x) + \partial_\mu\left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x)\,\delta\varphi_a(x)\right\} \qquad -(1)$$

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu \varphi_a)}\partial^\nu \varphi_a - g^{\mu\nu}$$

Lorentz transformation of fields

$$\varphi_a'(x) = \exp\left(\tfrac{1}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\right)\varphi_b(x-\omega x)$$

$$\mathcal{L}'(x) = \mathcal{L}(x-\omega x)$$

$$\delta S = 0$$

$$\delta\varphi_a(x) = -\omega^\mu{}_\nu x^\nu \partial_\mu(\varphi_a) + \tfrac{1}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\varphi_b(x)$$

$$\delta \mathcal{L} = -w^{\mu}{}_{\nu} x^{\nu} \partial_\mu (\mathcal{L})$$

insert into equation (1)

$$\delta \mathcal{L} = \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x) + \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} (x) \delta \varphi_a(x) \right\}$$

$$-w^{\mu}{}_{\nu} x^{\nu} \partial_\mu (\mathcal{L}) = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \left[-w^{\rho}{}_{\nu} x^{\nu} \partial_\rho (\varphi_a) + \frac{1}{2} w_{\rho\nu} (I^{\rho\nu})_{ab} \varphi_b(x) \right] \right\}$$

$$-\partial_\mu (w^{\mu}{}_{\nu} x^{\nu} \mathcal{L}) = -w^{\mu}{}_{\mu} \mathcal{L} - w^{\mu}{}_{\nu} x^{\nu} \partial_\mu (\mathcal{L})$$

$$= -w^{\mu}{}_{\nu} x^{\nu} \partial_\mu (\mathcal{L})$$

$$\Lambda^{\mu}{}_{\alpha} x^{\alpha} \Lambda_{\mu}{}^{\beta} x_{\beta} = x^{\mu} x_{\mu}$$

$$\Lambda^{\mu}{}_{\alpha} \Lambda_{\mu}{}^{\beta} = \delta^{\beta}_{\alpha}$$

$$\delta^{\beta}_{\alpha} + w_{\alpha}{}^{\beta} + w^{\beta}{}_{\alpha} = \delta^{\beta}_{\alpha}$$

$$w_{\alpha\beta} + w_{\beta\alpha} = 0 \quad \Rightarrow \quad w_{\alpha\alpha} = 0$$

$$\partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \left[-w^{\rho}{}_{\nu} x^{\nu} \partial_\rho (\varphi_a) + \frac{1}{2} w_{\rho\nu} (I^{\rho\nu})_{ab} \varphi_b(x) \right] + w^{\mu}{}_{\nu} x^{\nu} \mathcal{L} \right\} = 0$$

$$\partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \left[-w_{\rho\nu} x^{\nu} \partial^{\rho} (\varphi_a) + \frac{1}{2} w_{\rho\nu} (I^{\rho\nu})_{ab} \varphi_b(x) \right] + g^{\mu\rho} w_{\rho\nu} x^{\nu} \mathcal{L} \right\} = 0$$

$$\downarrow \quad \longleftrightarrow \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \partial^{\nu} \varphi_a - g^{\mu\nu} \mathcal{L}$$

$$\partial_\mu \left\{ -w_{\rho\nu} x^{\nu} T^{\mu\rho} + \frac{1}{2} w_{\rho\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} (I^{\rho\nu})_{ab} \varphi_b(x) \right\} = 0$$

$$\partial_\mu \left\{ -\frac{1}{2} w_{\rho\nu} x^{\nu} T^{\mu\rho} + \frac{1}{2} w_{\rho\nu} x^{\rho} T^{\mu\nu} + \frac{1}{2} w_{\rho\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} (I^{\rho\nu})_{ab} \varphi_b(x) \right\} = 0$$

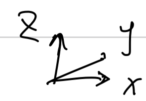
$$j^{\mu} = \underbrace{x^{\rho} T^{\mu\nu} - x^{\nu} T^{\mu\rho}}_{\text{Angular momentum}} + \underbrace{\frac{1}{2} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} (I^{\rho\nu})_{ab} \varphi_b(x)}_{\text{Spin momentum.}}$$

∘ Notes on Transformation between different generator in Weyl representation.

Left hand, Right hand representation (t^i: generator of rotation s^i, generator of boost)

$$R_L(\Lambda) = \exp\left(\left(-\tfrac{1}{2}s^i - \tfrac{i}{2}t^i\right)\sigma^i\right)$$

$$R_R(\Lambda) = \exp\left(\left(+\tfrac{1}{2}s^i - \tfrac{i}{2}t^i\right)\sigma^i\right)$$



$$W^\mu{}_\nu = t^i \tilde{J}_i + S^i K_i$$

$$= t^1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + t^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + t^3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ S^1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + S^2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + S^3 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$W_{\mu\nu} = \begin{pmatrix} 0 & W_{01} & W_{02} & W_{03} \\ -W_{01} & 0 & W_{12} & W_{13} \\ -W_{02} & -W_{12} & 0 & W_{23} \\ -W_{03} & -W_{13} & -W_{23} & 0 \end{pmatrix}$$

$$W^\mu{}_\nu = \begin{pmatrix} 0 & W_{01} & W_{02} & W_{03} \\ +W_{01} & 0 & -W_{12} & -W_{13} \\ +W_{02} & +W_{12} & 0 & -W_{23} \\ +W_{03} & +W_{13} & +W_{23} & 0 \end{pmatrix}$$

$$t^1 = W_{23} \quad t^2 = -W_{13} \quad t^3 = W_{12}$$

$$\boxed{t^i = \tfrac{1}{2}\varepsilon^{ijk} W_{jk}}$$

$$S^1 = W_{01} \quad S^2 = W_{02} \quad S^3 = W_{03}$$

$$S^i = W_{0i}$$

$$R_D(\Lambda) \sim \mathbb{I} - \tfrac{1}{2} s^i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} - \tfrac{i}{2} t^i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$= \mathbb{I} - \tfrac{1}{2} W_{0i} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} - \tfrac{i}{2}\tfrac{1}{2}\varepsilon^{ijk} W_{jk} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$= \mathbb{I} - \tfrac{1}{4} W_{0i} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} - \tfrac{1}{4} W_{i0} \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} - \tfrac{i}{4}\varepsilon^{ijk} W_{jk} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$= \mathbb{I} + W_{\mu\nu} \tilde{J}^{\mu\nu} \qquad (\tilde{J} \text{ Anti-Symmetric})$$

$$(\tilde{J})^{0i} = -\tfrac{1}{4} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$(\tilde{J})^{ij} = -\tfrac{i}{4}\varepsilon^{kij} \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} = -\tfrac{i}{4}\varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}$$

$$[\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k$$

$$= -\tfrac{1}{8} \begin{bmatrix} [\sigma^i, \sigma^j], & 0 \\ 0, & -[\sigma^i, \sigma^j] \end{bmatrix}$$

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{bmatrix} \qquad \sigma^\mu = (\mathbb{I}, \sigma^1, \sigma^2, \sigma^3) \quad \bar\sigma^\mu = (\mathbb{I}, -\sigma^1, -\sigma^2, -\sigma^3)$$

Suppose

$$[\gamma^\mu, \gamma^\nu] = \left[\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix}\right] = \begin{bmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{bmatrix}$$

Find That

$$\frac{1}{8}[\gamma^0, \gamma^i] = \frac{1}{8}\begin{bmatrix} -2\sigma^i & 0 \\ 0 & 2\sigma^i \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{bmatrix} = \widetilde{J}^{0i}$$

$$\frac{1}{8}[\gamma^i, \gamma^j] = \frac{1}{8}\begin{bmatrix} -\sigma^i\sigma^j + \sigma^j\sigma^i & 0 \\ 0 & -\sigma^i\sigma^j + \sigma^j\sigma^i \end{bmatrix} = \widetilde{J}^{ij}$$

In All

$$W_{\mu\nu}\widetilde{J}^{\mu\nu} = \frac{1}{8} W_{\mu\nu}[\gamma^\mu, \gamma^\nu]$$

Physical convention

$$W_{\mu\nu}\widetilde{J}^{\mu\nu} = -\frac{i}{2}\ \frac{i}{4}[\gamma^\mu, \gamma^\nu]W_{\mu\nu}$$

Notes on helicity

Field under lorentz transformation.

$$\psi'(x') = L\,\psi(x)$$

$$\psi'(x) = L(\Lambda)\psi(\Lambda^{-1}x)$$

Quantum Field under lorentz transformation

$$U^{-1}(\Lambda)\,\psi(x)\,U(\Lambda) = \psi'(x) = L(\Lambda)\,\psi(\Lambda^{-1}x)$$

$$U^{-1}(\Lambda)\,\psi(x)\,U(\Lambda) = L(\Lambda)\,\psi(\Lambda^{-1}x)$$

$$L(\Lambda) = 1 + \tfrac{1}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]$$

$$\equiv 1 - \tfrac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})$$

$$\varphi(\Lambda^{-1}x) = \varphi(x) - \tfrac{i}{2}\omega_{\mu\nu} L^{\mu\nu}\varphi(x)$$

$$\left(1+\tfrac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)\varphi_A(x)\left(1-\tfrac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) = \left(\delta_A{}^B - \tfrac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_A{}^B\right)\left(\varphi_B(x) - \tfrac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\varphi_B(x)\right)$$

$$\tfrac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\,\varphi_A(x) - \varphi_A(x)\tfrac{i}{2}\omega_{\mu\nu}J^{\mu\nu} = -\tfrac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\varphi_A(x) - \tfrac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_A{}^B\varphi_B(x)$$

$$[\varphi_A(x), J^{\mu\nu}] = L^{\mu\nu}\varphi_A(x) + (S^{\mu\nu})_A{}^B\varphi_B(x)$$

Rotation in $\hat{p}$ direction. for φ angle

$$L(\Lambda) = \exp\left(-\tfrac{i}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}\cdot\hat{p}\,\varphi\right)$$

Mode expansion

$$\psi(x) = \int\frac{d^3p}{(2\pi)^3 2E_p}\cdot\left(b_{p,s}u(p,s)e^{-ip\cdot x} + d^\dagger_{p,s}v(p,s)\,e^{ip\cdot x}\right)$$

$$\Sigma\cdot v(p,\pm) = \mp\, v(p,+)$$

$$L(\Lambda)\,d^\dagger_{p,+}\,v(p,+) = \exp(+\tfrac{i}{2}\varphi)\,d^\dagger_{p,+}\,v(p,+) \sim \exp(i J\cdot\hat{p}\varphi)\,\psi\,\exp(-iJ\cdot\hat{p}\,\varphi)$$

$$\Rightarrow\ (1+iJ\cdot\hat{p}\,\varphi)\,d^\dagger_{p,+}\,(1-iJ\cdot\hat{p}\,\varphi) = (1+\tfrac{i}{2}\varphi)\,d^\dagger_{p,+}$$

$$J\cdot\hat{p}\,d^\dagger_{p,+} - d^\dagger_{p,+}J\cdot\hat{p} = +\tfrac{1}{2}d^\dagger_{p,+}$$

$$[J\cdot\hat{p}, d^\dagger_{p,+}] = \tfrac{1}{2}d^\dagger_{p,+}$$

$$J\cdot\hat{p}\,d^\dagger_{p,+} = \tfrac{1}{2}d^\dagger_{p,+} + d^\dagger_{p,+}J\cdot\hat{p}$$

Means $d^\dagger_{p,+}$ generates a state with spin $\tfrac{1}{2}$ in the direction $\hat{p}$.

- Path Integral version of Symmetry

——— Generating Function unchange under Field shift

$$Z[J] = \int \mathcal{D}\varphi \exp(i(S + \int d^4y\, J_a \varphi_a))$$

Shift of Field $\varphi_a(x) \to \varphi_a(x) + \delta\varphi_a(x)$ Leaves Measure $\mathcal{D}\varphi$ invariant.

$$0 = \delta Z[J] = i\int \mathcal{D}\varphi \exp(i[S + \int d^4y\, J_a\varphi_a]) \int d^4x \left(\frac{\delta S}{\delta\varphi_a(x)} + J_a(x)\right)\delta\varphi_a(x)$$

Functional Derivative of $\frac{\delta}{i\delta J_{a_1}(x_1)} \frac{\delta}{i\delta J_{a_2}(x_2)} \cdots \frac{\delta}{i\delta J_{a_n}(x_n)}$; Then let $J=0$

$$0 = \int \mathcal{D}\varphi \exp(i[S + \int d^4y\, J_a\varphi_a]) \int d^4x \left(\frac{\delta S}{\delta\varphi_a(x)} \varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)\right.$$
$$\left. + \frac{1}{i}\sum_{j=1}^{n} \varphi_{a_1}(x_1)\cdots \delta_{a,a_j}\delta^{(4)}(x-x_j)\cdots\varphi_{a_n}(x_n)\right)\delta\varphi_a(x)$$

Since Path integral Computes vacuum expectation of time-order product.

$$0 = i\langle 0|T\frac{\delta S}{\delta\varphi_a(x)}\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)|0\rangle + \sum_{j=1}^{n}\langle 0|T\varphi_{a_1}(x_1)\cdots\delta_{aa_j}\delta^{(4)}(x-x_j)\cdots\varphi_{a_n}(x_n)|0\rangle$$

$\Uparrow$

Schwinger - Dyson Equation

——— Ward Takahashi Identity.

Choose $\delta\varphi_a(x)$ That leaves $\mathcal{L}$ invariant.

$$j^\mu(x) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x) \times \delta\varphi_a(x)$$
$$\partial_\mu(j^\mu(x)) = \delta\mathcal{L}(x) - \frac{\delta S}{\delta\varphi_a(x)}\delta\varphi_a(x).$$
$$\Downarrow$$
$$-\partial_\mu(j^\mu(x)) = \sum_a \frac{\delta S}{\delta\varphi_a(x)}\delta\varphi_a(x)$$

$$0 = i\langle 0|T\frac{\delta S}{\delta\varphi_a(x)}\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)|0\rangle + \sum_{j=1}^{n}\langle 0|T\varphi_{a_1}(x_1)\cdots\delta_{aa_j}\delta^{(4)}(x-x_j)\cdots\varphi_{a_n}(x_n)|0\rangle$$

$$0 = i\sum_a\langle 0|T\frac{\delta S}{\delta\varphi_a(x)}\delta\varphi_a(x)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)|0\rangle + \sum_{j=1}^{n}\sum_a\langle 0|T\varphi_{a_1}(x_1)\cdots(\delta\varphi_{a}(x))\delta_{a,a_j}\delta^{(4)}(x-x_j)\cdots\varphi_{a_n}(x_n)|0\rangle$$

$$0 = -i\partial_\mu\langle 0|T j^\mu(x)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)|0\rangle + \sum_{j=1}^{n}\langle 0|T\varphi_{a_1}(x_1)\cdots\delta\varphi_{a_j}(x)\delta^{(4)}(x-x_j)\cdots\varphi_{a_n}(x_n)|0\rangle$$

Ward Identity in QED.

$k^\mu T_\mu = 0$ Ward identity T

QED Lagrangian

$$\mathcal{L} = \bar{\psi}(i Z_2 \not{\partial} - Z_m m)\psi - Z_3 \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - Z_1 e \bar{\psi}\gamma^\mu A_\mu \psi - \frac{Z_3}{2\xi}(\partial_\mu A^\mu)^2 \quad (e<0)$$

- Global U(1) Symmetry and conserved currents

U(1) Transformation

$$\psi \longrightarrow \psi \exp(ie\Gamma)$$
$$\bar{\psi} \longrightarrow \bar{\psi} \exp(-ie\Gamma)$$

Noether conserved current

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(x) \times \delta\varphi_a(x)$$
$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\delta\bar{\psi}$$
$$= i Z_2 \bar{\psi}\gamma^\mu (ie\Gamma\psi)$$
$$= -e\Gamma Z_2 \bar{\psi}\gamma^\mu\psi$$

- Ward Takahashi Identity used in U(1) - Global Symmetry current.

$$0 = -i\partial_\mu \langle 0|T\, j^\mu(x)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)|0\rangle + \sum_{j=1}^{n}\langle 0|T\varphi_{a_1}(x_1)\cdots\varphi_{a_j}(x)\delta^{(4)}(x-x_j)\cdots\varphi_{a_n}(x_n)|0\rangle$$

- From LSZ Formula for scalar Fields. define F function.

$$\langle f|i\rangle = \langle 0|a_{q_m}(+\infty)\cdots a^\dagger_{p_1}(-\infty)\cdots a^\dagger_{p_n}(-\infty)|0\rangle$$
$$= (i)^{n+m}\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial^2_{x_1}+m^2)\cdots\int d^4y_1 \exp(iq_1 y_1)(\partial^2_{y_1}+m^2)\cdots$$
$$\langle 0|T\varphi(x_1)\varphi(x_2)\cdot\varphi(x_n)\varphi(y_1)\cdots\varphi(y_m)|0\rangle$$

Fourier Transform与自洽性.

$$\tilde{\varphi}(k) \equiv i\int d^4x\, e^{ik\cdot x}\varphi(x)$$
$$\varphi(x) = \frac{1}{(2\pi)^4}\frac{1}{i}\int d^4k\,\tilde{\varphi}(k)\,e^{-ik\cdot x}$$
$$= \frac{1}{(2\pi)^4}\frac{1}{i}\int d^4k\cdot i\cdot\int d^4y\, e^{ik\cdot y}\varphi(y)\,e^{-ik\cdot x}$$
$$= \frac{1}{(2\pi)^4}\int d^4y\cdot(2\pi)^4\delta^{(4)}(x-y)\varphi(y)$$
$$= \varphi(x)$$

$$\langle f|i\rangle = (i)^{n+m}\int d^4x_1\, \exp(-ip_1\cdot x_1)(\partial_{x_1}^2+m^2)\cdots\int d^4y_1\, \exp(+iq_1\cdot y_1)\cdot(\partial_{y_1}^2+m^2)$$
$$\left(\frac{1}{(2\pi)^4}\right)^{n+m}\left(\frac{1}{i}\right)^{n+m}\int d^4k_1\cdots d^4k_n\int d^4k'_1\cdots d^4k'_m$$
$$\langle 0|T\exp(-ik_1x_1)\tilde\varphi(k_1)\cdots\ \exp(-ik'_1\cdot y_1)\tilde\varphi(k'_1)\cdots|0\rangle$$

$$=\left(\frac{1}{(2\pi)^4}\right)^{n+m}\int d^4x_1\cdots\int d^4y_1\cdots\int d^4k_1\cdots d^4k_n\int d^4k'_1\cdots d^4k'_m$$
$$(-k_1^2+m^2)\cdots(-k_n^2+m^2)(-k_1'^2+m^2)\cdots(-k_m'^2+m^2)$$
$$\langle 0|T\exp(-i(k_1+p_1)x_1)\tilde\varphi(k_1)\cdots\ \exp(-i(k'_1-q_1)\cdot y_1)\tilde\varphi(k'_1)\cdots|0\rangle$$

不严格，因为有时序乘积，固定 $k_i\ k'_i$ 对 x，y 积分并不显然是 delta Function！

$$= (-p_1^2+m^2)\cdots(-p_n^2+m^2)(-q_1^2+m^2)\cdots(-q_m^2+m^2)$$
$$(2\pi)^4\delta^{(4)}(\Sigma p-\Sigma q)\,F(p_i^2,\ q_i^2,\ p_i\cdot p_j,\ q_i\cdot q_j,\ p_i\cdot q_j)$$

Srednicki (67.4); (67.5).

- Relation between Scattering Amplitude & F.

已知，$\langle f|i\rangle$ 的形式为

$$\langle f|i\rangle = (2\pi)^4\delta^{(4)}(\Sigma p-\Sigma q)\,(iT)$$

且 iT 是解析的，则，

$$(-p_1^2+m^2)(-p_2^2+m^2)\cdots F(p_i^2,\ q_i^2,\ p_i\cdot p_j,\ p_i\cdot q_j,\ q_i\cdot q_j) = iT$$

Suppose：

$$F(p_i^2,\ q_i^2,\ p_i\cdot p_j,\ p_i\cdot q_j,\ q_i\cdot q_j) = (\text{Singular part}) + (\text{Non Singular Part})$$

则，其与散射振幅连系.

$$(\text{Singular part})(-p_1^2+m^2)(-p_2^2+m^2)\cdots = iT$$

$$F = \frac{iT}{(-p_1^2+m^2)(-p_2^2+m^2)\cdots} + (\text{Non Singular Part})$$

- Schwinger Dyson Equation 引出 contact term 的 Definition.

Schwinger-Dyson Equation. (后一项叫作 contact term).

$$\langle 0|T\frac{\delta S}{\delta\varphi_a(x)}\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)|0\rangle = \sum_j i\,\langle 0|T\varphi_{a_1}(x_1)\cdots\delta_{aa_j}\delta^{(4)}(x-x_j)\cdots\varphi_{a_n}(x_n)|0\rangle$$

contact term.

若 LSZ formula $\langle 0|T\varphi(x_1)\cdots\varphi(y_n)|0\rangle$ 中有 term：记算其对 T 的 contribution

$$\langle 0|T\cdots\delta(x_1-x_2)\cdots|0\rangle$$

$$\Downarrow$$

x_1, x_2 都当作了入射

$$\langle f|i\rangle \sim \int d^4x_1\, d^4x_2\, \exp(-ip_1\cdot x_1)\exp(-ip_2\cdot x_2)(+\partial^2_{x_1}+m^2)(+\partial^2_{x_2}+m^2)$$

$$\int d^4k_1 \int d^4k_2$$

$\int d^4k_1\, d^4k_2\, \exp(ik_1\cdot x_1)\exp(ik_2\cdot x_2)\,\delta^{(4)}(k_1+k_2) \sim \delta^{(4)}(x_1-x_2)$

$$\langle 0|T\cdots\exp(ik_1(x_1))\exp(ik_2 x_2)\,\delta^4(k_1+k_2)\cdots|0\rangle$$

$$\sim (-p_1^2+m^2)(-p_2^2+m^2)\langle 0|T\cdots\delta^{(4)}(p_1+p_2)\cdots|0\rangle$$

$\sim \delta^{(4)}(\Sigma p-\Sigma q)\,F$ F 中无 singularity.

结论：contact term in correlation function do Not contribute to T

○ LSZ Reduction Formula for photon field (出射 $\vec{k}$ photon).

$$a_\lambda(k)_{out} \to i\varepsilon^\mu_\lambda(\vec{k})\int d^4x\, \exp(ik\cdot x)(+\partial^2)A_\mu(x)$$

$$\langle f|i\rangle \sim i\varepsilon^\mu\int d^4x\, \exp(ik\cdot x)(\partial_x^2)\cdots\langle 0|T A_\mu(x)\cdots|0\rangle$$

Equation of Motion of Vector Field.

$$\mathcal{L} = \bar\psi(iZ_2\not\partial - Z_m m)\psi - Z_3\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - Z_1 e\bar\psi\gamma^\mu A_\mu\psi - \frac{Z_3}{2\xi}(\partial_\mu A^\mu)^2 \quad (e<0)$$

Euler-lagrange Equation. ($\xi=1$)

$$\mathcal{L} \sim -Z_3\frac{1}{2}\left[(\partial_\mu A_\nu)(\partial^\mu A^\nu)-(\partial_\mu A_\nu)(\partial^\nu A^\mu)\right] - \frac{Z_3}{2}(\partial_\mu A^\mu)^2 - Z_1 e\bar\psi\gamma^\mu A_\mu\psi \qquad (1)$$

与 A 相关的 Term.

$$\partial_\nu\left(\frac{\partial(\mathcal{L})}{\partial(\partial_\nu A^\mu)}\right) = \frac{\partial\mathcal{L}}{\partial A}. \qquad (2)$$

Notice: $\frac{\partial((\partial_\rho A^\rho)^2)}{\partial(\partial_\nu A^\mu)} = 2(\partial_\rho A^\rho)g^\nu_\mu$

$$\partial_\nu\left(-\frac{1}{2}Z_3\partial^\nu A_\mu + \frac{1}{2}Z_3\partial_\mu A^\nu - Z_3(\partial_\rho A^\rho)g^\nu_\mu\right) = -Z_1 e\bar\psi\gamma_\mu\psi$$

$$-Z_3\partial^2 A_\mu + Z_3\partial_\mu(\partial_\nu A^\nu) - Z_3\partial_\mu(\partial_\nu A^\nu) = -Z_1 e\bar\psi\gamma_\mu\psi$$

$$-Z_3\partial^2 A_\mu = -Z_1 e\bar\psi\gamma_\mu\psi$$

$$\partial^2 A_\mu = \frac{Z_1}{Z_3}e\bar\psi\gamma_\mu\psi$$

由前文，Global U(1) Symmetry 的 conserved current 是 $j_\mu = e\bar\psi\gamma_\mu\psi$.

LSZ Reduction Formula for 出射光子.

$$\langle f|i\rangle \sim i\,\varepsilon^{\mu}\int d^4x\ \exp(ik\cdot x)\,(\partial_x^2)\cdots\ \langle 0|T A_\mu(x)\cdots|0\rangle$$

$$= i\frac{Z_1}{Z_3}\,\varepsilon^{\mu}\int d^4x\ \exp(ik\cdot x)\ \langle 0|T\, j_\mu(x)\cdots|0\rangle$$

○ 若：$T\sim \varepsilon^{\mu}T_\mu$，则 $k_\mu T^\mu=0$. 由上文对LSZ reduction Formula的分析.

$$\langle f|i\rangle \sim \varepsilon^{\mu}\int d^4x\ \exp(ik\cdot x)\langle 0|T j_\mu(x)\cdots|0\rangle$$

$$\delta^{(4)}(\Sigma k)\,k^\mu T_\mu \sim k^{\mu}\int d^4x\ \exp(ik\cdot x)\langle 0|T j_\mu(x)\cdots|0\rangle$$

↓ integrate by parts.

$$= \int d^4x\ \exp(ikx)\ \langle 0|T\partial^\mu j_\mu(x)\cdots|0\rangle$$

Ward-Takahashi Identity

$$0=-i\partial_\mu\langle 0|T\, j^{\mu}(x)\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)|0\rangle+\sum_{j=1}^{n}\langle 0|T\varphi_{a_1}(x_1)\cdots\delta\varphi_{a_j}(x)\delta^{(4)}(x-x_j)\cdots\varphi_{a_n}(x_n)|0\rangle$$

$$\delta^{(4)}(\Sigma k)\,k^\mu T_\mu \sim \int d^4x\ \exp(ik\cdot x)\cdot(\text{contact term.})$$

由前文分析, contact term 不对 T 有贡献，则 if $T=\varepsilon^{\mu}T_\mu$，则 $k^\mu T_\mu=0$.

Ward Identity II.

○ Define Correlation function

$$C^{\mu}_{\alpha\beta}(k,p',p)=iZ_1\int d^4x\,d^4y\,d^4z\ \exp(-ikx+ip'y-ipz)\,\langle 0|T\, j^\mu(x)\,\psi_\alpha(y)\,\bar\psi_\beta(z)|0\rangle$$

$$j^\mu(x)\equiv e\,\bar\psi\gamma^\mu\psi$$

$$C^{\mu}_{\alpha\beta}(k,p',p)=iZ_1\int d^4x\,d^4y\,d^4z\ \exp(-ikx+ip'y-ipz)\,\langle 0|T\ e\bar\psi(x)\gamma^\mu\psi(x)\,\psi_\alpha(y)\,\bar\psi_\beta(z)|0\rangle$$

Contraction in the R-H-S.

回顾, Dirac Field Feynman Rules.

$$x'\ \otimes\!\longrightarrow\!\otimes\ x = \int d^4x\,d^4x'\ i\bar\eta(x)\,\frac{1}{i}S(x-x')\,i\eta(x')$$

$$= \int d^4x\,d^4x'\ i\bar\eta(x)\,\underbrace{\langle\psi(x)\bar\psi(x')\rangle}\,i\eta(x')$$

$$\langle \underbrace{\psi_\alpha(x)}\,\bar\psi_\beta(x')\rangle \sim \frac{1}{i}\frac{\delta}{\delta\bar\eta(x)}\times i\frac{\delta}{\delta\eta(x')}\ Z[\eta,\bar\eta]$$

$$= \frac{1}{i}S_{\alpha\beta}(x-x')$$

$$\underset{\beta}{x'}\longrightarrow\underset{\alpha}{x}$$

$$S(x-y)_{\alpha,\beta}=\int\frac{d^4p}{(2\pi)^4}\exp(-ip\cdot(x-y))\frac{(\not p+m)}{-p^2+m^2-i\varepsilon}.$$

过程 $p_1 \to p_2$ LSZ Reduction (只是用于回顾,无实际作用).

$$\langle f|i\rangle = (i)^2 \int d^4x\, d^4y \;\langle 0|T\, \bar\psi(x)(i\overleftarrow{\not\partial}_x + m) u_s(p_1)\, e^{-ip_1\cdot x}$$
$$e^{ip_2\cdot y}\, \bar u_s(p_2)(-i\not\partial_y + m)\psi(y)\,|0\rangle$$

$$= (i)^2 \int d^4x\, d^4y \; e^{-ip_1\cdot x}\, e^{ip_2\cdot y}\, (i\not\partial_x + m)(-i\not\partial_y + m)\,\langle 0|T\,\bar\psi(x) u_s(p_1)\bar u_s(p_2)\psi(y)|0\rangle$$

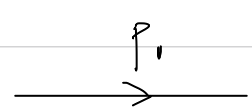
$$= (i)^2 \int d^4x\, d^4y\; u_{\alpha'}(p_1)\bar u_{\beta'}(p_2)\, e^{-ip_1\cdot x}\, e^{ip_2\cdot y}\,(i\not\partial_x + m)_{\alpha\alpha'}(-i\not\partial_y + m)_{\beta'\beta}\,\underbrace{\langle 0|T\,\bar\psi_\alpha(x)\psi_\beta(y)|0\rangle}_{\frac{1}{i}S_{\beta\alpha}(y-x)}$$

$$= (i)\int d^4x\, d^4y\, d^4p\;\; u_{\alpha'}(p_1)\bar u_{\beta'}(p_2)\, e^{-ip_1\cdot x}\, e^{ip_2\cdot y}\,(i\not\partial_x + m)_{\alpha\alpha'}(-i\not\partial_y + m)_{\beta'\beta}$$
$$\frac{1}{(2\pi)^4}\cdot \exp(-ip\cdot(y-x))\,\frac{(\not p + m)_{\beta\alpha}}{-p^2+m^2-i\epsilon}$$

$$= (i)\int d^4x\, d^4y\, d^4p\;\; u_{\alpha'}(p_1)\bar u_{\beta'}(p_2)\, e^{-i(p_1-p)\cdot x}\, e^{i(p_2-p)\cdot y}\,(-\not p + m)_{\alpha\alpha'}(-\not p + m)_{\beta'\beta}\,\frac{(\not p + m)_{\beta\alpha}}{-p^2+m^2-i\epsilon}$$

$$= (i)(2\pi)^4\cdot\delta^{(4)}(p_1-p_2)\cdot u_{\alpha'}(p_1)\bar u_{\beta'}(p_1)\,(-\not p_1 + m)_{\alpha\alpha'}(-\not p_1 + m)_{\beta'\beta}\,\frac{(\not p_1 + m)_{\beta\alpha}}{-p_1^2+m^2-i\epsilon}$$

$$= (i)(2\pi)^4\,\delta^{(4)}(p_1-p_2)\;\bar u(p_1)(-\not p_1 + m)\frac{(\not p_1 + m)}{-p_1^2+m^2-i\epsilon}(-\not p_1 + m)\,u(p_1)$$



Notice (虽然不用,但也说下). $(-\not p + m)(\not p + m) = m^2 - \not p\not p = m^2 - p^2$

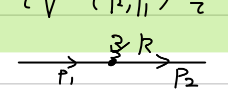
back To relation

$$C^\mu_{\alpha\beta}(k,p',p) \equiv iZ_1\int d^4x\, d^4y\, d^4z\;\exp(-ikx + ip'y - ipz)$$
$$\langle 0|T\; e\bar\psi(x)\gamma^\mu\psi(x)\,\psi_\alpha(y)\,\bar\psi_\beta(z)|0\rangle$$

$$= iZ_1\int d^4x\, d^4y\, d^4z\;\exp(-ik\cdot x + ip'\cdot y - ip\cdot z) \quad \text{in coming}$$
$$\langle 0|T\; e\;\psi_\alpha(y)\;\bar\psi(x)\gamma^\mu\psi(x)\;\bar\psi_\beta(z)|0\rangle$$

$$= -\int d^4p_1\, d^4p_2\;\; d^4x\, d^4y\, d^4z\;\;\exp(-ik\cdot x + ip'\cdot y - ipz)$$
$$\frac{1}{(2\pi)^8}\;\frac{1}{i}\tilde S(p_2)\, iV^\mu(p_2,p_1)\,\frac{1}{i}\tilde S(p_1)\;\exp(-ip_1\cdot(x-z))\exp(-ip_2\cdot(y-x))$$

这个表达式有点没道理!



$$= \frac{-1}{i}\;(2\pi)^4\int d^4p_1\, d^4p_2\cdot\delta^{(4)}(k+p_1-p_2)\,\delta^{(4)}(p'-p_2)\,\delta^{(4)}(p_1-p)\cdot$$
$$\tilde S(p_2)\,V^\mu(p_2,p_1)\,\tilde S(p_1)$$

$$= i\;(2\pi)^4\,\delta^{(4)}(k+p-p')\;\;\tilde S(p')\;V^\mu(p',p)\;\tilde S(p)$$

∘ Consider The Term.

$$k_\mu C^\mu_{\alpha\beta}(k, p', p) = k_\mu i Z_1 \int d^4x\, d^4y\, d^4z \; \exp(-ikx + ip'y - ipz)$$
$$\langle 0 | T \; e\bar{\psi}(x)\gamma^\mu\psi(x)\, \psi_\alpha(y)\, \bar{\psi}_\beta(z) | 0 \rangle$$
$$j^\mu(x) \equiv e\bar{\psi}(x)\gamma^\mu\psi(x)$$
$$= i Z_1 \int d^4x\, d^4y\, d^4z \; \exp(-ik\cdot x + ip'\cdot y - ip\cdot z)$$
$$k_\mu \langle 0 | T\, j^\mu(x)\, \psi_\alpha(y)\, \bar{\psi}_\beta(z) | 0 \rangle$$
$$= -Z_1 \int d^4x\, d^4y\, d^4z \left[\partial_{\mu_x} \exp(-ik\cdot x + ip'\cdot y - ip\cdot z)\right]$$
$$\langle 0 | T\, j^\mu(x)\, \psi_\alpha(y)\, \bar{\psi}_\beta(z) | 0 \rangle$$
$$= Z_1 \int d^4x\, d^4y\, d^4z \; \exp(-ik\cdot x + ip'\cdot y - ip\cdot z)$$
$$\partial_{\mu x} \langle 0 | T\, j^\mu(x)\, \psi_\alpha(y)\, \bar{\psi}_\beta(z) | 0 \rangle$$

Ward – Takahashi Identity.

$$0 = -i\partial_\mu \langle 0 | T\, j^\mu(x)\, \varphi_{a_1}(x_1) \cdots \varphi_{a_n}(x_n) | 0 \rangle$$
$$+ \sum_{j=1}^{n} \langle 0 | T \varphi_{a_1}(x_1) \cdots \delta\varphi_{a_j}(x)\, \delta^{(4)}(x - x_j) \cdots \varphi_{a_n}(x_n) | 0 \rangle$$

Conserved current from $\delta\psi$

$$\mathcal{L} = \bar{\psi}(iZ_2 \not{\partial} - Z_m m)\psi - Z_3 \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$
$$- Z_1 e \bar{\psi}\gamma^\mu A_\mu \psi - \frac{Z_3}{2\xi}(\partial_\mu A^\mu)^2$$

U(1) Transformation

$$\psi \longrightarrow \psi \exp(ie\Gamma)$$
$$\bar{\psi} \longrightarrow \bar{\psi} \exp(-ie\Gamma)$$

Noether conserved current

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x) \times \delta\varphi_a(x)$$
$$= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta\bar{\psi}$$
$$= iZ_2\bar{\psi}\gamma^\mu (ie\Gamma\psi)$$
$$= -e\Gamma Z_2 \bar{\psi}\gamma^\mu\psi$$

Relation between conserved current & $\delta\psi$

$$j^\mu(x) = -e\Gamma Z_2 \bar{\psi}\gamma^\mu\psi \iff \delta\psi = ie\Gamma\psi$$
$$j^\mu(x) = e\bar{\psi}\gamma^\mu\psi \iff \delta\psi = -ie\frac{1}{Z_2}\psi$$
$$\delta\bar{\psi} = +ie\frac{1}{Z_2}\bar{\psi}$$

$$k_\mu C^\mu_{\alpha\beta}(k, p', p) = Z_1 \int d^4x\, d^4y\, d^4z \; \exp(-ik\cdot x + ip'\cdot y - ip\cdot z)$$
$$\partial_{\mu x} \langle 0 | T\, j^\mu(x)\, \psi_\alpha(y)\, \bar{\psi}_\beta(z) | 0 \rangle$$
$$= \frac{1}{i} Z_1 \int d^4x\, d^4y\, d^4z \; \exp(-ik\cdot x + ip'\cdot y - ip\cdot z)$$
$$\Big(\langle 0 | T \,(-ie\tfrac{1}{Z_2})\psi_\alpha(x)\, \delta^{(4)}(x-y)\; \bar{\psi}_\beta(z) | 0 \rangle$$
$$+ \langle 0 | T\, \psi_\alpha(y)\, (ie\tfrac{1}{Z_2})\, \bar{\psi}_\beta(x)\, \delta^{(4)}(x-z) | 0 \rangle \Big)$$

$$= -\frac{e}{Z_2} Z_1 \int d^4x\, d^4y\, d^4z \quad \exp(-ik\cdot x + ip'\cdot y - ip\cdot z)$$
$$\Big(\langle 0| T\, \psi_\alpha(x) \bar\psi_\beta(z) |0\rangle\, \delta^{(4)}(x-y)$$
$$- \langle 0| T\, \psi_\alpha(y) \bar\psi_\beta(x) |0\rangle\, \delta^{(4)}(x-z) \Big)$$

$$= -\frac{e}{Z_2} Z_1 \Big(\int d^4x \int d^4z\, \langle 0|T \psi_\alpha(x)\bar\psi_\beta(z)|0\rangle \exp(i(p'-k)\cdot x - ip\cdot z)$$
$$- \int d^4x\, d^4y\, \langle 0|T\, \psi_\alpha(y)\bar\psi_\beta(x)|0\rangle \exp(-i(p+k)\cdot x + ip'\cdot y) \Big)$$

$$= -\frac{1}{i}\frac{e}{Z_2} Z_1 \Big(\int d^4x \int d^4z \int d^4p_1\, \frac{1}{(2\pi)^4} S_{\alpha\beta}(p_1)\, e^{-ip_1\cdot(x-z)} \exp(i(p'-k)\cdot x - ip\cdot z)$$
$$- \int d^4x\, d^4y \int d^4p_1 \frac{1}{(2\pi)^4} S_{\alpha\beta}(p_1)\, e^{-ip_1\cdot(y-x)} \exp(-i(p+k)\cdot x + ip'\cdot y) \Big)$$

$$= i\, \frac{e}{Z_2} Z_1 \Big(\int d^4p_1\, (2\pi)^4 \cdot S_{\alpha\beta}(p_1)\, \delta^{(4)}(p'-k-p_1)\, \delta^{(4)}(p_1-p)$$
$$- \int d^4p_1\, (2\pi)^4\, S_{\alpha\beta}(p_1)\, \delta^{(4)}(p_1-p-k)\, \delta^{(4)}(p'-p_1) \Big)$$

$$= i\, \frac{e}{Z_2} Z_1\, (2\pi)^4 \cdot \big(S_{\alpha\beta}(p) - S_{\alpha\beta}(p') \big)\, \delta^{(4)}(p'-k-p)$$

On the other hand.

$$k_\mu C^\mu_{\alpha\beta}(k,p',p) = -i \quad (2\pi)^4\, \delta^{(4)}(k+p-p') \quad \hat S(p')\, V^\mu(p',p)\, \hat S(p)$$

Relation between exact propagator & Vertex

$$i(2\pi)^4\, \delta^{(4)}(k+p-p')\, \hat S(p')\, k_\mu V^\mu(p',p)\, \hat S(p)$$
$$= i\, \frac{e}{Z_2} Z_1\, (2\pi)^4 \cdot \big(S_{\alpha\beta}(p) - S_{\alpha\beta}(p') \big)\, \delta^{(4)}(p'-k-p)$$

$$\hat S(p')(p'-p)_\mu\, V^\mu(p',p)\, \hat S(p) = \frac{e}{Z_2} Z_1\, \big(S_{\alpha\beta}(p) - S_{\alpha\beta}(p') \big)$$
$$(p'-p)_\mu\, \hat S(p') V^\mu(p',p)\, \hat S(p) = e Z_2^{-1} Z_1\, \big(\hat S(p) - \hat S(p') \big)$$
$$(p'-p)_\mu\, V^\mu(p',p) = e Z_2^{-1} Z_1\, \big(\hat S^{-1}(p') - \hat S^{-1}(p) \big)$$

有点迷了呢： $V, \hat S$ are finite $\Rightarrow Z_1/Z_2$ finite $\Rightarrow$ 由于又 Z_1, Z_2 correction are infinite $\Rightarrow Z_1 = Z_2$

- Better understand： when $Z_1 = Z_2 \Rightarrow (-Z_1 e \bar\psi \not{A} \psi),\ (iZ_2 \bar\psi \not\partial \psi) \Rightarrow iZ_2 \bar\psi \not{D} \psi.$

$$D^\mu = \partial^\mu + ieA^\mu$$

practice, counterterm Feynman Rules from lagrangian

- QED lagrangian And Basic information

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^{\mu}A_{\mu}\psi$$

Generating function and functional derivative

$$\psi \Rightarrow \frac{1}{i}\frac{\delta}{\delta\bar{\eta}}$$
$$\bar{\psi} \Rightarrow i\frac{\delta}{\delta\eta}$$

$$Z_{spin-1} = \exp\left\{ i\frac{1}{2}\int d^4x\, d^4x'\; J_{\mu}(x)\Delta_F^{\mu\nu}(x-x')J_{\nu}(x')\right\} \qquad \bar{\psi} \Rightarrow i\frac{\delta}{\delta\eta} \quad \psi \Rightarrow \frac{1}{i}\frac{\delta}{\delta\bar{\eta}}$$

$$Z_{spin-\frac{1}{2}} = \exp\left\{ \int d^4x\, d^4y\; i\bar{\eta}(x)\frac{1}{i}S(x-y)i\eta(y)\right\}$$

$$\frac{1}{i}S(x-y)_{\alpha\beta} = \langle 0|\psi_{\alpha}(x)\bar{\psi}_{\beta}(y)|0\rangle = \frac{1}{i}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{(\not{p}+m)}{-p^2+m^2-i\epsilon}$$

$$\int d^4x\, d^4y\; i\bar{\eta}(x)\frac{1}{i}S(x-y)i\eta(y) = \int d^4x\, d^4y\; i\bar{\eta}_{\alpha}(x)\langle 0|\psi_{\alpha}(x)\bar{\psi}_{\beta}(y)|0\rangle i\eta_{\beta}(y)$$

- QED lagrangian with counter term.

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - Z_1 e\,\tilde{\mu}^{\varepsilon/2}\bar{\psi}\gamma^{\mu}A_{\mu}\psi - \frac{1}{2}(\partial_{\mu}A^{\mu})^2$$
$$+ i(Z_2-1)\bar{\psi}\not{\partial}\psi - (Z_m-1)m\bar{\psi}\psi - \frac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu}$$

Generating Function

$$\int d^4x\, d^4y\; i\bar{\eta}(x)\frac{1}{i}S(x-y)i\eta(y)$$

1° Term $\quad i(Z_2-1)\bar{\psi}\not{\partial}\psi$

Generating function

$$\exp\left\{ \int d^4x\, d^4y\; i\bar{\eta}(x)\frac{1}{i}S(x-y)i\eta(y)\right\}$$

propagator with counter term

$$\int d^4y_0\, d^4x\; i\bar{\eta}(y_0)\frac{1}{i}S(y_0-x)i\eta(x)\int d^4x\, d^4y\; i\bar{\eta}(x)\frac{1}{i}S(x-y)i\eta(y)$$

Counterterm functional derivative. $\qquad \boxed{\exp(iS)}$

$$(i)\int d^4x\; i(Z_2-1)\, i\frac{\delta}{\delta\eta}\not{\partial}\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}$$

$\frac{1}{2!}$ factor from exponential expansion of Generating Function Canceled by two term caused by different derivative mode.

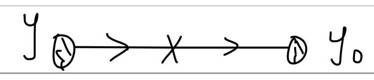
Term :

$$(i)\int d^4x\; i(Z_2-1)\, i\frac{\delta}{\delta\eta}\not{\partial}\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\left\{\int d^4y_0\, d^4x\; i\bar{\eta}(y_0)\frac{1}{i}S(y_0-x)i\eta(x)\int d^4x\, d^4y\; i\bar{\eta}(x)\frac{1}{i}S(x-y)i\eta(y)\right\}$$

$$= (i)\int d^4x\; i(Z_2-1)\not{\partial}_x\int d^4y_0\, i\bar{\eta}(y_0)\frac{1}{i}S(y_0-x)\int d^4y\;\frac{1}{i}S(x-y)i\eta(y)$$

$$= (i)\int d^4x\int d^4y\int d^4y_0\; i(Z_2-1)\; i\bar{\eta}(y_0)\frac{1}{i}S(y_0-x)\not{\partial}_x\frac{1}{i}S(x-y)i\eta(y)$$

Denote As



Propagator with counterterm

$$(i)\int d^4x \quad i(Z_2-1)\quad \frac{1}{i}S(y_0-x)\,\not\partial_x\,\frac{1}{i}S(x-y) \;=\; y \longrightarrow \times \longrightarrow y_0$$

$$= \langle \psi(y_0)\,\bar\psi(y)\rangle$$

In momentum space (coordinate space propagator represented by momentum propagator)

$$(i)\int d^4x\,\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\; i(Z_2-1)\,\frac{1}{i}\frac{\not p_1+m}{-p_1^2+m^2-i\epsilon}\exp\{-ip_1\cdot(y_0-x)\}\;\not\partial_x\;\frac{1}{i}\frac{\not p_2+m}{-p_2^2+m^2-i\epsilon}\exp\{-ip_2\cdot(x-y)\}$$

$$= (i)\,\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\; i(Z_2-1)\,\frac{1}{i}\frac{\not p_1+m}{-p_1^2+m^2-i\epsilon}\exp\{-ip_1 y_0\}\,(-i\not p_2)\,\frac{1}{i}\frac{\not p_2+m}{-p_2^2+m^2-i\epsilon}\exp\{ip_2 y\}\,(2\pi)^4\delta^{(4)}(p_1-p_2)$$

$$= (i)\,\frac{d^4p_1}{(2\pi)^4}\qquad i(Z_2-1)\,\frac{1}{i}\frac{\not p_1+m}{-p_1^2+m^2-i\epsilon}\exp\{-ip_1\cdot(y_0-y)\}\,(-i\not p_1)\,\frac{1}{i}\frac{\not p_1+m}{-p_2^2+m^2-i\epsilon}$$

$$= \int\frac{d^4p_1}{(2\pi)^4}\quad(-1)\,(Z_2-1)\cdot\frac{1}{i}\frac{\not p_1+m}{-p_1^2+m^2-i\epsilon}\quad(-i\not p_1)\quad\frac{1}{i}\frac{\not p_1+m}{-p_1^2+m^2-i\epsilon}\quad\exp\{-ip_1\cdot(y_0-y)\}$$

Vertex in momentum space

$$(-1)\,(Z_2-1)\,(-i\not p) = i(Z_2-1)\not p$$

▽ Another way, wick contraction methdo $\mathcal{L}=\mathcal{L}_0+\mathcal{L}$

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \langle\phi(x_1)\cdots\phi(x_n)\exp(-i\int\mathcal{H}_I\;)\rangle$$

$$\sim\langle\phi(x_1)\cdots\phi(x_n)\exp(i\int\mathcal{L}_I)\rangle$$

$$\mathcal{L}_I \sim i(Z_2-1)\bar\psi\not\partial\psi$$

$$\langle\psi(x')\,\bar\psi(x'')\rangle = \langle\psi(x')(i)\,i(Z_2-1)\bar\psi\not\partial\psi\;\;\bar\psi(x'')\rangle$$

$$= \int d^4x\,\frac{1}{i}S(x'-x)\;\; i(Z_2-1)\not\partial_x\;\frac{1}{i}S(x-x'')$$

In momentum space

$$\frac{1}{i}\;\frac{\not p+m}{-p^2+m^2-i\epsilon}\,(i)\,i(Z_2-1)\,(-i\not p)\;\frac{1}{i}\;\frac{\not p+m}{-p^2+m^2-i\epsilon}$$

Counterterm $\sim i(Z_2-1)\not p$

Term $\mathcal{L}_I = -(Z_m-1)m\bar\psi\psi$

propagator with counterterm

$$\langle\psi(x')\bar\psi(x'')\rangle = \langle\underbrace{\psi(x')\int d^4x\,-(i)\,(Z_m-1)m\bar\psi}\underbrace{\psi\;\;\bar\psi}(x'')\rangle$$

$$= \int d^4x\;(-i)(Z_m-1)\;\frac{1}{i}S(x'-x)\,\frac{1}{i}S(x-x'')$$

Momentum space

$$(-i)(Z_m-1)\qquad = -i(Z_m-1)$$

—— Term $\int d^4x\,(Z_3-1) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$

$$\int d^4x\; -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int d^4x\,(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\frac{1}{4}\int d^4x\,(\partial_\mu A_\nu\,\partial^\mu A^\nu + \partial_\nu A_\mu\,\partial^\nu A^\mu - \partial_\nu A_\mu\,\partial^\mu A^\nu - \partial_\mu A_\nu\,\partial^\nu A^\mu)$$

$$= \frac{1}{2}\int d^4x\,(\partial_\nu A_\mu\,\partial^\mu A^\nu - \partial_\mu A_\nu\,\partial^\mu A^\nu)$$

$$= \frac{1}{2}\int d^4x\,(-\partial_\nu\partial^\mu (A_\mu)\,A^\nu + A_\nu\,\partial^2 A^\nu)$$

$$= \frac{1}{2}\int d^4x\; A^\mu(-\partial_\mu\partial_\nu + g_{\mu\nu}\partial^2)A^\nu$$

$$= \frac{1}{2}\int d^4x\; A^\mu(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu$$

$$\int d^4x\,(Z_3-1)\,\frac{1}{2}A^\mu(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\nu$$

propagator with counterterm

$$\langle A^\mu(x')A^\nu(x'')\rangle = \langle A^\mu(x')A^\nu(x'')\exp(i\textstyle\int \mathcal{L}_I)\rangle$$

$$= \langle \underbrace{A^\mu(x')\int d^4x\; i(Z_3-1)\frac{1}{2}A^\alpha}(g_{\alpha\beta}\partial^2 - \partial_\alpha\partial_\beta)\underbrace{A^\beta\;A^\nu}(x'')\rangle$$

$$= \frac{1}{2}i(Z_3-1)\cdot\frac{1}{i}S^{\mu\alpha}(x'-x)(g_{\alpha\beta}\partial^2 - \partial_\alpha\partial_\beta)\frac{1}{i}S^{\beta\nu}(x-x'')$$

Momentum space counterterm

$$\frac{1}{2}i(Z_3-1)(-g_{\alpha\beta}p^2 + p_\alpha p_\beta)$$

$$= -i(Z_3-1)\frac{1}{2}(g_{\alpha\beta}p^2 - p_\alpha p_\beta)$$

反常磁矩

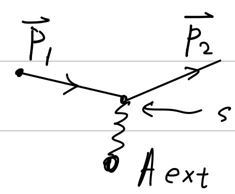
- Lagrangian with External Field

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^{\mu}\psi(A_{ext} + A)_{\mu}$$

Normalized lagrangian

$$\mathcal{L} = \bar{\psi}(iZ_2\not{\partial} - Z_m m)\psi - \frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} - Z_1 e\bar{\psi}\gamma^{\mu}\psi (A_{ext} + A)_{\mu}$$

Consider process, electron scattered by external Field.



← Normalised propagator

← skelton expansion (Normalised 3- vertex).

$$\langle P_2|S|P_1\rangle = -\int d^4x\, d^4y\; \bar{u}(p')(-i\overleftarrow{\not{\partial}} + m)\langle 0|T\,\psi(y)\bar{\psi}(x)|0\rangle(i\overleftarrow{\not{\partial}} + m)u(p)\, e^{iP_2 y - iP_1 X}$$

$$= -\int d^4x d^4y d^4z\, d^4p\, d^4p'\;\bar{u}(p')(-i\overleftarrow{\not{\partial}}_y + m)\frac{1}{(2\pi)^4}\frac{\not{p}' + m}{-(p')^2 + m^2}\exp(-ip'\cdot(y - z))\, e^{ip'y - ipX}$$
$$A^{ext}_{\mu}(z)\;\left(\frac{1}{i}\right)^2 i\Gamma^{\mu}(p', p)\frac{1}{(2\pi)^4}\frac{\not{p} + m}{-p^2 + m^2}(i\overrightarrow{\not{\partial}}_x + m)\;\exp(-ip\cdot(z - x))u(p)$$

$$= -\int d^4x d^4y d^4z\; d^4p\, d^4p'\;\bar{u}(p')(-\not{p}' + m)\frac{1}{(2\pi)^4}\frac{\not{p}' + m}{-(p')^2 + m^2}\exp(-ip'\cdot(y - z))\, e^{ip_2 y - iP_1 X}$$
$$A^{ext}_{\mu}(z)\;\left(\frac{1}{i}\right)^2 i\Gamma^{\mu}(p', p)\frac{1}{(2\pi)^4}\frac{\not{p} + m}{-p^2 + m^2}(-\not{p} + m)\;\exp(-ip\cdot(z - x))u(p)$$

$$= \frac{1}{(2\pi)^8}\frac{1}{i}\cdot\int d^4x\, d^4y\, d^4p\, d^4p' d^4z\;\bar{u}(p')\;A^{ext}_{\mu}(z)\;\Gamma^{\mu}(p', p)\;u(p)\;e^{iP_2 y - iP_1 X}$$
$$\exp(-ip'\cdot(y - z))\quad\exp(-ip\cdot(z - x))$$

$$= -i\int d^4p\, d^4p'\, d^4z\,\delta^{(4)}(p - P_1)\,\delta^{(4)}(P_2 - p')\;\bar{u}(p')\;A^{ext}_{\mu}(z)\;\Gamma^{\mu}(p', p)\;u(p)$$
$$\exp(i(p' - p)\cdot z)$$

$$= -i\int d^4z\;A^{ext}_{\mu}(z)\;\bar{u}(P_2)\;\Gamma^{\mu}(P_2, P_1)u(P_1)\;\exp(i(P_2 - P_1)\cdot z)$$

> Fourier Transformation of external Field.(外场只与空间有关).
> $$\int d^4z\;A^{ext}_{\mu}(z)\;\exp(i(P_2 - P_1)\cdot z)$$
> $$= \int dz^0\int d^3\vec{z}\quad\exp(-i(\vec{P}_2 - \vec{P}_1)\cdot\vec{z})\,A^{ext}_{\mu}(\vec{z})\quad\exp(i(P_2^0 - P_1^0)z^0)$$
> $$= (2\pi)\,\delta(P_2^0 - P_1^0)\quad A^{ext}_{\mu}(\vec{P}_2 - \vec{P}_1)$$

$$= -i(2\pi)\,\delta(P_2^0 - P_1^0)\;A^{ext}_{\mu}(\vec{P}_2 - \vec{P}_1)\;\bar{u}(P_2)\Gamma^{\mu}(P_2, P_1)u(P_1)$$

Scattering Probability.

$$P = \frac{|\langle P_2|S|P_1\rangle|^2}{\langle P_2|P_2\rangle\langle P_1|P_1\rangle} = \frac{(2\pi)^2\,\delta(P_2^0 - P_1^0)\,\delta(0)\cdot\left|A^{ext}_{\mu}(\vec{P}_2 - \vec{P}_1)\;\bar{u}(P_2)\Gamma^{\mu}(P_2, P_1)u(P_1)\right|^2}{2E_1 V\; 2E_2 V}$$

$$dP = \frac{(2\pi)^2 \delta(P_2^0 - P_1^0)\,\delta(0)\cdot \left| A_\mu^{ext}(\vec{P}_2 - \vec{P}_1)\ \bar{u}(P_2)\Gamma^\mu(P_2, P_1)u(P_1)\right|^2}{2E_1 V\ 2E_2 V} \frac{V}{(2\pi)^3} d^3P_2$$

$$= \frac{(2\pi)\,\delta(P_2^0 - P_1^0)\, T \left| A_\mu^{ext}(\vec{P}_2 - \vec{P}_1)\ \bar{u}(P_2)\Gamma^\mu(P_2, P_1)u(P_1)\right|^2}{(2E_1)(2E_2)V} \frac{1}{(2\pi)^3} d^3P_2$$

$$d\sigma = \frac{V}{T}\frac{E_1}{|\vec{P}_1|} dP$$

$$= \frac{(2\pi)\,\delta(P_2^0 - P_1^0)\left| A_\mu^{ext}(\vec{P}_2 - \vec{P}_1)\ \bar{u}(P_2)\Gamma^\mu(P_2, P_1)u(P_1)\right|^2}{2\,|\vec{P}_1|} \frac{1}{(2\pi)^3} \frac{d^3P_2}{2E_2}$$

$$= \frac{(2\pi)\,\delta(P_2^0 - P_1^0)\left| A_\mu^{ext}(\vec{P}_2 - \vec{P}_1)\ \bar{u}(P_2)\Gamma^\mu(P_2, P_1)u(P_1)\right|^2}{2\,|\vec{P}_1|} \frac{1}{(2\pi)^3} \frac{P_2^2\, dP_2\, d\Omega}{2E_2}$$

$$= \frac{(2\pi)\left| A_\mu^{ext}(\vec{P}_2 - \vec{P}_1)\ \bar{u}(P_2)\Gamma^\mu(P_2, P_1)u(P_1)\right|^2}{2\,|\vec{P}_1| \times \left(\frac{dP_2^0}{dP_2}\right)} \frac{1}{(2\pi)^3} \frac{|\vec{P}_1|^2\, d\Omega}{2E_2}$$

$$\frac{d\sqrt{m^2+P_2^2}}{dP_2} = \frac{P_2}{E_2} = \frac{|\vec{P}_1|}{E_1}$$

$$= \frac{1}{16\pi^2} \left| A_\mu^{ext}(\vec{P}_2 - \vec{P}_1)\ \bar{u}(P_2)\Gamma^\mu(P_2, P_1)u(P_1)\right|^2$$

- boost spinor. Lorentz Transform of spinor.

$$R_L = \exp\left(\tfrac{1}{2}(-s^i - i t^i)\sigma^i\right)$$

$$R_R = \exp\left(\tfrac{1}{2}(s^i - i t^i)\sigma^i\right)$$

Lorentz

$$R = \exp(t^i \underbrace{\tilde{J}_i}_{\text{rotate}} + s^i \underbrace{\tilde{K}_i}_{\text{boost}})$$

Lorentz Trans of boost

$$\Lambda = \exp(s^i \tilde{K}_i) \simeq (1 + s^i \tilde{K}_i)$$

Lorentz Trans of Spinor

$$R = \begin{pmatrix} \exp(-\frac{1}{2}s^i\sigma^i) & 0 \\ 0 & \exp(\frac{1}{2}s^i\sigma^i) \end{pmatrix}$$

—— s^i 与 boost 后的动量的关系.

$$\exp\left\{ 1 + s^1\begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} + s^2\begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix} + s^3\begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix} \right\}\begin{pmatrix} m\\0\\0\\0 \end{pmatrix} = \begin{pmatrix} m\\P_1\\P_2\\P_3 \end{pmatrix}$$

—— boost 不影响 Helicity 本征态. $\left\{ u_s(0) = \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}\quad v(0) = \begin{pmatrix} \xi^{-s} \\ -\xi^{-s} \end{pmatrix} \right\} \Rightarrow$ Boost 后仍是 Helicity 本征态.

boost 时

$$R = \exp\left[-\tfrac{1}{2}s^i\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}\right]$$

Helicity op:

$$h = \frac{1}{2}\frac{p^i}{|\vec{p}|}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}$$

$[h, R] = 0$.（$\vec{p}$与$\vec{s}$同方向 ⇐ Lorentz Transform 性质）.

—— boost/velocity 很小时.

$$\exp\left\{ 1 + s^1\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + s^2\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix} + s^3\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}\right\}\begin{pmatrix}m\\0\\0\\0\end{pmatrix} = \begin{pmatrix}m\\p_1\\p_2\\p_3\end{pmatrix}$$

$$s^1 = \frac{p^1}{m} \quad s^2 = \frac{p^2}{m} \quad s^3 = \frac{p^3}{m}.$$

—— small velocity boost for Spinor $u(p)$

$$u(\vec{p}) = \left\{1 - \frac{1}{2}\frac{\vec{p}}{m}\cdot\begin{pmatrix}\vec{\sigma} & 0\\ 0 & -\vec{\sigma}\end{pmatrix}\right\}\sqrt{m}\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix}$$

$$= \sqrt{m}\begin{pmatrix}(1 - \frac{\vec{p}\cdot\vec{\sigma}}{2m})\xi^s\\ (1 + \frac{\vec{p}\cdot\vec{\sigma}}{2m})\xi^s\end{pmatrix}$$

$$\bar{u}(\vec{p}) = u^\dagger(\vec{p})\gamma^0 = \sqrt{m}(\xi^{\dagger s}, \xi^{\dagger s})\cdot\left(1 - \frac{\vec{p}}{2m}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & -\vec{\sigma}\end{pmatrix}\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

$$= \sqrt{m}\left(\xi^{\dagger s}(1 - \frac{\vec{p}}{2m}\cdot\vec{\sigma}),\ \xi^{\dagger s}(1 + \frac{\vec{p}}{2m}\cdot\vec{\sigma})\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= \sqrt{m}\left(\xi^{\dagger s}(1 + \frac{\vec{p}}{2m}\cdot\vec{\sigma}),\ \xi^{\dagger s}(1 - \frac{\vec{p}}{2m}\cdot\vec{\sigma})\right)$$

—— Term

$$\bar{u}_{s'}(p_2)\gamma^\mu u_s(p_1) = m\left(\xi^{\dagger s}(1 + \frac{\vec{p}_2}{2m}\cdot\vec{\sigma}),\ \xi^{\dagger s}(1 - \frac{\vec{p}_2}{2m}\cdot\vec{\sigma})\right)\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}(1 - \frac{\vec{p}_1\cdot\vec{\sigma}}{2m})\xi^s\\ (1 + \frac{\vec{p}_1\cdot\vec{\sigma}}{2m})\xi^s\end{pmatrix}$$

$$= m\left[\xi^{\dagger s'}(1 + \frac{\vec{p}_2}{2m}\cdot\vec{\sigma})\sigma^\mu(1 + \frac{\vec{p}_1\cdot\vec{\sigma}}{2m})\xi^s + \xi^{\dagger s'}(1 - \frac{\vec{p}_2}{2m}\cdot\vec{\sigma})\bar{\sigma}^\mu(1 - \frac{\vec{p}_1\cdot\vec{\sigma}}{2m})\xi^s\right]$$

1° $\gamma^\mu = \gamma^0$.

$$\bar{u}_{s'}(p_2)\gamma^\mu u_s(p_1) \doteq 2m\,\xi^{\dagger s'}\xi^s \doteq 2m\,\delta_{s,s'}$$

2° $\gamma^\mu = \gamma^i$

$$\bar{u}_{s'}(p_2)\gamma^\mu u_s(p_1) = m(\xi^{\dagger s'})\left((1 + \frac{\vec{p}_2}{2m}\cdot\vec{\sigma})\sigma^i(1 + \frac{\vec{p}_1\cdot\vec{\sigma}}{2m}) - (1 - \frac{\vec{p}_2}{2m}\cdot\vec{\sigma})\sigma^i(1 - \frac{\vec{p}_1\cdot\vec{\sigma}}{2m})\right)\xi^s$$

$$= m(\xi^{\dagger s'})\left(\frac{\vec{p}_2}{m}\cdot\vec{\sigma}\,\sigma^i + \sigma^i\frac{\vec{p}_1}{m}\cdot\vec{\sigma}\right)\xi^s$$

$$\left\{ \frac{\vec{p}_2}{m}\cdot\vec{\sigma}\,\sigma^i + \sigma^i\,\frac{\vec{p}_1}{m}\cdot\vec{\sigma} \right.$$
$$= \frac{1}{m}\left((p_2)^j \sigma^j \sigma^i + \sigma^i (p_1)^k \sigma^k \right)$$

$$= \frac{\xi^{+s'}}{2}\left\{ (p_2)^j \left(\{\sigma^j, \sigma^i\} + [\sigma^j, \sigma^i] \right) + (p_1)^j \left(\{\sigma^i, \sigma^j\} + [\sigma^i, \sigma^j] \right) \right\} \xi^s$$

$$= \frac{\xi^{+s'}}{2}\left\{ (p_2^j + p_1^j)\{\sigma^j, \sigma^i\} + (p_2^j - p_1^j)[\sigma^j, \sigma^i] \right\} \xi^s$$

$$\{\sigma^1, \sigma^2\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$
$$= 0$$
$$[\sigma^1, \sigma^2] = 2\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\{\sigma^1, \sigma^1\} = 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\{\sigma^i, \sigma^j\} = 2\delta_{ij}$$
$$[\sigma^i, \sigma^j] = 2i\,\varepsilon^{ijk}\sigma^k$$

$$= \xi^{+s'}\left\{ (\vec{p}_2 + \vec{p}_1)^i + (p_2^j - p_1^j)\, i\,\varepsilon^{jik}\sigma^k \right\} \xi^s$$

$$= \left(\xi^{+s'}\right) \left\{ (\vec{p}_2 + \vec{p}_1) - i(\vec{p}_2 - \vec{p}_1)\times\vec{\sigma} \right\}^i \left(\xi^s\right)$$

$$p_1 \to p_2 \quad q = p_2 - p_1$$

◦ 电石兹顶点与形状因子 $(p' \Leftrightarrow p_2,\ p \Leftrightarrow p_1)$

$$\Gamma^\mu(p, p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}}{2m} q_\nu F_2(q^2) \qquad \text{(Peskin 6.33)}$$

$$\bar{u}(p')\Gamma^\mu(p', p)u(p) = \bar{u}(p')\left[\tilde{F}_1(q^2)\gamma^\mu + \frac{i}{m} S^{\mu\nu} q_\nu \tilde{F}_2(q^2) \right] u(p)$$

Gordon Identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p)$$

$$= \bar{u}(p')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\gamma^\mu - F_2(q^2)\frac{p'^\mu + p^\mu}{2m} \right] u(p)$$

$$= (F_1(0) + F_2(0))\,\bar{u}(p')\gamma^\mu u(p)$$
$$- F_2(0)\frac{(p'^\mu + p^\mu)}{2m}\bar{u}(p')u(p)$$

1° $\mu = 0$

$$(F_1(0) + F_2(0))\,\bar{u}_{s'}(p')\gamma^0 u_s(p) - F_2(0)\frac{2m}{2m}\bar{u}_{s'}(p')u_s(p)$$
$$= 2m F_1(0)\,\xi_{s'}^{+}\xi_s$$
$$\approx 2m F_1(0)\,\delta_{s's}$$

2° $\mu = i$

$$(F_1(0)+F_2(0))\,\bar{u}(p')\gamma^{i}u(p) - F_2(0)\frac{(p'+p)^i}{2m}\bar{u}(p')u(p)$$

$$= (F_1(0)+F_2(0))\quad \xi^{\dagger s'}\{(\vec{p}'+\vec{p}) - i(\vec{p}'-\vec{p})\times\vec{\sigma}\}^i\,\xi^s$$

$$- F_2(0)\,\frac{(\vec{p}'+\vec{p})^i}{2m}\,\bar{u}_{s'}(p')u_s(p)$$

$$= F_1(0)\quad(\vec{p}'+\vec{p})^i\,\xi^{\dagger s'}\xi^s$$

$$-(F_2(0)+F_1(0))\,\xi^{\dagger s'}\,i\left((\vec{p}'-\vec{p})\times\vec{\sigma}\right)^i\,\xi^s$$

—— 散射截面

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\left|A^{ext}_\mu(\vec{p}_2-\vec{p}_1)\,\bar{u}(p_2)\Gamma^\mu(p_2,p_1)u(p_1)\right|^2$$

$$= \frac{e^2}{16\pi^2}\Big[\xi^{\dagger s'}\{2m\,F_1(0)A^0 - [F_1(0)\quad(\vec{p}'+\vec{p}) - (F_1(0)+F_2(0))\,i\,((\vec{p}'-\vec{p})\times\vec{\sigma})]\cdot\vec{A}\quad\}\xi^s\Big]^2$$

$$= \frac{m^2e^2}{4\pi^2}\left[\xi^{\dagger s'}\{F_1(0)A^0 - [F_1(0)\tfrac{1}{2m}(\vec{p}+\vec{p}') - i\frac{(F_1(0)+F_2(0))}{2m}(\vec{p}'-\vec{p})\times\vec{\sigma}]\cdot\vec{A}\}\xi^s\right]^2$$

—— compare with born Approx

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{m^2}{4\pi^2}\left|\xi^{\dagger s'}\,V(\vec{q})\,\xi^s\right|^2$$

$$V(\vec{q}) = eF_1(0)A^0 - e\left(\frac{F_1(0)}{2m}(\vec{p}+\vec{p}') - i\frac{(F_1(0)+F_2(0))}{2m}(\vec{p}'-\vec{p})\times\vec{\sigma}\right)\cdot\vec{A}$$

$$A^0 = \phi$$

$$\vec{A}(\vec{p}_2-\vec{p}_1) = \int d^3\vec{x}\quad\exp(-i(\vec{p}_2-\vec{p}_1)\cdot\vec{x})\,\vec{A}^{ext}(\vec{x})$$

$$\vec{A}^{ext}(\vec{x}) = \int d^3\vec{q}\quad\exp(i\vec{q}\cdot\vec{x})\vec{A}(\vec{q})\cdot\frac{1}{(2\pi)^3}$$

$$B = \nabla\times\vec{A}^{ext} = \int d^3\vec{q}\;(i\vec{q})\times\vec{A}(\vec{q})\,\exp(i\vec{q}\cdot\vec{x})\frac{1}{(2\pi)^3}$$

$$B(\vec{p}_2-\vec{p}_1) = B(\vec{p}'-\vec{p}) = i(\vec{p}'-\vec{p})\times\vec{A}^{ext}(\vec{p}'-\vec{p})$$

$$-iA\cdot((\vec{p}'-\vec{p})\times\vec{\sigma}) = i(\vec{\sigma})\cdot((\vec{p}'-\vec{p})\times\vec{A}) \sim i\vec{\sigma}\cdot B$$

$$= eF_1(0)\phi - e\left(\frac{F_1(0)}{2m}(p\cdot A + p'\cdot A) + i\frac{(F_1(0)+F_2(0))}{2m}\vec{\sigma}\cdot\vec{B}\right)$$

Anomaly Magnetic Moment

$$\mu = \frac{e}{2m}\left(2F_1(0) + 2F_2(0)\right).$$

Form Factor from Dirac equation

- Dirac Equation and magnetic momentum

$$(i\not{D} - M)\psi = 0$$

$D_\mu = \partial_\mu + ieA_\mu$

$$(i\not{D}+M)(i\not{D}-M)\psi = 0$$

$$\{-\not{D}\not{D} - M^2\}\psi = 0$$

Evoluate

$$\not{D}\not{D} = \gamma^\mu(\partial_\mu + ieA_\mu)\gamma^\nu(\partial_\nu + ieA_\nu)$$

$$= \gamma^\mu\gamma^\nu(\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)$$

$$= \tfrac{1}{2}(\{\gamma^\mu,\gamma^\nu\} - [\gamma^\nu,\gamma^\mu])(\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)$$

$$= \tfrac{1}{2}(2\eta^{\mu\nu} - [\gamma^\nu,\gamma^\mu])(\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)$$

$$= (\partial + ieA)^2 - \tfrac{1}{2}[\gamma^\nu,\gamma^\mu](\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)$$

$$= (\partial + ieA)^2 - \tfrac{1}{2}[\gamma^\nu,\gamma^\mu](\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)$$

$$= (\partial + ieA)^2 + \tfrac{1}{4}[\gamma^\mu,\gamma^\nu]\{(\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) - (\partial_\nu + ieA_\nu)(\partial_\mu + ieA_\mu)\}$$

$$= (\partial + ieA)^2 + \tfrac{1}{4}[\gamma^\mu,\gamma^\nu](ie)(\partial_\mu(A_\nu) - \partial_\nu(A_\mu))$$

$$\{-(\partial + ieA)^2 - \tfrac{1}{4}[\gamma^\mu,\gamma^\nu](ie)(\partial_\mu(A_\nu) - \partial_\nu(A_\mu)) - M^2\}\psi = 0$$

$$(i\partial + eA)^2\psi = \{M^2 + \tfrac{1}{4}[\gamma^\mu,\gamma^\nu](ie)F_{\mu\nu}\}\psi$$

$$= \{M^2 + eS^{\mu\nu}F_{\mu\nu}\}\psi$$

Physical Meaning. Energy of dirac particle.

$$E \sim \frac{e}{2M}S^{\mu\nu}F_{\mu\nu} \sim \frac{e}{4M}\,\frac{i}{2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}$$

z direction Magnetic Field. $\underline{B_z = \partial_x A_y - \partial_y A_x}$ $\quad B^z = \partial_x A_y - \partial_y A_x$

$$\vec{A} = (0,\ By,\ 0,\ 0)$$

$B^\mu = (0,0,0,B) \Rightarrow \vec{B} = \nabla\times\vec{A} \Rightarrow B^3 = \partial_1 A^2 - \partial_2 A^1 \Rightarrow F_{12} = -B$

$B_\mu = (0,0,0,-B)$ $\qquad B = -\partial_1 A_2 + \partial_2 A_1$

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & +B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E \sim \frac{e}{4M}\,i[\gamma^1,\gamma^2]F_{12}$$

$$= -\frac{e}{4M}\,i\left[\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\right]B$$

$$= -i\frac{e}{4M}\left\{\begin{pmatrix} -\sigma^1\sigma^2 & 0 \\ 0 & -\sigma^1\sigma^2 \end{pmatrix} - \begin{pmatrix} -\sigma^2\sigma^1 & 0 \\ 0 & -\sigma^2\sigma^1 \end{pmatrix}\right\}B$$

$$= -i\frac{e}{4M}(2)\begin{bmatrix} -[\sigma^1,\sigma^2] & 0 \\ 0 & -[\sigma^1,\sigma^2] \end{bmatrix}B$$

$$= +i\frac{e}{4M}\cdot 4(i)\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix}B$$

$$= -\frac{e}{M}\begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix}B \sim -\frac{eB}{M}\times\frac{1}{2}$$

$\uparrow$ Spin

○ Pauli Form Factor

Lagrangian

$$\mathcal{L} = \bar{\psi}(iD_\mu\gamma^\mu - m)\psi - \frac{e}{4m}F_2(0)\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

Equation of motion from lagrangian

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) = \frac{\partial\mathcal{L}}{\partial(\bar{\psi})} = 0$$

$$(iD_\mu\gamma^\mu - m)\psi = \frac{e}{4m}F_2(0)\sigma^{\mu\nu}F_{\mu\nu}\psi$$

$$(iD_\mu\gamma^\mu - m)(iD_\mu\gamma^\mu + m)\psi = \frac{e}{4m}F_2(0)\sigma^{\mu\nu}F_{\mu\nu} \quad (iD_\mu\gamma^\mu + m)\psi$$

$$(i\not{D} - m)(i\not{D} + m)\psi = \frac{e}{2}F_2(0)\sigma^{\mu\nu}F_{\mu\nu}\psi$$

$$(-\not{D}\not{D} - m^2)\psi = \frac{e}{2}F_2(0)\sigma^{\mu\nu}F_{\mu\nu}\psi$$

由前文结论

$$(i\partial + eA)^2\psi = \{M^2 + eS^{\mu\nu}F_{\mu\nu} + eF_2(0)S^{\mu\nu}F_{\mu\nu}\}\psi$$

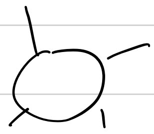
$$E \sim \frac{1}{2M}|e|(1 + F_2(0))S^{\mu\nu}F_{\mu\nu}$$

Form Factor : $1 + F_2(0)$

Dimensional Analysis & Superficial degree of Divergence.

o

$$\mathcal{L} = \frac{1}{2} Z_\varphi \partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2} Z_m m^2 \varphi^2 - \sum_{n=3}^{+\infty} \frac{1}{n!} Z_n g_n \varphi^n$$



E external Lines

I internal lines $\longrightarrow \frac{1}{p^2+m^2} \sim -2I$

L closed Loops $\longrightarrow \frac{d^d l}{(2\pi)^d} \sim dL$

V_n n-type vertex

$-\times- \sim (Z_m-1)m^2 + k^2(Z_\varphi-1)$

$\sim [g_2] = 2$

$-\times-$ ⇒叫作1个Internal Line

Superficial Degree of Divergence

$D \equiv dL - 2I$

有E个外腿的项 $g_E\cdot\varphi^E$

树图结论. $[\text{diagram}] = [g_E]$ 它的树图量纲是$[g_E]$ ⇒ 于是 All 有E个外腿的图量纲为$[g_E]$.

$$[\text{diagram}] = dL - 2I + \sum_{n=3}^{\infty} V_n [g_n]$$

$$D = [\text{diagram}] - \sum_{n=3}^{+\infty} V_n [g_n]$$

$$D = -\sum_{n=3}^{+\infty} V_n [g_n] + [g_E]$$

V_n 代表有 V_n 个 n-type vertex.

如果 $\exists\, [g_i] < 0 \Rightarrow$ 一定发散

Renormalisation procedure

$$\mathcal{L} = \frac{1}{2} Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} Z_m m^2 \varphi^2 - \frac{1}{6} Z_g g \tilde{\mu}^{\varepsilon/2} \varphi^3 + Y \varphi$$
$$\mathcal{L} = \frac{1}{2} \partial^\mu \varphi_0 \partial_\mu \varphi_0 - \frac{1}{2} m_0^2 \varphi_0^2 + Y_0 \varphi_0 - \frac{1}{6} g_0 \varphi_0^3$$

$$\varphi_0 = Z_\varphi^{1/2} \varphi \qquad m_0 = Z_m^{1/2} Z_\varphi^{-1/2} m \qquad g_0 = Z_g \cdot Z_\varphi^{-3/2} g \tilde{\mu}^{\varepsilon/2}$$
$$Y_0 = Y \cdot Z_\varphi^{-1/2}$$

$$Z_\varphi = 1 + \left(-\frac{1}{6}\alpha + O(\alpha^2)\right) \frac{1}{\varepsilon} + \sum_{n=2} \frac{a_n(\alpha)}{\varepsilon^n}$$
$$Z_m = 1 + (-\alpha + O(\alpha^2)) \frac{1}{\varepsilon} + \sum_{n=2} \frac{b_n(\alpha)}{\varepsilon^n}$$
$$Z_g = 1 + (-\alpha + O(\alpha^2)) \frac{1}{\varepsilon} + \sum_{n=2} \frac{c_n(\alpha)}{\varepsilon^n}$$

$$\alpha_0 \equiv \frac{g_0^2}{(4\pi)^3} = Z_g^2 Z_\varphi^{-3} \tilde{\mu}^\varepsilon \alpha$$

$$\ln(\alpha_0) = \underbrace{\ln(Z_g^2 Z_\varphi^{-3})}_{G(\alpha,\varepsilon) = \sum_{n=1}^{+\infty} \frac{G_n(\alpha)}{\varepsilon^n}} + \ln(\alpha) + \varepsilon \ln(\mu) + \varepsilon \ln\left(\frac{1}{\sqrt{4\pi}\, e^{\gamma/2}}\right)$$

$$\frac{\partial G(\alpha,\varepsilon)}{\partial \alpha} \frac{d\alpha}{d\ln\mu} + \frac{1}{\alpha} \frac{d\alpha}{d\ln\mu} + \varepsilon$$

$$\frac{1}{\alpha} \Bigg/ \left(\left(1 + \frac{\alpha G_1'(\alpha)}{\varepsilon} + \frac{\alpha G_2'(\alpha)}{\varepsilon^2} + \cdots\right) \frac{d\alpha}{d\ln\mu} + \varepsilon\alpha \right)$$
$$(-\varepsilon\alpha + \beta(\alpha))$$

$$-\alpha^2 G_1'(\alpha) + \beta(\alpha) = 0$$

m_0 Invariance.

$$\gamma_m(\alpha) \equiv \frac{1}{m} \frac{dm}{d\ln\mu}.$$

Callan - Symmansik - Equation. ...

Relation between QFT & Statistic mechanics

- Generating Functional

$$Z[J] = \int \mathcal{D}\phi \, \exp\left(i\int d^4x\,(\mathcal{L} + J\phi)\right)$$

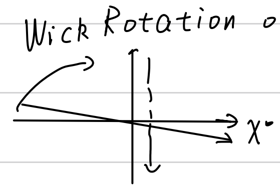
$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = Z^{-1}[J]\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)Z[J]$$

Action

$$S = \int d^4x\left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m\phi^2 - \frac{\lambda}{4!}\phi^4\right)$$

Wick Rotation

Wick Rotation of x （路径积分中 $T\to\infty(1-i\varepsilon)$，取 ε 大时，相当于Rotate积分区间）



$-\infty \to +\infty$ $\overset{\text{等价于}}{\Longleftrightarrow}$ $+i\infty \downarrow -i\infty$

$$x^0 \longrightarrow -i\,x^d$$

$$\left.\begin{aligned}\frac{\partial}{\partial x^0} &\longrightarrow i\frac{\partial}{\partial x^d}\\ \frac{\partial}{\partial x_0} &\longrightarrow i\frac{\partial}{\partial x^d}\end{aligned}\right\} \Rightarrow (\partial_\mu\phi)(\partial^\mu\phi) = -(\partial_E\phi)^2 = -(\partial_E^i\phi)(\partial_E^i\phi)$$

$$d^4x \longrightarrow -i\,d^4x_E$$

Action After wick Rotation

$$\begin{aligned}S &= -i\int d^4x_E\left(-\frac{1}{2}(\partial_{E\mu}\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right)\\ &= i\int d^4x_E\left(\frac{1}{2}(\partial_{E\mu}\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)\\ &\equiv i\int d^4x_E\,\mathcal{L}_E\end{aligned}$$

Generating Function After Wick Rotation.

$$\begin{aligned}Z[J] &= \exp\left\{(i)(-i)\int d^4x_E\left(-\mathcal{L}_E + J\phi\right)\right\}\\ &= \exp\left\{\int d^4x_E\left(-\frac{1}{2}(\partial_{E\mu}\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi\right)\right\}\\ &= \exp\left\{-\int d^4x_E\left(\mathcal{L}_E - J\phi\right)\right\}\end{aligned}$$

- Green's Function After Wick rotation

$$Z[J] = \exp\left\{-\int d^4x_E\left(\frac{1}{2}(\partial_{E\mu}\phi)(\partial_{E\mu}\phi) + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - J\phi\right)\right\}$$

$$\Big\uparrow \quad \phi(x) = \int\frac{d^4k}{(2\pi)^4}\exp(ikx)\,\phi(k)$$

All in Euclidian space! $\Downarrow$

$$= \exp\Big\{-\int d^4x\,d^4k_1\,d^4k_2\Big(\frac{1}{2}(ik_1)\cdot(ik_2)\,\phi(k_1)\phi(k_2) + \frac{1}{2}m^2\phi(k_1)\phi(k_2) - \frac{1}{2}J(k_1)\phi(k_2) - \frac{1}{2}J(k_2)\phi(k_1)\Big)\frac{1}{(2\pi)^8}\cdot e^{i(k_1+k_2)\cdot x}\Big\}$$

$$= \exp\Big\{-\int d^4x\,d^4k_1\,d^4k_2\,\frac{e^{i(k_1+k_2)\cdot x}}{(2\pi)^8}\cdot\Big(-\frac{1}{2}k_1k_2\,\phi(k_1)\phi(k_2) + \frac{1}{2}m^2\phi(k_1)\phi(k_2) - \frac{1}{2}J(k_1)\phi(k_2) - \frac{1}{2}J(k_2)\phi(k_1)\Big)\Big\}$$

$$= \exp\left\{ - \int d^4k_1\, d^4k_2 \, \frac{\delta^{(4)}(k_1+k_2)}{(2\pi)^4} \cdot \left(-\frac{1}{2} k_1 k_2 \phi(k_1)\phi(k_2) + \frac{1}{2} m^2 \phi(k_1)\phi(k_2) - \frac{1}{2} J(k_1)\phi(k_2) - \frac{1}{2} J(k_2)\phi(k_1) \right) \right\}$$

$$= \exp\left\{ - \int d^4k \, \frac{1}{(2\pi)^4} \left(\frac{1}{2} k^2 \phi(k)\phi(-k) + \frac{1}{2} m^2 \phi(k)\phi(-k) - \frac{1}{2} J(k)\phi(-k) - \frac{1}{2} J(-k)\phi(k) \right) \right\}$$

$$\chi(k) \equiv \phi(k) - \frac{J(k)}{k^2+m^2}$$

$$= \exp\left\{ - \int d^4k \, \frac{1}{(2\pi)^4} \left(\frac{1}{2} k^2 \chi(k)\chi(-k) + \frac{1}{2} m^2 \chi(k)\chi(-k) + \frac{1}{2}(k^2+m^2) \frac{J(k)J(-k)}{(k^2+m^2)^2} + \frac{1}{2} J(k)\phi(-k) + \frac{1}{2} J(-k)\phi(k) - \frac{1}{2} J(k)\phi(-k) - \frac{1}{2} J(-k)\phi(k) \right) \right\}$$

$$= \exp\left\{ - \int d^4k \, \frac{1}{(2\pi)^4} \left(\frac{1}{2}(k^2+m^2)\chi(k)\chi(-k) + \frac{1}{2} \frac{J(k)J(-k)}{k^2+m^2} \right) \right\}$$

$$\sim \exp\left\{ - \int d^4k \, \frac{1}{(2\pi)^4} \, \frac{1}{2} \frac{J(k)J(-k)}{k^2+m^2} \right\}$$

$$= \exp\left\{ - \frac{1}{(2\pi)^4} \frac{1}{2} \int d^4k\, d^4k_1\, d^4k_2 \, \delta^{(4)}(k_1-k)\, \delta^{(4)}(k_2+k) \frac{J(k_1)J(k_2)}{k^2+m^2} \right\}$$

$$= \exp\left\{ - \frac{1}{(2\pi)^4} \frac{1}{2} \int d^4k\, d^4k_1\, d^4k_2\, d^4x_1\, d^4x_2 \, \frac{e^{-i x_1\cdot(k_1-k)} \, e^{-i(k_2+k)\cdot x_2}}{(2\pi)^8} \frac{J(k_1)J(k_2)}{k^2+m^2} \right\}$$

$$= \exp\left\{ - \frac{1}{2} \int d^4k\, d^4x_1\, d^4x_2 \, \frac{e^{-ik\cdot(x_2-x_1)}}{(2\pi)^4} \frac{J(x_1)J(x_2)}{k^2+m^2} \right\}$$

$$= \exp\left\{ - \frac{1}{2} \int d^4x_1\, d^4x_2 \, J(x_1)\, \Delta_E(x_2-x_1)\, J(x_2) \right\}$$

Euclidian propagator

$$\Delta_E(x_2-x_1) = \int d^4k \, \frac{1}{(2\pi)^4} \frac{e^{-ik\cdot(x_2-x_1)}}{k^2+m^2}$$

- Use Functional Derivative attain correlation Function in Euclidian Space

$$\langle \phi(x_1)\phi(x_2) \rangle = Z[J]^{-1} \left(-\frac{\delta}{\delta J(x_1)}\right)\left(-\frac{\delta}{\delta J(x_2)}\right)$$

Irreducible Diagram and Effective Action

- Connected-Diagram generating function

$$Z[J] = \exp(-i W[J])$$ ← Connected generating Function normalized by $W[0]=0$.

Generating Function normalized by $Z[0]=1$

- Classical field

$$\phi(x,J] \equiv \frac{\delta W[J]}{\delta J(x)} \Rightarrow J(x) = J(x,\phi]$$

- Effective Action

$$\Gamma[\phi] \equiv -W[J] + \int d^4y\, J(y)\phi(y)$$

—— Derivative of Effective Action

$$\frac{\delta \Gamma[\phi]}{\delta\phi(x)} = -\frac{\delta W[J]}{\delta\phi(x)} + \int d^4y \frac{\delta J(y,\phi]}{\delta\phi(x)}\phi(y) + J(x)$$

$$= -\int d^4y \frac{\delta W[J]}{\delta J(y)} \frac{\delta J(y,\phi]}{\delta\phi(x)} + \int d^4y \frac{\delta J(y,\phi]}{\delta\phi(x)}\phi(y) + J(x)$$

$$= -\int d^4y \left.\frac{\delta W[J]}{\delta J(y)}\right|_{J(x)=J(x,\phi]} \frac{\delta J(y,\phi]}{\delta\phi(x)} + \int d^4y \frac{\delta J(y,\phi]}{\delta\phi(x)} \left.\frac{\delta W[J]}{\delta J(y)}\right|_{J(x)=J(x,\phi]} + J(x)$$

$$= J(x,\phi]$$

—— Averaged field

$$J=0 \Rightarrow W[0]=0 \Rightarrow \Gamma[\phi[J=0]]=0$$

$$\Downarrow$$

$$\phi(x,J=0] = \bar{\Phi}(x) \Rightarrow \Gamma[\bar{\Phi}]=0$$

—— Define Modified Effective Action

$$\hat{\Gamma}[\phi] \equiv \Gamma[\phi+\bar{\Phi}] - \Gamma[\bar{\Phi}]$$

$$\hat{\Gamma}[0] = 0$$

$$\frac{\delta\hat{\Gamma}[\phi]}{\delta\phi(x)} = J(x,\phi+\bar{\Phi}] \Rightarrow \left.\frac{\delta\hat{\Gamma}[\phi]}{\delta\phi(x)}\right|_{\phi=0} = 0$$

—— Note：当 $W[J]$ 对应的 Theory 无 Tadopole diagram 时，不用定义 $\hat{\Gamma}[\phi]$，$\left.\frac{\delta\Gamma[\phi]}{\delta\phi}\right|_{\phi=0}=0$

—— Expansion

$$\hat{\Gamma}[\phi] = \sum_{n=2}^{+\infty} \int d^4x_1\cdots d^4x_n \frac{1}{n!}(i)^{n-1} \frac{\delta\hat{\Gamma}[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)} \left(\frac{1}{i}\right)^{n-1}\phi(x_1)\cdots\phi(x_n)$$

$$W[J] = \sum_{n=1}^{+\infty} \int d^4x_1\cdots d^4x_n \frac{1}{n!}\left(\frac{1}{i}\right)^{n-1} \frac{\delta W[J]}{\delta J(x_1)\cdots\delta J(x_n)} (i)^{n-1} J(x_1)\cdots J(x_n)$$

$$\frac{\delta\phi(x_1)}{\delta\phi(x_2)} = \delta^{(4)}(x_1-x_2)$$

$$\int d^4y \frac{\delta\phi(x_1,J]}{\delta J(y)} \frac{\delta J(y,\phi]}{\delta\phi(x_2)} = \delta^{(4)}(x_1-x_2)$$

$$\Big\downarrow \longleftarrow \Big\} \quad \phi(x_1,J] \equiv \frac{\delta W[J]}{\delta J(x_1)}, \quad \frac{\delta\hat{\Gamma}[\phi-\bar{\Phi}]}{\delta\phi(y)} = J(y,\phi]$$

$$\int d^4y \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(y)} \frac{\delta^2\hat{\Gamma}[\phi-\bar{\Phi}]}{\delta\phi(x_2)\delta\phi(y)} = \delta^{(4)}(x_1-x_2)$$

Variable Substitution $y \to y_1$, $x_2 \to y_2$,

Apply $\frac{\delta}{\delta J(x_3)} = \int d^4y_3 \frac{\delta \phi(y_3, J]}{\delta J(x_3)} \frac{\delta}{\delta \phi(y_3)}$

$= \int d^4 y_3 \frac{\delta^2 W[J]}{\delta J(x_3)\delta J(y_3)} \frac{\delta}{\delta \phi(x_3)}$

$$\int d^4 y_1 \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(y_1)} \frac{\delta^2 \Gamma[\phi - \bar{\phi}]}{\delta \phi(y_2)\delta \phi(y_1)} = \delta^{(4)}(x_1 - y_2)$$

$$\int d^4 y_1 \frac{\delta^3 W[J]}{\delta J(x_1)\delta J(y_1)\delta J(x_3)} \frac{\delta^2 \Gamma[\phi - \bar{\phi}]}{\delta \phi(y_2)\delta \phi(y_1)} + \int d^4y_1 d^4y_3 \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(y_1)} \frac{\delta^2 W[J]}{\delta J(x_3)\delta J(y_3)} \frac{\delta^3 \Gamma[\phi - \bar{\phi}]}{\delta \phi(x_3)\delta \phi(y_2)\delta \phi(y_1)} = 0$$

Apply $\int d^4 y_2 \frac{\delta^2 W[J]}{\delta J(y_2)\delta J(x_2)}$

Use $\int d^4 y \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(y)} \frac{\delta^2 \Gamma[\phi - \bar{\phi}]}{\delta \phi(x_2)\delta \phi(y)} = \delta^{(4)}(x_1 - x_2)$

$$\int d^4 y_1 d^4 y_2 \frac{\delta^3 W[J]}{\delta J(x_1)\delta J(y_1)\delta J(x_3)} \frac{\delta^2 \Gamma[\phi - \bar{\phi}]}{\delta \phi(y_2)\delta \phi(y_1)} \frac{\delta^2 W[J]}{\delta J(y_2)\delta J(x_2)}$$

$$= -\int d^4y_1 d^4y_2 d^4y_3 \frac{\delta^2 W[J]}{\delta J(y_2)\delta J(x_2)} \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(y_1)} \frac{\delta^2 W[J]}{\delta J(x_3)\delta J(y_3)} \frac{\delta^3 \Gamma[\phi - \bar{\phi}]}{\delta \phi(x_3)\delta \phi(y_2)\delta \phi(y_1)}$$

$$\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)}$$

$$= -\int d^4y_1 d^4y_2 d^4y_3 \frac{\delta^2 W[J]}{\delta J(y_2)\delta J(x_2)} \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(y_1)} \frac{\delta^2 W[J]}{\delta J(x_3)\delta J(y_3)} \frac{\delta^3 \Gamma[\phi - \bar{\phi}]}{\delta \phi(x_3)\delta \phi(y_2)\delta \phi(y_1)}$$

(···) (详细见latex笔记. 不算了先). (核心思想: 可用 Effective Action 生成 connected Diagram).

Calculation of Effective Action

- **Quantum Action Definition**

$$\Gamma(\varphi) \equiv \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\, \tilde{\varphi}(-k)\left(-k^2+m^2-\Pi(k^2)\right)\tilde{\varphi}(k)$$
$$+\sum_{n=3}^{+\infty}\frac{1}{n!}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\cdots\frac{d^dk_n}{(2\pi)^d}(2\pi)^d\,\delta^{(d)}(k_1+k_2\cdots+k_n)$$
$$\times\,\tilde{V}_n(k_1,k_2,\cdots k_n)\,\tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\cdots\tilde{\varphi}(k_n)$$

↑ $\tilde{V}$ 中动量都是出射的!

(类似于 Skeleton Expansion.)　　注意，srednicki中 $\mathcal{L}$ 的 interacting term 系数是"+"的.

- **Quantum Action As Lagrangian.**

$$Z_\Gamma[J] \equiv \int \partial\varphi \exp\left\{i\Gamma(\varphi)+i\int d^4x\, J\varphi\right\}$$
$$=\exp\left\{iW_\Gamma[J]\right\}=\exp\left\{i\sum_{L=0}^{+\infty}W_{\Gamma,L}[J]\right\}$$

但 $Z_\Gamma[J]$ 也会生成圈图，(目标.计算 $W_{\Gamma,0}[J]$) ⟶ 它能生成正确 Feynman Diagram.

$$Z_{\Gamma,\hbar}[J] \equiv \int\partial\varphi\ \exp\left\{\frac{i}{\hbar}\left(\Gamma[\varphi]+\int d^dx\, J\varphi\right)\right\}$$

其中，$\hbar$ 是一个标量无量纲数字.

$$Z_{\Gamma,\hbar}[J] = \exp\left(iW_{\Gamma,\hbar}[J]\right)$$

—— $Z_{\Gamma,\hbar}[J]$ 生成图形性质:

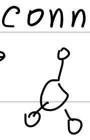
$$Z_{\Gamma,\hbar}[J]=\exp\left\{\frac{i}{\hbar}\frac{1}{n!}\int d^dx_1\,d^dx_2\cdots V(x_1,x_2\cdots)\left(\frac{\hbar}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{\hbar}{i}\frac{\delta}{\delta J(x_2)}\right)\cdots\right\}$$
$$\times\exp\left\{i\frac{1}{2}\int d^dx_1\,d^dx_2\,\frac{J(x_1)}{\hbar}\hbar\Delta(x_1-x_2)\frac{1}{\hbar}J(x_2)\right\}$$

Each vertex ⟶ $\frac{1}{\hbar}$
Each source ⟶ $\frac{1}{\hbar}$　　⟹ Total factor $\hbar^{P-V-E} = \hbar^{L-1}$
Each propagator ⟶ $\hbar$
Relation between L, P, E, V.

结论：$Z_{\Gamma,\hbar}[J]$ 所表示的生成函数 是在
是用 Real-Propagator, Real vertex 的描述场论的相应生成函数上
给 External Source ⟶ $\frac{1}{\hbar}$, vertex ⟶ $\frac{1}{\hbar}$, propagator ⟶ $\hbar$.

—— $iW_{\Gamma,\hbar}[J]$ 是 $Z_{\Gamma,\hbar}[J]$ 对应场论的 connected diagram，现在观察每个 connected diagram 的 $\hbar$ 系数

$$\underbrace{P-V}_{\text{图中要确定的动量数}}=\underbrace{L+E-1}_{\text{自由动量数}}$$

⟶ 
$E=3$　$V=3$　$L=1$
$P=6$
$P-V=3$　$L+E-1=3$

$$P=\underbrace{I}_{\text{Internal Line}}+\underbrace{E}_{\text{External Line/ External Source}}$$

$$W_{\Gamma,\hbar}[J]=\sum_{L=0}\hbar^{L-1}W_{\Gamma,L}[J]\xrightarrow{W_{\Gamma,L}\text{的定义}}W_\Gamma[J]=\sum_{L=0}^{+\infty}W_{\Gamma,L}[J]$$ ⟶ 对应无 $\hbar$ modify 的 Γ

Letting $\hbar\to 0$,

$$W_{\Gamma,\hbar}[J]=\frac{1}{\hbar}W_{\Gamma,0}[J]$$

process: 1° Calculate $W_{\Gamma,\hbar}[J]$ 2° $W_{\Gamma,0}[J] = \lim_{\hbar\to 0} \hbar W_{\Gamma,\hbar}[J]$

—— 用 Saddle point Approximation 方法近似.

$$Z_{\Gamma,\hbar}[J] \equiv \int \partial\varphi \ \exp\left\{ \frac{i}{\hbar} \left(\Gamma[\varphi] + \int d^d x \, J\varphi \right) \right\}$$

Saddle Point Approximation. ($\hbar \to 0$, 任何偏离 from stationary point leads to $+\infty$)

$$Z_{\Gamma,\hbar}[J] = \exp\left\{ \frac{i}{\hbar} \left(\Gamma[\varphi_J] + \int d^d y \, J(y)\varphi_J(y) \right) + O(\hbar^0) \right\} \quad —(1)$$

φ_J Satisfies

$$\frac{\delta}{\delta\varphi(x)} \left[\Gamma[\varphi] + \int d^4 y \, J(y)\varphi(y) \right] \Bigg|_{\varphi=\varphi_J(x)} = \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} + J(x) \Bigg|_{\varphi(x)=\varphi_J(x)} = 0$$

From (1)

$$Z_{\Gamma,\hbar}[J] = \exp\left\{ \frac{i}{\hbar} \left(\Gamma[\varphi_J] + \int d^d y \, J(y)\varphi_J(y) \right) + O(\hbar^0) \right\}$$

$$= \exp\left\{ i \sum_{L=0}^{+\infty} \hbar^{L-1} W_{\Gamma,L}[J] \right\}$$

$$W_{\Gamma,0}[J] = \Gamma[\varphi_J] + \int d^d y \, J(y)\varphi_J(y)$$

$W[J] \equiv W_{\Gamma,0}[J]$ $\longrightarrow$ 真实的, after renormalized theory 的

$= \Gamma[\varphi_J] + \int d^d y \, J(y)\varphi_J(y)$ Connected Generating function.

—— $\varphi_J(x)$ 的意义.

Vacuum Expectation value. With Source Define as.

$$\langle 0|\varphi(x)|0\rangle\big|_J \equiv \frac{\delta W[J]}{\delta J(x)}$$

$$= \int d^4 y \, \frac{\delta\Gamma[\varphi]}{\delta\varphi(y)}\Bigg|_{\varphi=\varphi_J} \frac{\delta\varphi_J(y,J)}{\delta J(x)} + \varphi_J(x) + \int d^4 y \, \frac{\delta\varphi_J(y,J)}{\delta J(x)} J(y)$$

$$= \int d^4 y \, (-J(y)) \frac{\delta\varphi_J(y)}{\delta J(x)} + \int d^4 y \, J(y) \frac{\delta\varphi_J(y,J)}{\delta J(x)} + \varphi_J(x)$$

$$= \varphi_J(x).$$

o Derivative Expansion.

Quantum Action 可以写为:

$$\Gamma[\varphi] = \int d^d x \left\{ -\underbrace{U(\varphi)}_{\text{Quantum Potential.}} + \frac{1}{2} Z(\varphi)\partial^\mu\varphi\partial_\mu\varphi + \underbrace{\cdots}_{\text{Higher order derivative}} \right\}$$

Evaluate Quantum action from Lagrangian.

Connected diagram and generating function

$$Z[J] = \exp(iW[J]) = \int D\phi \exp\left\{ i\left(S[\phi] + \int d^4x\, J(x)\phi(x)\right)\right\}$$

Expand near ϕ_{cl} $\quad S = S_1[\phi] + S_2[\phi] \quad S_1$: normalised $\quad S_2$: counterterm.

$J = J_1(x) + J_2(x)$

$J_1(x)$ satisfy $\quad \frac{\delta S_1}{\delta\phi(x)}\Big|_{\phi=\phi_{cl}} + J_1(x) = 0$. $\;|$ lowest order of interaction, $\frac{\delta S[\phi]}{\delta\phi(x)} \approx \frac{\delta\Gamma[\phi]}{\delta\phi(x)}$.

$(J_1 + J_2)(\phi) = (J_1 + J_2)(\phi_{cl} + \rho)$

（Γ相当于skelton expansion，与S的差距 high order）

$$Z[J] = \int D\phi \exp\left\{ i\left(S_1[\phi_{cl}] + \int d^4x\, J_1(x)\phi_{cl}(x) + \int d^4x \frac{\delta S_1[\phi]}{\delta\phi(x)}\Big|_{\phi=\phi_{cl}}\rho(x) + \int d^4x\, J_1(x)\rho(x) + \frac{1}{2!}\int d^4x_1 d^4x_2 \frac{\delta^2 S_1[\phi]}{\delta\phi(x_1)\delta\phi(x_2)}\Big|_{\phi=\phi_{cl}}\rho(x_1)\rho(x_2) + \cdots\right)\right\}$$

$$\exp\left\{ i\left(S_2[\phi] + \int d^4x\, J_2\phi\right)\right\}$$

$$= \int D\phi \exp\left\{ i\left(S_1[\phi_{cl}] + \int d^4x\, J_1(x)\phi_{cl}(x) + \int d^4x \frac{\delta S_1[\phi]}{\delta\phi(x)}\Big|_{\phi=\phi_{cl}}\rho(x) + \int d^4x\, J_1(x)\rho(x) + \frac{1}{2!}\int d^4x_1 d^4x_2 \frac{\delta^2 S_1[\phi]}{\delta\phi(x_1)\delta\phi(x_2)}\Big|_{\phi=\phi_{cl}}\rho(x_1)\rho(x_2) + \cdots\right)\right\}$$

$$\exp\left\{ i\left(S_2[\phi_{cl}] + \int d^4x\, J_2(x)\phi_{cl}(x) + \left(S_2[\phi_{cl}+\rho] - S_2[\phi_{cl}] + \int d^4x\, J_2(x)\rho(x)\right)\right)\right\}$$

$$= \exp\left\{ i\left(S_1[\phi_{cl}] + \int d^4x\, J_1(x)\phi_{cl}(x) + S_2[\phi_{cl}] + \int d^4x\, J_2(x)\phi_{cl}(x)\right)\right\}$$

$$\int D\rho \cdot \exp\left(\frac{i}{2}\int d^4x_1 d^4x_2\, \rho(x_1)\frac{\delta^2 S[\phi]}{\delta\phi(x_1)\delta\phi(x_2)}\Big|_{\phi=\phi_{cl}}\rho(x_2) + \text{others } \rho \text{ term}\right\}$$

| Gauss integral.

$$\int d^n x \exp\left(-\tfrac{1}{2}x^T A x\right) = \sqrt{\frac{(2\pi)^n}{\det A}}$$

$$= \exp\left\{ i\left(S_1[\phi_{cl}] + \int d^4x\, J_1(x)\phi_{cl}(x) + S_2[\phi_{cl}] + \int d^4x\, J_2(x)\phi_{cl}(x)\right)\right\}$$

$$\times\left(\det\left(-\frac{\delta^2 S[\phi]}{\delta\phi(x_1)\delta\phi(x_2)}\right)^{-\frac{1}{2}} + \text{other terms}\right)$$

$$Z[J] = \exp(iW[J])$$

$$W[J] = S_1[\phi_{cl}] + \int d^4x\, J_1(x)\phi_{cl}(x) + S_2[\phi_{cl}] + \int d^4x\, J_2(x)\phi_{cl}(x)$$

$$- i\ln\left(\det\left(-\frac{\delta^2 S_1[\phi]}{\delta\phi(x_1)\delta\phi(x_2)}\right)^{-\frac{1}{2}} + \text{other terms}\right)$$

$$= S_1[\phi_{cl}] + \int d^4x\, J_1(x)\phi_{cl}(x) + S_2[\phi_{cl}] + \int d^4x\, J_2(x)\phi_{cl}(x)$$

$$- i\ln\left(\det\left(-\frac{\delta^2 S[\phi]}{\delta\phi(x_1)\delta\phi(x_2)}\right)^{-\frac{1}{2}} + \text{other terms}\right)$$

$$\approx S_1[\phi_{cl}] + \int d^4x\, J(x)\phi_{cl}(x) + S_2[\phi_{cl}] + \frac{i}{2}\ln\left(\det\left(-\frac{\delta^2 S_1}{\delta\phi(x_1)\delta\phi(x_2)}\Big|_{\phi=\phi_{cl}}\right)\right) + \text{other terms}$$

$$\Gamma[\phi_{cl}] = W[J] - \int d^d y\, J(y)\varphi_J(y)$$

$$= S[\phi_{cl}] + \frac{i}{2}\ln\left(\det\left(-\frac{\delta^2 S_1}{\delta\varphi(x_1)\delta\varphi(x_2)}\bigg|_{\varphi=\phi_{cl}}\right)\right) + \text{other terms.}$$

Spontaneous Symmetry broken.

φ^4 Theory with negative mass.

- φ^4 Theory With negative mass.

Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4$$

$$= \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{24}\lambda\varphi^4$$

$$= \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{24}\lambda\left(\varphi^4 + \frac{12m^2}{\lambda}\varphi^2\right)$$

$$= \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{24}\lambda\left(\left(\varphi^2 + \frac{6m^2}{\lambda}\right)^2 - \left(\frac{6m^2}{\lambda}\right)^4\right)$$

$$v^2 = \frac{-6m^2}{\lambda} \Rightarrow v = \sqrt{\frac{6|m^2|}{\lambda}}$$

$$= \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{24}\lambda\left((\varphi^2 - v^2)^2 - v^4\right)$$

$$= \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + \frac{1}{24}\lambda v^4 - \frac{1}{24}\lambda(\varphi^2 - v^2)^2$$

平移

$$\varphi(x) = \rho(x) + v.$$

$$\mathcal{L} = \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{24}\lambda v^4 - \frac{1}{24}\lambda\left((\rho+v)^2 - v^2\right)^2$$

$$= \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{24}\lambda v^4 - \frac{1}{24}\lambda(\rho^2 + 2\rho v)^2$$

$$= \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{24}\lambda v^4 - \frac{1}{24}\lambda\rho^4 - \frac{1}{6}\lambda v^2\rho^2 - \frac{1}{6}\lambda v\rho^3$$

$$= \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{24}\lambda v^4 - \frac{1}{24}\lambda\rho^4 + \frac{1}{2}(2m^2)\rho^2 - \frac{1}{6}\lambda v\rho^3$$

- 两个真空.

$\varphi(x) = v$ or $\varphi(x) = -v$ leads to Minimized Energy

$$\langle 0+|\varphi(x)|0+\rangle = v$$

$$\langle 0-|\varphi(x)|0-\rangle = -v$$

—— Z transformation (与经典 $\mathbb{Z}_2$ transformation $\varphi \to -\varphi$ 对应).

$$\langle 0+|Z\varphi(x)Z|0+\rangle = -v$$

$$Z|0+\rangle = |0-\rangle$$

—— 真空态正交 $\Longrightarrow$ 从经典类比.(场在 x 处类似于谐振子 $V(\varphi) = -\frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4$

$$H = \frac{1}{2}p^2 + \frac{1}{24}\lambda(x^2 - v^2)^2$$

基态有2个解 $\psi(x) \sim \exp(-\omega(x \mp v)^2/2) \Rightarrow \langle 0+|0-\rangle \neq 0, \ll 1.$

$|0-\rangle \doteq |0-\rangle \otimes |0-\rangle \cdots$, $\langle 0+|0-\rangle \sim (\langle 0+|0-\rangle)^{+\infty}.$

$$\langle 0+|0-\rangle = 0.$$

U(1) Symmetry.

o Lagrangian

$$\mathcal{L} = \partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \frac{1}{4}\lambda (\varphi^\dagger \varphi)^2$$

Minimal potential point

$$-2m^2\varphi - \lambda \varphi^3 = 0$$

$$\varphi^2 = -\frac{2m^2}{\lambda}$$

$$= \frac{1}{2}\frac{4|m^2|}{\lambda}$$

$$\varphi = \frac{1}{\sqrt{2}} v e^{i\theta} \qquad v = \sqrt{\frac{4|m^2|}{\lambda}}$$

—— Vacuum

$$\langle \theta | \varphi(x) | \theta \rangle = \frac{1}{\sqrt{2}} v e^{i\theta}$$

$$\langle \theta | \theta' \rangle = 0$$

—— 平移

1°

$$\varphi(x) = \frac{1}{\sqrt{2}}(v + a(x) + i b(x))$$

$$\varphi^\dagger(x) = \frac{1}{\sqrt{2}}(v + a(x) - i b(x))$$

$$\mathcal{L}(x) = +\frac{1}{2}\partial^\mu a \partial_\mu a + \frac{1}{2}\partial^\mu b \partial_\mu b - |m^2| a^2 - \frac{1}{2}\lambda^{1/2}|m| a(a^2+b^2) - \frac{1}{16}\lambda (a^2+b^2)$$

2°

$$\varphi(x) = \frac{1}{\sqrt{2}}(v + \rho(x)) e^{-i\chi(x)/v}$$

$$\mathcal{L} = +\frac{1}{2}\partial^\mu \rho \partial_\mu \rho + \frac{1}{2}\left(1+\frac{\rho}{v}\right)^2 \partial^\mu \chi \partial_\mu \chi - |m^2|\rho^2 - \frac{1}{2}\lambda^{1/2}|m|\rho^3 - \frac{1}{16}\lambda\rho^4$$

3° χ / b 不会由圈图得到质量.

$\boxed{\text{SO(N)}}$

$$\mathcal{L} = \frac{1}{2}\partial^\mu \varphi_i \partial_\mu \varphi_i - \frac{1}{2}m^2 \varphi_i \varphi_i - \frac{1}{16}\lambda(\varphi_i \varphi_i)^2$$

$$= T - \frac{1}{16}\lambda(\varphi_i\varphi_i - v^2)^2 \qquad v = \sqrt{\frac{4|m^2|}{\lambda}}$$

若 $\varphi(x) = (\varphi_1(x) \ldots \varphi_{N-1}(x), \rho(x) + v) \Leftarrow (\vec{0}, v)$ 时, potential minimal.

$$\Rightarrow \begin{cases} V = \frac{1}{4}\lambda v^2 \rho^2 + \underbrace{\frac{1}{4}\lambda \rho(\rho^2 + \varphi_i\varphi_i) + \frac{1}{16}\lambda(\rho^2+\varphi_i\varphi_i)^2}_{} \\ T = \frac{1}{2}\partial^\mu \varphi_i \partial_\mu \varphi_i + \frac{1}{2}\partial^\mu \rho \partial_\mu \rho. \qquad i: 1 \to N-1 \end{cases}$$

o GoldSton Theorem:

$(0 \cdots, 0, v)$ 在 SO(N-1) 下不变

$\Downarrow$

无质量场也有 SO(N-1) 对称性

$\Downarrow$

无质量粒子数.

$$N-1 = \underbrace{\frac{N(N-1)}{2}}_{\text{SO(N)生成元}} - \underbrace{\frac{(N-1)(N-2)}{2}}_{\text{SO(N-1)生成元}}$$

Goldston Theorem.

- Potential:

$$V = V(x_1, x_2 \cdots x_N)$$

Minimal potential:

$$\frac{\partial V}{\partial x_i} = 0 \Rightarrow x_i = x_{0,i}$$

Transformation (Transformation A leave potential V invariant)

$$x_i \longrightarrow A_{ij} x_j = (\delta_{ij} + T_{ij}) x_j$$

$$\delta x_i = T_{ij} x_j$$

定义 Function (因为 potential has A invariant 性).

$$F(x) = \frac{\partial V}{\partial x_i} T_{ij} x_j = 0 \qquad \boxed{\text{Note: } \left.\frac{\partial V}{\partial x_i}\right|_{x = x_0} = 0}$$

$$\frac{\partial F}{\partial x_i} = 0 = \frac{\partial^2 V}{\partial x_i \partial x_\beta} T_{\alpha\beta} x_\beta + \frac{\partial V}{\partial x_\alpha} T_{\alpha i} = 0.$$

Set $x = x_0$

$$\left.\frac{\partial^2 V}{\partial x_i \partial x_j} T_{j\alpha} x_\alpha\right|_{x = x_0} = 0.$$

Note: T 是 Transformation A 的 Generator.

1° $T^{(i)}_{j\alpha} x_{0,\alpha} = 0$. Symmetry unbroken.

2° $T^{(i)}_{j\alpha} x_{0,\alpha} \neq 0$, Symmetry broken: Exist generator keeps potential invariant while leaves minimal point Exchanged!

$\Rightarrow$ $T^{(i)}_{j\alpha} x_{0,\alpha}$ is Eigenvector with eigenvalue 0.

$\Rightarrow$ Exists massless particle.

Goldston's Theorem: Number of broken Symmetry = Number of Massless Particle.

Higgs Mechanism.

- Complex scalar field with $U(1)$ Symmetry.

$$\mathcal{L} = +(D^\mu\phi)^\dagger(D_\mu\phi) - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$V(\phi) = \frac{1}{4}\lambda(\phi^\dagger\phi - \frac{1}{2}v^2)^2$$

Minimal potential

$$\phi(x) = \frac{1}{\sqrt{2}} v e^{i\theta}$$

一系列 Vacuum：

$$\langle\theta|\phi(x)|\theta\rangle = \frac{1}{\sqrt{2}} v \exp(i\theta)$$

Set vacuum be $\langle\Omega|\phi(x)|\Omega\rangle = \frac{1}{\sqrt{2}}v$ ，场展开为：

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \rho(x)) e^{i\chi(x)/v}$$

Lagrangian：

$$\mathcal{L} = \frac{1}{2}\partial^\mu\rho\,\partial_\mu\rho + \frac{1}{2}(v+\rho)^2(\frac{1}{v}\partial_\mu\chi - eA_\mu)(\frac{1}{v}\partial^\mu\chi - eA^\mu) - \frac{1}{4}\lambda^2 v^2\rho^2 + \frac{1}{4}\lambda v\rho^3 + \frac{1}{16}\lambda\rho^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

χ Field Canceled by Gauge Transformation of A field.

$$\mathcal{L} = \frac{1}{2}\partial^\mu\rho\,\partial_\mu\rho + \frac{1}{2}(v+\rho)^2\, eA_\mu\, eA^\mu - \frac{1}{4}\lambda^2 v^2\rho^2 + \frac{1}{4}\lambda v\rho^3 + \frac{1}{16}\lambda\rho^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$\Downarrow$

A-field 由 v 得到质量.

BRST Symmetry

BRST Invariant.

- Lagrangian of Yang-Mills field with ghost field

$$+\ \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b \Downarrow$$

$$\mathcal{L} = \boxed{\bar\psi(i\not{D}-m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}} + (\partial^\mu \bar{c}^a(x))\, D^{ab}_\mu c^b(x) - \frac{1}{2\xi}\partial^\mu A^a_\mu(x)\,\partial^\mu A^a_\mu(x)$$

Gauge Invariant

$$D_\mu = \partial_\mu - i g A^i_\mu(x) R^i$$

$$D^{ij}_\mu = \partial_\mu \delta^{ij} - g A^k_\mu(x) f^{kij}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g A^k_\mu A^j_\nu f^{kji}$$

$$\boxed{\begin{aligned}\psi(x) &\longrightarrow \exp(i\alpha^i(x)R^i)\psi(x) = V(x)\psi(x)\\ A^i_\mu(x) &\longrightarrow A^i_\mu(x) + \frac{1}{g}D^{ij}_\mu\alpha^j(x).\end{aligned}}$$

Gauge Transformation.

$$\mathcal{L} = \bar\psi(i\not{D}-m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b \boxed{- \frac{1}{2\xi}\partial^\mu A^a_\mu(x)\,\partial^\mu A^a_\mu(x)}$$

$$\Downarrow$$

$$= \bar\psi(i\not{D}-m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b + \boxed{\frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu}$$

—— 引入 B^a field 的正确性.

$$\int \mathcal{D}B\cdot \exp\left\{i\left(\frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu\right)\right\}$$

$$= \int \mathcal{D}B \ \exp\left\{i\frac{\xi}{2}\left((B^a)^2 + 2\frac{1}{\xi}B^a\partial^\mu A^a_\mu + \frac{1}{\xi^2}(\partial^\mu A^a_\mu)^2 - \frac{1}{\xi^2}(\partial^\mu A^a_\mu)^2\right)\right\}$$

$$= \int \mathcal{D}B \ \exp\left\{i\frac{\xi}{2}\left[\left(B^a + \frac{1}{\xi}\partial^\mu A^a_\mu\right)^2 - \frac{1}{\xi^2}(\partial^\mu A^a_\mu)^2\right]\right\}$$

$$= \int \mathcal{D}\tilde{B} \ \exp\left\{i\frac{\xi}{2}(\tilde{B})^2 - i\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right\}$$

$$\sim \exp\left\{-i\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right\}$$

—— BRST Transformation.(其中 ε 是 grassmann number)

$$\left.\begin{aligned}\delta A^a_\mu &= \varepsilon D^{ab}_\mu c^b\\ \delta\psi &= i g\varepsilon c^a t^a\psi\end{aligned}\right\} \Longrightarrow \left\{\begin{aligned}\psi(x) &\longrightarrow \exp(i\alpha^i(x)R^i)\psi(x) = V(x)\psi(x)\\ A^i_\mu(x) &\longrightarrow A^i_\mu(x) + \frac{1}{g}D^{ij}_\mu\alpha^j(x).\end{aligned}\right.$$

相当于 $\alpha(x) = \varepsilon g c(x)$

$$\Downarrow$$

$$\boxed{\begin{aligned}\delta c^a &= -\frac{1}{2}g\varepsilon f^{abc}c^b c^c\\ \delta\bar{c}^a &= \varepsilon B^a\\ \delta B^a &= 0\end{aligned}}$$

$\bar\psi(i\not{D}-m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}$ is BRST Invariant.

—— $\boxed{\bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b + \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu}$ Under BRST Transform

1° $\delta\left(\frac{\xi}{2}(B^a)^2\right) = 0$

2° $\delta\left(\bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b + B^a\partial^\mu A^a_\mu\right)$ ← ∂ 也作用在c上. ← partial 也作用在 c 上

$$= B^a\partial^\mu(\delta A^a_\mu) + (\delta\bar{c}^a)(-\partial^\mu D^{ab}_\mu)c^b + \bar{c}^a\,\delta(-\partial^\mu D^{ab}_\mu c^b)$$

3° First two

$$= B^a \partial^\mu (\delta A^a_\mu) + (\delta \bar{c}^a)(-\partial^\mu D^{ab}_\mu) c^b$$

$$\Big\downarrow \leftarrow \begin{cases} \delta A^a_\mu = \varepsilon D^{ab}_\mu c^b \\ \delta \bar{c}^a = \varepsilon B^a \end{cases}$$

$$= B^a \partial^\mu (\varepsilon D^{ab}_\mu c^b) + (\varepsilon B^a)(-\partial^\mu D^{ab}_\mu c^b)$$

$$= 0$$

4° Last Term

$$\bar{c}^a \, \delta(-\partial^\mu D^{ab}_\mu c^b)$$

$$\begin{cases} D^{ij}_\mu = \partial_\mu \delta^{ij} - g A^k_\mu(x) f^{kij} \\ \delta A^a_\mu = \varepsilon D^{ab}_\mu c^b \end{cases}$$

$$= \bar{c}^a \delta \left\{ -\partial^\mu \left((\partial_\mu \delta^{ab} - g A^c_\mu(x) f^{cab})(c^b) \right) \right\}$$

$$= \bar{c}^a (-\partial^\mu) \, \delta\left((\partial_\mu \delta^{ab} - g A^c_\mu f^{cab}) c^b \right)$$

For The last term

作用在右側全部 $(\partial_\mu \cdots) x \cdots$, or $\partial_\mu \cdots$

作用在右側括号内 $(\partial_\mu \cdots)(\cdots)$ or $\partial_\mu(\;)$

$$\delta\left((\partial_\mu \delta^{ab} - g A^c_\mu f^{cab})(c^b) \right)$$

$$\Big\downarrow \leftarrow \begin{cases} \delta c^a = -\frac{1}{2} g \varepsilon f^{abc} c^b c^c \\ \delta A^a_\mu = \varepsilon D^{ab}_\mu c^b \end{cases}$$

$$= (\partial_\mu \delta^{ab} - g A^c_\mu f^{cab})\left(-\tfrac{1}{2} g \varepsilon f^{bde} c^d c^e\right) - g \, \varepsilon D^{bd}_\mu (c^d) \, f^{bac} \, c^c$$

$$= (\partial_\mu \delta^{ab} - g A^c_\mu f^{cab})\left(-\tfrac{1}{2} g \varepsilon f^{bde} c^d c^e\right) + g \, \varepsilon D^{bd}_\mu (c^d) \, f^{abc} \, c^c$$

$$= -\tfrac{1}{2} g \varepsilon f^{bde} \delta^{ab} \partial_\mu (c^d c^e) + \tfrac{1}{2} g^2 \varepsilon f^{cab} f^{bde} A^c_\mu c^d c^e$$

$$+ g \varepsilon f^{abc} (\partial_\mu \delta^{bd} - g A^e_\mu f^{ebd})(c^d) c^c$$

$$= g \varepsilon \left(-\tfrac{1}{2} f^{ade} \partial_\mu (c^d c^e) + f^{abc} \partial_\mu (c^b) c^c \right)$$
$$+ g^2 \varepsilon \left(+\tfrac{1}{2} f^{cab} f^{bde} A^c_\mu c^d c^e - f^{abc} f^{ebd} A^e_\mu c^d c^c \right)$$

First term

$$-\tfrac{1}{2} f^{ade} \partial_\mu (c^d c^e) + f^{abc} \partial_\mu (c^b) c^c$$

$$= -\tfrac{1}{2} f^{ade} \partial_\mu (c^d c^e) - f^{abc} \partial_\mu (c^b) c^c$$

$$= -\tfrac{1}{2} f^{ade} \partial_\mu (c^d) c^e - \tfrac{1}{2} f^{ade} c^d \partial_\mu (c^e) + f^{abc} \partial_\mu (c^b) c^c$$

$$= -\tfrac{1}{2} f^{ade} \partial_\mu (c^d) c^e + \tfrac{1}{2} f^{ade} \partial_\mu (c^e) c^d + f^{abc} \partial_\mu (c^b) c^c$$

$$= -\tfrac{1}{2} f^{abc} \partial_\mu (c^b) c^c - \tfrac{1}{2} f^{abc} \partial_\mu (c^b) c^c + f^{abc} \partial_\mu (c^b) c^c$$

$$= 0$$

Second Term : $f^{abc} f^{ebd} A^e_\mu c^d c^c = f^{abe} f^{cbd} A^c_\mu c^d c^e$

$$= f^{abd} f^{cbe} A^c_\mu c^e c^d \quad \Downarrow \text{ grassmann variable } c.$$

$$= - f^{abd} f^{cbe} A^c_\mu c^d c^e$$

$= 0$

$+ g^2 \varepsilon \left(+ \frac{1}{2} f^{cab} f^{bde} A_\mu^c c^d c^e - \frac{1}{2} (f^{abe} f^{cbd} - f^{abd} f^{cbe}) A_\mu^c c^d c^e \right)$

$= - \frac{1}{2} g^2 \varepsilon (- f^{cab} f^{bde} + f^{abe} f^{cbd} - f^{abd} f^{cbe}) A_\mu^c c^d c^e$

$= - \frac{1}{2} g^2 \varepsilon (- f^{bae} f^{edc} + f^{aec} f^{bed} - f^{aed} f^{bec}) A_\mu^b c^d c^c$

$= - \frac{1}{2} g^2 \varepsilon A_\mu^b c^d c^e (- f^{bac} f^{cde} + f^{ace} f^{bcd} - f^{acd} f^{bce})$

$= - \frac{1}{2} g^2 \varepsilon A_\mu^b c^d c^e (- f^{bac} f^{cde} + f^{bdc} f^{cae} + f^{adc} f^{bce})$

Jacobi identity

$f_{ij}{}^{l} f_{lk}{}^{n} + f_{jk}{}^{l} f_{li}{}^{n} + f_{ki}{}^{l} f_{lj}{}^{n} = 0$

$f_{bac} f_{cde} + f_{adc} f_{cbe} + f_{dbc} f_{cae} = 0$

$= 0$

Nil potency

- Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b + \boxed{\frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu}$$

BRST Trans

$$\delta A^a_\mu = \varepsilon D^{ab}_\mu c^b \qquad \delta c^a = -\frac{1}{2}g\varepsilon f^{abc}c^bc^c$$

$$\delta\psi = ig\varepsilon c^a t^a\psi \qquad \delta\bar{c}^a = \varepsilon B^a$$

$$\delta B^a = 0$$

$U_B = \exp(-i\varepsilon Q_B)$, U_B: $\boxed{\text{unitary}}$, Q_B $\boxed{\text{hermite.}}$ $U_B^\dagger = U_B^{-1} = \exp(-i\varepsilon Q_B)$, $Q_B^\dagger = Q_B$

ε is grassman number.

1° Boson

$$U_B^{-1}\phi U_B = \phi + \delta\phi$$

$$(1 + i\varepsilon Q_B)\phi(1 - i\varepsilon Q_B) = \phi + \delta\phi$$

$$i\varepsilon[Q_B, \phi] = \delta\phi$$

2° Fermi

$$U_B^{-1}\psi U_B = \psi + \delta\psi$$

$$(1 + i\varepsilon Q_B)\psi(1 - i\varepsilon Q_B) = \psi + \delta\psi$$

$$+i\varepsilon\psi Q_B + i\varepsilon Q_B\psi = \delta\psi$$ ψ与ε反对易.

$$i\varepsilon\{Q_B, \psi\} = \delta\psi$$

$$\delta_B\delta_B O = \sum_{\alpha\neq\beta}\left(\frac{\partial}{\partial\phi_\alpha}\frac{\partial}{\partial\phi_\beta}O\right)\delta_B\phi_\alpha\,\delta_B\phi_\beta + \sum_\alpha\frac{\partial^2 O}{\partial\phi_\alpha^2}\delta_B\delta_B\phi_\alpha$$

1° 第一项, for boson field

交换对称. $\delta_B\phi$ 中有 grassman number ε.

$$\sum_{\alpha,\beta}\left(\frac{\partial}{\partial\phi_\alpha}\frac{\partial}{\partial\phi_\beta}O\right)\delta_B\phi_\alpha\,\delta_B\phi_\beta = -\sum_{\alpha,\beta}\left(\frac{\partial}{\partial\phi_\alpha}\frac{\partial}{\partial\phi_\beta}O\right)\delta_B\phi_\beta\,\delta_B\phi_\alpha$$

$$= 0$$

2° 第一项, for Fermion Field

交换反对称. $\delta_B\psi \sim \varepsilon\psi \sim$ ordinary number.

$$\sum_{\alpha,\beta}\left(\frac{\partial}{\partial\psi_\alpha}\frac{\partial}{\partial\psi_\beta}O\right)\delta_B\psi_\alpha\,\delta_B\psi_\beta = \sum_{\alpha,\beta}\left(\frac{\partial}{\partial\psi_\alpha}\frac{\partial}{\partial\psi_\beta}O\right)\delta_B\psi_\beta\,\delta_B\psi_\alpha$$

$$= 0$$

3° $\delta_B\delta_B\psi = \delta_B(ig\varepsilon c^a t^a\psi)$

$$= -ig\varepsilon_1\delta_B(c^a)t^a\psi + ig\varepsilon_1 c^a t^a\delta_B(\psi)$$ (δ_B acts like Anti commuting number).

$$\delta c^a = -\frac{1}{2}g\varepsilon_2 f^{abc}c^bc^c$$

$$\delta\psi = ig\varepsilon_2 c^a t^a\psi$$

$$= -ig\varepsilon_1\left(-\frac{1}{2}\right)g\varepsilon_2 f^{abc}c^bc^c t^a\psi + ig\varepsilon_1 c^a t^a\, ig\varepsilon_2 c^a t^a\psi$$

$$= \frac{i}{2}g^2\varepsilon_1\varepsilon_2 f^{abc}c^bc^c t^a\psi - g^2\varepsilon_1\varepsilon_2 c^a t^a c^b t^b\psi$$

$$= g^2\varepsilon_1\varepsilon_2\left(\frac{i}{2}f^{abc}c^bc^c t^a - c^a t^a c^b t^b\right)\psi$$

$$c^a t^a c^b t^b = -c^b c^a t^a t^b$$
$$= -c^a c^b t^b t^a$$
$$= \frac{1}{2} c^a c^b (t^a t^b - t^b t^a)$$
$$= \frac{1}{2} c^a c^b [t^a, t^b]$$
$$= \frac{-i}{2} c^a c^b f^{abc} t^c$$

$$= g^2 \varepsilon_1 \varepsilon_2 \left(\frac{-i}{2} f^{abc} c^b c^c t^a - \frac{-i}{2} c^a c^b f^{abc} t^c \right) \psi$$
$$= 0$$

4° $\delta_B \delta_B A^a_\mu = \delta_B \left(\varepsilon_1 D^{ab}_\mu c^b \right)$

前文证明了 $\delta_B \left((\partial_\mu \delta^{ab} - g A^c_\mu f^{cab})(c^b) \right) = 0$

5° $\delta_B \delta_B C^a = \delta_B \left(-\frac{1}{2} g \varepsilon_1 f^{abc} c^b c^c \right)$

$$= +\frac{1}{2} g \varepsilon_1 f^{abc} \delta_B(c^b) c^c - \frac{1}{2} g \varepsilon_1 f^{abc} c^b \delta_B(c^c)$$

$$\delta c^a = -\frac{1}{2} g \varepsilon_2 f^{abc} c^b c^c$$

$$= \frac{1}{2} g \varepsilon_1 f^{abc} \left(-\frac{1}{2}\right) g \varepsilon_2 f^{bde} c^d c^e c^c - \frac{1}{2} g \varepsilon_1 f^{abc} c^b \left(-\frac{1}{2}\right) g \varepsilon_2 f^{cde} c^d c^e$$

$$= -\frac{1}{4} g^2 \varepsilon_1 \varepsilon_2 f^{abc} f^{bde} c^d c^e c^c - \frac{1}{4} g^2 \varepsilon_1 \varepsilon_2 f^{abc} f^{cde} c^b c^d c^e$$

$$= -\frac{1}{4} g^2 \varepsilon_1 \varepsilon_2 \left(f^{abc} f^{bde} c^d c^e c^c + f^{abc} f^{cde} c^b c^d c^e \right)$$

$$= -\frac{1}{4} g^2 \varepsilon_1 \varepsilon_2 \left(f^{abc} f^{bde} c^d c^e c^c + f^{acb} f^{bde} c^c c^d c^e \right)$$

$$= -\frac{1}{4} g^2 \varepsilon_1 \varepsilon_2 \left(f^{abc} f^{bde} c^d c^e c^c - f^{abc} f^{bde} c^d c^e c^c \right)$$

$$= 0$$

6° $\delta_B \delta_B \bar{c}^a = \delta_B (\varepsilon_1 B^a) = 0.$

$Q_B^2 \neq 1$

7° $\delta_B \delta_B O = 0$, $\delta_B \delta_B O \propto [Q_B, [Q_B, O]_\pm]_\mp = [Q_B^2, O] = 0 \Rightarrow \boxed{Q_B^2 = 0}$

physical State

° physical state：形成 Hilbert 空间的子空间. $\mathcal{H}_{ph}=\{|\Phi\rangle \mid |\Phi\rangle \text{ is physical state}\}$

Requirement： a' $\langle\Phi'|O|\Phi\rangle = \langle\Phi'|e^{+i\varepsilon Q_B} O e^{-i\varepsilon Q_B}|\Phi\rangle$ (Expectation 不随 BRST 变)

b' $\langle\Phi|\Phi\rangle > 0$ (态内积大于0)

1° $|\Phi\rangle \in \ker Q_B$

Invariant of matrix element

$$\delta_B\langle\Phi'|O|\Phi\rangle = \langle\Phi'|i\varepsilon[Q_B, O]_\pm|\Phi\rangle = 0.$$

$$\Rightarrow Q_B|\Phi\rangle = 0, \quad Q_B|\Phi'\rangle = 0$$

$$|\Phi\rangle \in \ker Q_B$$

2° $|\Phi\rangle \notin \mathrm{Img}\, Q_B$

if $|\Phi\rangle \in \mathrm{Img}\, Q_B$ $|\Phi\rangle = Q_B|\varphi\rangle$

$$\langle\Phi|\Phi\rangle = \langle\varphi|Q_B^2|\varphi\rangle = 0$$

Contradiction with $\langle\Phi|\Phi\rangle > 0$.

3° $|\Phi\rangle + Q_B|\varphi\rangle \in \ker Q_B$

4° $|\Phi\rangle + Q_B|\varphi\rangle \notin \mathrm{Img}\, Q_B$

5° $|\Phi\rangle \in \mathcal{H}_{ph}$, $|\Phi\rangle + Q_B|\varphi\rangle \notin \mathcal{H}_{ph}$. (物理态加非物理态不是物理态)

6° $\mathrm{Img}\, Q_B \subset \ker Q_B$

if $|\varphi\rangle \in \mathrm{Img}\, Q_B$, $|\varphi\rangle = Q_B|\psi\rangle$ $Q_B|\varphi\rangle = Q_B^2|\psi\rangle = 0 \Rightarrow |\varphi\rangle \in \ker Q_B$

7° $\mathrm{Img}\, Q_B$, $\ker Q_B$ 是 linear space

8. $\mathcal{H}_{ph} = \ker Q_B / \mathrm{Img}\, Q_B = \{[u], \}$

$[u] = \{|\Phi\rangle, |\Phi\rangle + Q_B|\varphi\rangle, \cdots\}$

↑ cohomology.

$A/B = \{[u]\}$, $B \subset A$, B, A 是线性空间

$[u] = \{v \mid v \in A, \exists b \in B, \text{s.t. } v = u + b\}$

$[u] = [u']$ iff $\exists b \in B$ s.t. $u' = u + u$.

$[u] + [u'] \equiv [u + u']$

$a[u] = [au]$

Example： $A = \mathbb{R}^3$ $B = \mathrm{Span}\{(1,0,0)\}_{\mathbb{R}}$

$A/B = \mathrm{Span}\{(1,0), (0,1)\}$.

° Mode Expansion and physical state

Suppose ghost fields are Hermitian.

$$A_\mu^a(x) = \sum_\lambda \int \widetilde{dk}\,[\varepsilon^*_{\lambda,\mu}(k)\, a_\lambda^a(k)\, e^{-ik\cdot x} + \varepsilon_{\lambda,\mu}(k)\, a_\lambda^{a\dagger}(k)\, e^{ik\cdot x}]$$

$$c^a(x) = \sum_\lambda \int \widetilde{dk}\,[b^a(k)\, e^{-ik\cdot x} + b^{a\dagger} e^{ik\cdot x}] \qquad \bar{c}^a(x) = \sum_\lambda \int \widetilde{dk}\,[d^a e^{-ik\cdot x} + d^{a\dagger} e^{ik\cdot x}]$$

Translation

$$\delta A^a_\mu = \varepsilon D^{ab}_\mu c^b \qquad \delta c^a = -\frac{1}{2} g\varepsilon f^{abc} c^b c^c$$

$$\delta\psi = ig\varepsilon c^a t^a \psi \qquad \delta\bar{c}^a = \varepsilon B^a$$

$$\delta B^a = 0$$

Commutation Relation

$$i\varepsilon\{Q_B,\psi\} = \delta\psi \qquad i\varepsilon\{Q_B, c^a\} = \delta c^a$$

$$i\varepsilon[Q_B, A^a_\mu] = \delta A^a_\mu \qquad i\varepsilon\{Q_B,\bar{c}^a\} = \delta\bar{c}^a$$

$$D^{ab}_\mu = \partial_\mu\delta^{ab} - g A^c_\mu f^{cab}$$

1° Vector boson

$$i\varepsilon[Q_B, a^a_\lambda(k)]\,\varepsilon^*_{\lambda,\mu} = \varepsilon(-ik_\mu)\,b^a \qquad i\varepsilon[Q_B, a^{a\dagger}_\lambda(k)]\,\varepsilon_{\lambda\mu} = \varepsilon(ik_\mu)\,b^{a\dagger}$$

$$\downarrow \leftarrow \{\varepsilon^*_\mu(k,\lambda)\,\varepsilon^\mu(k,\lambda') = -\delta_{\lambda,\lambda'} \qquad \downarrow$$

$$[Q_B, a^a_\lambda(k)] = +\,k_\mu\varepsilon^\mu_\lambda\, b^a \qquad [Q_B, a^{a\dagger}_\lambda(k)] = -(\varepsilon^{*\mu}_\lambda k_\mu)\,b^{a\dagger}$$

纵向极化 $k\cdot\varepsilon\neq0$, $[Q_B, a^{a\dagger}_\lambda(k)]\neq0 \Rightarrow Q_B a^{a\dagger}_\lambda|0\rangle\neq0 \Rightarrow a^{a\dagger}_\lambda|0\rangle\notin\ker Q_B \Rightarrow$ 非物理.

2° Ghost

$$\{Q_B, b^a(k)\}\neq0\cdots \qquad \{Q_B, b^{a\dagger}(k)\}\neq0\cdots$$

由于 $\{Q_B, b^a(k)\}\neq0$, 非物理.

3° Anti-Ghost

B^a 与 A^a 的关系.

$$\frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu = -\frac{1}{2\xi}(\partial^\mu A^a_\mu)(\partial^\mu A^a_\mu) \Rightarrow B^a = -\frac{1}{\xi}\partial^\mu A^a_\mu .$$

Anti-Ghost under Transformation.

$$\delta\bar{c}^a = \varepsilon B^a$$

$$= -\frac{\varepsilon}{\xi}\partial^\mu A^a_\mu$$

$$i\varepsilon\{Q_B,\bar{c}^a\} = -\frac{\varepsilon}{\xi}\partial^\mu A^a_\mu$$

$$i\{Q_B, d^a(k)\} = -\frac{1}{\xi}(-ik^\mu)\cdot\varepsilon^*_{\lambda\mu}(k) \qquad i\{Q_B, d^{a\dagger}(k)\} = -\frac{1}{\xi}(ik^\mu)\,\varepsilon_{\lambda,\mu}(k)$$

$$\{Q_B, d^a(k)\} = \frac{1}{\xi}k\cdot\varepsilon^*_{\lambda\mu}(k) \qquad \{Q_B, d^{a\dagger}(k)\} = -\frac{1}{\xi}k\cdot\varepsilon_\lambda(k)$$

反鬼粒子态非物理.

$\{Q_B, d^{a\dagger}(k)\}\neq0 \Rightarrow Q_B d^{a\dagger}(k)|0\rangle\neq0 \Rightarrow d^{a\dagger}(k)|0\rangle$ 非物理.

perturbation Theory Anomalies

Axial Current

o

$$Z[J] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\{ i \int d^4x \ \bar{\psi}(i\not{\partial})\psi \}$$

$$\mathcal{L} = \bar{\psi}(i\not{\partial})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Transform

$$\psi \to e^{-i\beta\gamma^5}\psi$$

$$\bar{\psi} \longrightarrow \psi^\dagger e^{i\beta(\gamma^5)^\dagger}\gamma^0$$

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\{\gamma^5, \gamma^\mu\} = 0$$

$$= \psi^\dagger \exp(i\beta(\gamma^5)^\dagger)\gamma^0 = \psi^\dagger \exp(i\beta\gamma^5)\gamma^0$$

$$= \psi^\dagger \ \gamma^0 \ \exp(-i\beta\gamma^5)$$

$$\bar{\psi} \longrightarrow \bar{\psi}\exp(-i\beta\gamma^5)$$

$\mathcal{L}$ Invariance under transform.

$$\mathcal{L} \longrightarrow \bar{\psi}\exp(-i\beta\gamma^5)(i\not{\partial})\exp(-i\beta\gamma^5)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= \bar{\psi}\exp(-i\beta\gamma^5)(i\partial_\mu\gamma^\mu)\exp(-i\beta\gamma^5)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= \bar{\psi}(i\partial_\mu\gamma^\mu)\exp(i\beta\gamma^5)\exp(-i\beta\gamma^5)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= \mathcal{L}$$

Conserved Current (Axial Current).

$$j^\mu(x) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(x) \times \delta\varphi_a(x)$$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)}(x) \times \delta\psi_a(x)$$

$$= \bar{\psi} i\gamma^\mu(-i\beta\gamma^5)\psi$$

$$\propto \bar{\psi}\gamma^\mu\gamma^5\psi$$

o path integral 測度変換

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\{iS[\psi]\} = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'\exp\{S[\psi']\}$$

Question: $\int \mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \boxed{?} \cdot \int \mathcal{D}\psi\mathcal{D}\bar{\psi}$

Trival way

$$\psi' = \exp\{-i\beta\gamma^5\}\psi \qquad \bar{\psi}'(x) = \bar{\psi}(x)\exp\{-i\beta(\gamma^5)\}$$

$$d\psi'(x) = \exp\{-i\beta\gamma^5\}d\psi(x) \qquad d\bar{\psi}'_a(x) = d\bar{\psi}_b(x)\exp\{-i\beta(\gamma^5)_{ba}\}$$

$$d\psi'_a(x) = \exp\{-i\beta(\gamma^5)_{ab}\}d\psi_b(x)$$

$$\frac{\delta\psi'_a(y)}{\delta\psi_b(x)} = \exp\{-i\beta(\gamma^5)_{ab}\}\delta^{(4)}(x-y) \qquad \frac{\delta\bar{\psi}'_a(y)}{\delta\bar{\psi}_b(x)} = \exp\{-i\beta(\gamma^5)_{ba}\}\delta^{(4)}(x-y)$$

$$dy_1 dy_2 \cdots dy_n = dx_1 dx_2 \cdots dx_n \det\left(\frac{\partial y_i}{\partial x_j}\right)$$

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \mathcal{D}\psi\mathcal{D}\bar{\psi} \ \det\left(\exp(-i\beta(\gamma^5)_{ab}\delta^{(4)}(x-y))\right)\det\left(\exp(-i\beta(\gamma^5)_{ab}\delta^{(4)}(x-y))\right)$$

$$= \mathcal{D}\psi\mathcal{D}\bar{\psi}\det\left(\exp(-2i\beta(\gamma^5)_{ab})\delta^{(4)}(x-y)\right)$$

$$\det(e^{A}) = \exp(\mathrm{Tr} A) = \exp\left(\int d^4x \sum_{a} \delta^{(4)}(x-x) \exp(-2i\beta(\gamma^5)_{aa}) \right)$$

—— Problem : $\sum_{a} -(\gamma^5)_{aa} = 0$. $\int d^4x\, \delta^{(4)}(x-x) = +\infty$.

—— Eigenstate of operator $i\not{D}$

$$\not{D} = \gamma^{\mu}(\partial_{\mu} + ieA_{\mu})$$

Wick rotation $x^0 = ix^4$.

$$\not{D} = \gamma^0(-i\partial_4 + ieA_0) + \gamma^1(\partial_1 + ieA_1) + \gamma^2(\partial_2 + ieA_2) + \gamma^3(\partial_3 + ieA_3)$$

$i\not{D}$ 作用在空间 $L^4 \otimes \mathbb{C}^4$ 上，L^4 是由全体4元函数 $f(x^1, x^2, x^3, x^4)$ 函数组成的空间.

$$\langle \phi_b | \not{D} | \phi_a \rangle =$$

$$\int d^4x_E\, \phi_b^*(x)\, \not{D}\phi_a(x) = \int d^4x_E\, \phi_b^*(x) \Big(\gamma^0_{ba}(-i\partial_4 + ieA_0) + \gamma^1_{ba}(\partial_1 + ieA_1) + \gamma^2_{ba}(\partial_2 + ieA_2) + \gamma^3_{ba}(\partial_3 + ieA_3) \Big)(\phi_a(x))$$

$$= \int d^4x_E\, \phi_b^*(x) \Big(\gamma^0_{ba}(+i\overleftarrow{\partial}_4 + ieA_0) + \gamma^1_{ba}(-\overleftarrow{\partial}_1 + ieA_1) + \gamma^2_{ba}(-\overleftarrow{\partial}_2 + ieA_2) + \gamma^3_{ba}(-\overleftarrow{\partial}_3 + ieA_3) \Big)\phi_a(x)$$

$$\Big\} \gamma^{0\dagger} = \gamma^0 \qquad (\gamma^i)^{\dagger} = -\gamma^i$$

$$= \int d^4x_E\, \phi_b^*(x) \Big(\gamma^{0*}_{ab}(+i\overleftarrow{\partial}_4 + ieA_0) - (\gamma^1)^*_{ab}(-\overleftarrow{\partial}_1 + ieA_1) - (\gamma^2)^*_{ab}(-\overleftarrow{\partial}_2 + ieA_2) - (\gamma^3)^*_{ab}(-\overleftarrow{\partial}_3 + ieA_3) \Big)\phi_a(x)$$

$$= \int d^4x_E\, \phi_a(x) \Big(\gamma^{0*}_{ab}(+i\partial_4 + ieA_0) - (\gamma^1)^*_{ab}(-\partial_1 + ieA_1) - (\gamma^2)^*_{ab}(-\partial_2 + ieA_2) - (\gamma^3)^*_{ab}(-\partial_3 + ieA_3) \Big)\phi_b^*(x)$$

$$= \left(\int d^4x_E\, \phi_a^*(x) \Big(\gamma^0_{ab}(-i\partial_4 + ieA_0) + (\gamma^1)_{ab}(+\partial_1 + ieA_1) + (\gamma^2)_{ab}(+\partial_2 + ieA_2) + (\gamma^3)_{ab}(+\partial_3 + ieA_3) \Big)\phi_b(x) \right)^*$$

$$= \left(\langle \phi_a | \not{D} | \phi_b \rangle \right)^* = \left(\langle \phi_a | \not{D}^{\dagger} | \phi_b \rangle \right)^* = \langle \not{D}^{\dagger}\phi_b | \phi_a \rangle$$

$$\not{D}^{\dagger} = \not{D}$$

$\not{D}$ is Hermit operator in space $L(4) \otimes \mathbb{C}^4$

$\Rightarrow$ 1° $\not{D}$ has real eigen value

2° $\not{D}$ 的不同本征值的本征态相互正交.

—— $\not{D}$ 的本征态与本征值（$x\in\mathbb{R}^4$，是 Euclidian space 中的向量）.

$$\not{D}\,\phi_m(x) = \lambda_m \phi_m(x)$$ (ϕ is pure imaginary number, $\not{D}=(\not\partial+ie\not{A})$)

—— 积分测度变换.

$\psi,\bar\psi$ 在基底上展开. $\psi(x) = \sum_m a_m \phi_m(x) \quad m=1\cdots+\infty$

$\bar\psi(x) = \sum_m b_m \phi_m^\dagger(x) \quad m=1\cdots+\infty$

$$\frac{\delta\psi_i(x)}{\delta a_m} = \phi_{m,i}(x)$$

$$\frac{\delta\bar\psi_i(x)}{\delta b_m} = \phi_{m,i}^\dagger(x)$$ （$x,i \Rightarrow$ 行指标，m，列指标）.

$$\mathcal{D}\psi\mathcal{D}\bar\psi = \prod_m da_m db_m \prod_{i=1}^4 \det(\phi_{m,i}(x))\det(\phi_{m,i}^\dagger(x))$$

$\langle x,i|\phi_n\rangle$ x,i：行指标 $\langle x,i|\phi_m\rangle^*$

x,i：行指标，m 列指标

$$=\left(\prod_m da_m db_m\right)\det(\phi_{m,i}(x))\det(\phi^*_{m,i}(x))$$

$$\det(A)\det(B)=\det(B^TA)$$

$$=\left(\prod_m da_m db_m\right)\det\left(\int d^4x_E \sum_i \phi^*_{m,i}(x)\phi_{n,i}(x)\right)$$

$$=\prod_m da_m db_m \det(\delta_{mn})$$

单位阵行列式为1.

$$=\prod_m da_m db_m$$

—— Axial U(1) Transformation.

$$\psi\to\exp(-i\beta\gamma^5)\psi \qquad \bar\psi\to\bar\psi\exp(-i\beta\gamma^5)$$

1°
$$\sum_m a_m'\phi_m = \exp(-i\beta\gamma^5)\sum_m a_m\phi_m(x)$$

$$a_n' = \int d^4x_E \sum_m \phi_n^\dagger(x)\exp(-i\beta\gamma^5)a_m\phi_m(x)$$

$$= \int d^4x_E \sum_m \phi_n^\dagger(x)(1-i\beta\gamma^5)a_m\phi_m(x)$$

$$=\sum_m(\delta_{nm}+C_{nm})a_m \qquad C_{nm}=\int d^4x_E\,\phi_n^\dagger(x)(-i\beta)\gamma^5\phi_m(x)$$

$$= -i\beta\langle\phi_n|\gamma^5|\phi_m\rangle = -i\beta\langle\phi_m|\gamma^5|\phi_n\rangle^*$$

$$\sum_m b_m'\phi_m^\dagger(x) = \sum_m b_m\phi_m^\dagger(x)\exp(-i\beta\gamma^5)$$

2°
$$b_n' = \int d^4x \sum_m b_m\phi_m^\dagger(x)(1-i\beta\gamma^5)\phi_n(x)$$

$$=\sum_m b_m(\delta_{mn}-i\beta\langle\phi_m|\gamma^5|\phi_n\rangle)$$

$$=\sum_m b_m(\delta_{mn}-C_{mn})$$

$$=\sum_m b_m(\delta_{mn}+C^*_{nm})$$

$$=\sum_m(\delta_{nm}+C^*_{nm})b_m.$$

认为 $\phi_m(x)$ 是 Pure Imaginary.

$$b_n' = \sum_m(\delta_{nm}+C_{nm})b_m.$$

n：行指标，m，列指标.

3° $\prod_m da'_m\, db'_m = \prod_m da_m\, db_m \quad \det(\delta_{nm}+C_{nm})\det(\delta_{nm}+C_{nm})$

$= \prod_m da_m\, db_m \quad \det(\delta_{nm}+2C_{nm}+\underbrace{C^2_{nm}}_{0})$

$= \prod_m da_m\, db_m \quad \det(\delta_{nm}+2C_{nm})$

$\det(A) = \exp(\mathrm{Tr}(\ln A))$

$C_{nm} = -i\beta\langle\phi_n|\gamma^5|\phi_m\rangle$

$= \left(\prod_m da_m\, db_m\right)\exp(\mathrm{Tr}(2C_{nm}))$

$= \left(\prod_m da_m\, db_m\right)\exp\mathrm{Tr}\left(-2i\ \beta\langle\phi_n|\gamma^5|\phi_m\rangle\right)$ 这里严格应写为 $\langle\phi_n|x,i\rangle(\gamma^5)_{ij}\langle x,j|\phi_m\rangle$

$= \left(\prod_m da_m\, db_m\right)\exp\mathrm{Tr}\left(-2i\ \beta\int d^4x_E\langle\phi_n|x\rangle\gamma^5\ \langle x|\phi_m\rangle\right)$

$= \left(\prod_m da_m\, db_m\right)\exp\left(-2i\sum_n\beta\int d^4x_E\langle\phi_n|x\rangle\gamma^5\ \langle x|\phi_n\rangle\right)$

$= \left(\prod_m da_m\, db_m\right)\ \exp\left(-2i\sum_n\beta\boxed{\int d^4x_E\ \phi_n^\dagger(x)\gamma^5\phi_n(x)}\right)$

Matrix product property.

$u^\dagger A v = u^\dagger_i A_{ij} v_j = \mathrm{Tr}(v u^\dagger A) = v_j u^\dagger_i A_{ij}$

$= \left(\prod_m da_m\, db_m\right)\ \exp\left(-2i\sum_n\beta\int d^4x_E\ \mathrm{Tr}(\phi_n(x)\phi_n^\dagger(x)\gamma^5)\right)$

$= \left(\prod_m da_m\, db_m\right)\ \exp\left(-2i\ \beta\int d^4x_E\ \delta^{(4)}(0)\,\mathrm{Tr}(\gamma^5)\right)$

与之前简单分析相同，$\mathrm{Tr}(\gamma^5)=0$，但有 $\int d^4x_E\ \delta^{(4)}(0)$ 发散.

Fujikawa Regularisation.

$\sum_n \phi_n^\dagger(x)\gamma^5 e^{-\lambda_n^2/M^2}\phi_n(x) = \sum_n\langle\phi_n|x\rangle\gamma^5 e^{-\lambda_n^2/M^2}\langle x|\phi_n\rangle$

$= \sum_n \phi_n^\dagger(x)\gamma^5\ \exp(-\not{D}^2/M^2)\phi_n(x)$

$\partial^2 - e^2A^2 + ie\partial A + ieA\partial$

Simplyfy $\not{D}$ Term. $\quad (\partial+ieA)^2$

$\not{D} = \not{\partial} + ie\not{A} \qquad D^2 - \frac{e}{2}\frac{i}{2}[\gamma^\mu,\gamma^\nu]$

$\not{D}^2 = (\not{\partial}+ie\not{A})(\not{\partial}+ie\not{A})$

$= \not{\partial}\not{\partial} + ie\not{A}\not{\partial} + ie\not{\partial}\not{A} - e^2\not{A}\not{A}$

$= \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + ie\gamma^\mu\gamma^\nu A_\mu\partial_\nu + ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu - e^2A_\mu A_\nu\gamma^\mu\gamma^\nu$

$= \partial^2 + \frac{ie}{2}\{\gamma^\mu,\gamma^\nu\}A_\mu\partial_\nu + \frac{ie}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu A_\nu - e^2A^2$

$+ \frac{ie}{2}[\gamma^\mu,\gamma^\nu]A_\mu\partial_\nu + \frac{ie}{2}[\gamma^\mu,\gamma^\nu]\partial_\mu A_\nu$

$= \partial^2 + ie\,A\cdot\partial + ie\,\partial\cdot A - e^2A^2$

$+ \frac{ie}{2}[\gamma^\mu,\gamma^\nu]A_\mu\partial_\nu + \frac{ie}{4}[\gamma^\mu,\gamma^\nu](\partial_\mu A_\nu - \partial_\nu A_\mu)$

$= \partial^2 + ie\,A\cdot\partial + ie\,\partial\cdot A - e^2A^2$

$+ \frac{ie}{4}[\gamma^\mu,\gamma^\nu](A_\mu\partial_\nu - A_\nu\partial_\mu) + \frac{ie}{4}[\gamma^\mu,\gamma^\nu](\partial_\mu(A_\nu) - \partial_\nu(A_\mu))$

$+ \frac{ie}{4}[\gamma^\mu,\gamma^\nu](A_\nu\partial_\mu - A_\mu\partial_\nu)$

$$= \partial^2 + ie A\cdot\partial + ie\,\partial\cdot A - e^2A^2 + \frac{ie}{4}[\gamma^\mu,\gamma^\nu]\big(\partial_\mu(A_\nu) - \partial_\nu(A_\mu)\big)$$

$$= D^2 + \frac{ie}{2} S^{\mu\nu} F_{\mu\nu} \qquad S^{\mu\nu} = \frac{1}{2}[\gamma^\mu,\gamma^\nu]$$

Fujikawa regularised term.

$$\sum_n \phi_n^\dagger(x)\gamma^5 \exp(-\not{D}^2/M^2)\phi_n(x) = \sum_n \phi_n^\dagger(x)\gamma^5 \exp\left(-(D^2 + \frac{ie}{2}S^{\mu\nu}F_{\mu\nu})/M^2\right)\phi_n(x)$$

$D_\mu = \partial_\mu + ieA_\mu$

$D^2 = (\partial + ieA)\cdot(\partial + ieA)$

$= \partial^2 + ieA\partial + ie\partial A - e^2A^2$

$$= \sum_n \phi_n^\dagger(x)\gamma^5 \exp\left(-\partial^2/M^2\right)\exp\left(-(ieA\cdot\partial + ie\partial\cdot A - e^2A^2 + \frac{ie}{2}S^{\mu\nu}F_{\mu\nu})/M^2\right)\phi_n(x)$$

Expand in momentum space

$\phi_n^\dagger(x) = \int \frac{d^4k_1}{(2\pi)^4} \exp(-ik_1\cdot x)\phi_n^\dagger(k_1)$

x 是 wick rotate 后 Euclidian space parameter.

$\phi_n(x) = \int \frac{d^4k_2}{(2\pi)^4}\exp(ik_2\cdot x)\phi_n(k_2)$

$$= \sum_n \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\cdot\phi_n^\dagger(k_1)\exp(ik_1\cdot x)\gamma^5 \exp\left(-\partial^2/M^2\right)\exp\left(-(ieA\cdot\partial + ie\partial\cdot A - e^2A^2 + \frac{ie}{2}S^{\mu\nu}F_{\mu\nu})/M^2\right)\exp(ik_2\cdot x)\phi_n(k_2)$$

$$= \sum_n \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\cdot\phi_n^\dagger(k_1)\exp(-ik_1\cdot x)\exp(ik_2\cdot x)\gamma^5 \exp\left(+(k_2)^2/M^2\right)\exp\left(+(2eA\cdot k_2 + ie\partial(A) + e^2A^2 - \frac{ie}{2}S^{\mu\nu}F_{\mu\nu})/M^2\right)\phi_n(k_2)$$

$$= \sum_n \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\cdot \mathrm{Tr}\left\{\phi_n(k_2)\phi_n^\dagger(k_1)\exp(-ik_1\cdot x)\exp(ik_2\cdot x)\gamma^5 \exp\left(+(k_2)^2/M^2\right)\exp\left[+(2eA\cdot k_2 + ie\partial(A) + e^2A^2 - \frac{ie}{2}S^{\mu\nu}F_{\mu\nu})/M^2\right]\right\}$$

Completely relation

$$\sum_n \phi_n(x_1)\phi_n^\dagger(x_2) = \delta^{(4)}(x_1 - x_2)\mathbb{I}$$

$$\sum_n \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\cdot\phi_n(k_1)\phi_n^\dagger(k_2)\exp(ik_1\cdot x_1)\exp(-ik_2\cdot x_2) = \delta^{(4)}(x_1 - x_2)\mathbb{I}$$

$$\sum_n \phi_n(k_1)\phi_n^\dagger(k_2)\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\exp(ik_1x_1 - ik_2x_1) = \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\exp(ik_1(x_1-x_2)) \times (2\pi)^4\delta(k_1 - k_2)$$

$$\sum_n \phi_n(k_1)\phi_n^\dagger(k_2) = (2\pi)^4\delta(k_1 - k_2)\mathbb{I}$$

$$= \int\frac{d^4k_E}{(2\pi)^4}\mathrm{Tr}\left\{\gamma^5\exp(k^2/M^2)\exp\left((2eA\cdot k + ie\partial(A) + e^2A^2 - \frac{ie}{2}S^{\mu\nu}F_{\mu\nu})/M^2\right)\right\}$$

$$= \int\frac{d^4k_E}{(2\pi)^4}\mathrm{Tr}\left\{\gamma^5\exp\left[\frac{2eA\cdot k + ie\partial(A) + e^2A^2 - \frac{ie}{2}S^{\mu\nu}F_{\mu\nu}}{M^2} + \frac{k^2}{M^2}\right]\right\}$$

$$= M^4 \int \frac{d^4 k_E}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma^5 \exp\left[\frac{ie\partial(A) + e^2 A^2 - \frac{ie}{2} S^{\mu\nu} F_{\mu\nu}}{M^2} + k^2 + \frac{2eA\cdot k}{M^2} \right] \right\}$$

Expand analysis,

$\frac{1}{M^n}$ $n>4$, abort

$\frac{1}{M^n}$ $n \leq 4$ save

γ number < 4 abort (与 γ^5 trace为0).

$$= M^4 \int \frac{d^4 k_E}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma^5 \frac{1}{2!} \frac{1}{M^4} \left(\frac{ie}{2} S^{\mu\nu} F_{\mu\nu}\right)^2 \exp(k^2) \right\}$$

$$\phi(x^0, x^1, x^2, x^3) = \int_-^+ \frac{d^4 k}{(2\pi)^4} \exp(-ik\cdot x)\, \phi(k^0, k^1, k^2, k^3)$$

$$k^0 = ik^4 \qquad x^0 = ix^4$$

$$x^E = (x^1, x^2, -ix^0) \equiv (x^1, x^2, x^3, x^4) \qquad k^E \equiv (k^1, k^2, k^3, -ik^0) = (k^1, k^2, k^3, k^4)$$

$$\phi_E(k_E) \equiv \phi(ik^4, k^1, k^2, k^3)$$

$$\phi_E(x_E) \equiv \phi(ix^4, x^1, x^2, x^3)$$

$$\phi(-ix^4, x^1, x^2, x^3)\Big|_{x^4\ img} = i \int_-^+ \frac{d^4 k_E}{(2\pi)^4} \exp(ik^0 x^4 + ik^1 x^1 + ik^2 x^2 + ik^3 x^3) \underbrace{}_{\phi_E(k)}$$

$$= i \int_-^+ \frac{d^4 k_E}{(2\pi)^4} \exp(i k_E \cdot x_E)\, \phi_E(k_E)$$

$$\phi_E(x^4)\Big|_{x^4\ real} = i \int_-^+ \frac{d^4 k_E}{(2\pi)^4} \exp(i k_E\cdot x_E)\, \phi_E(k_E)$$

$\downarrow$

$$\phi_E(k_E) = \left(\frac{1}{i}\right) \int_-^+ \frac{d^4 x_E}{(2\pi)^4} \exp(-ik_E\cdot x_E)\, \phi_E(x_E)$$

$$x^4 = -ix^0 \qquad x^0 = ix^4$$

$$= (-1) \int_{-i\infty}^{+i\infty} \frac{d^4 x}{(2\pi)^4} \exp(-k^4 x^0 - i\vec{k}\cdot\vec{x})\, \phi_E(x_E)$$

$$k^4 = -ik^0$$

$$\phi(k) = (-1) \int_{-i\infty}^{+i\infty} \frac{d^4 x}{(2\pi)^4} \exp(ik^0 x^0 - i\vec{k}\cdot\vec{x})\, \phi(x)$$

又:

$$\phi(k) = (+1) \int_{-\infty}^{+\infty} \frac{d^4 x}{\square} (\cdots)$$

故. $x^0 : (-\infty, +\infty) \Leftrightarrow x^0 : (+i\infty, -i\infty)$.

$$= M^4 \int \frac{d^4 k_E}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma^5 \frac{1}{2!} \frac{1}{M^4} \left(\frac{ie}{2} S^{\mu\nu} F_{\mu\nu}\right)^2 \exp(k_E^2) \right\}$$

$$k^0 = ik^4, \quad k^4 = -ik^0$$

$$= M^4 (-i) \int_{-\infty}^{+\infty} \frac{d^4 k}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma^5 \frac{1}{2!} \frac{1}{M^4} \left(\frac{ie}{2} S_{\mu\nu} F^{\mu\nu}\right)^2 \exp(-k^2) \right\}$$

$$= (-i) \int_{-\infty}^{+\infty} \frac{d^4 k}{(2\pi)^4} \mathrm{Tr}\left\{ \frac{1}{2!} \gamma^5 \left(\frac{-e^2}{4}\right) S_{\mu\nu} S_{\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \exp(-k^2) \right\}$$

$$= \frac{ie^2}{8} \int_{-\infty}^{+\infty} \frac{d^4 k}{(2\pi)^4} \exp(-k^2)\, \mathrm{Tr}\left\{ \gamma^5 S^{\mu\nu} S^{\rho\sigma} \right\} F_{\mu\nu} F_{\rho\sigma}$$

$$\int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} \exp(-k^2) = \frac{\pi^2}{16\pi^4}$$

$$= \frac{1}{16\pi^2}$$

$$\mathrm{Tr}\{\gamma^5 S^{\mu\nu} S^{\rho\sigma}\} F_{\mu\nu}F_{\rho\sigma} = \mathrm{Tr}\{\gamma^5 \tfrac{i}{2}[\gamma^\mu,\gamma^\nu]\}\,\mathrm{Tr}\{\tfrac{i}{2}[\gamma^\rho,\gamma^\sigma]\} F_{\mu\nu}F_{\rho\sigma}$$

$$= -\frac{1}{4} F_{\mu\nu}F_{\rho\sigma}\,\mathrm{tr}\{\gamma^5(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho + \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho)\}$$

$$= -\frac{1}{4} F_{\mu\nu}F_{\rho\sigma}\,\{-4i(\varepsilon^{\mu\nu\rho\sigma} - \varepsilon^{\nu\mu\rho\sigma} - \varepsilon^{\mu\nu\sigma\rho} + \varepsilon^{\nu\mu\sigma\rho})\}$$

$$= i4 F_{\mu\nu}F_{\rho\sigma}\,\varepsilon^{\mu\nu\rho\sigma}$$

$$= -\frac{ie^2}{2}\,\frac{1}{16\pi^2} F_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$$

—— Gauge Transformation

$$\prod_m da'_m\, db'_m = \left(\prod_m da_m\, db_m\right) \exp\left(-2i\sum_n \beta \int d^4x_E\, \phi_n^\dagger(x)\gamma^5\phi_n(x)\right)$$

$$= \left(\prod_m da_m\, db_m\right) \exp\left[-2i\, \beta\int d^4x_E \cdot \left(-\frac{ie^2}{32\pi^2} F_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)\right]$$

$$i x^4 = x^0 \qquad x^4 = -ix^0$$

$$\to \prod_m da_m\, db_m \exp\left(-2i\beta\int d^4x\, \frac{e^2}{32\pi^2} F_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)$$

—— local

$$\prod_m da'_m\, db'_m = \prod_m da_m\, db_m \exp\left(-i\int d^4x\, \frac{e^2}{16\pi^2}\beta F_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)$$

—— Local Gauge Transformation.

$$\mathcal{L} = \bar\psi(i\not{D})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\psi \to e^{-i\beta\gamma^5}\psi$$

$$\bar\psi \to \bar\psi\exp(-i\beta\gamma^5)$$

$$\mathcal{L}' = \bar\psi\exp(i\beta\gamma^5)\, i\not{D}\, \exp(-i\beta\gamma^5)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= \bar\psi\exp(i\beta\gamma^5)\, i(\not\partial + ie\not{A})\exp(-i\beta\gamma^5)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= \mathcal{L} + \bar\psi\, i\not\partial(\beta)(-i)\gamma^5\psi$$

$$= \mathcal{L} + \bar\psi\not\partial(\beta)\gamma^5\psi$$

$$= \mathcal{L} + \partial_\mu(\beta)\,\bar\psi\gamma^\mu\gamma^5\psi$$

$$= \mathcal{L} - \beta\,\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi)$$

Consider 积分测度变化

$$\mathcal{L}' = \mathcal{L} - \beta\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi) - \frac{e^2}{16\pi^2}\beta F_{\mu\nu}F_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}$$

Conservation law. $(Z[\psi'] = Z[\psi])$

$$\partial_\mu j^{5\mu} = -\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

Anomaly in Chiral Gauge Theory.

○ Lagrangian

$$\mathcal{L} = \bar{\Psi}_L (i\not{D})\Psi_L - \frac{1}{4}F^{a\mu\nu}F_{a\mu\nu}$$

$$\Psi_L \to \exp(i\Gamma^a T_R^a P_L)\Psi_L \qquad \bar{\Psi}_L \to \bar{\Psi}_L \exp(-i\Gamma^a T_R^a P_R)$$

$$A_\mu^a \to A_\mu^a - g f^{abc}\Gamma^b A_\mu^c + O(\Gamma^2)$$

○ 积分测度变换.

$$\mathcal{D}\Psi_L \mathcal{D}\bar{\Psi}_L \longrightarrow \mathcal{D}\Psi_L' \mathcal{D}\bar{\Psi}_L' = \det\left(\exp(-i\Gamma^a(x) T_R^a P_L)\ \det\left(\exp(i\Gamma^a(x) T_R^a P_R)\right)\right.$$

$$\mathcal{D}\Psi_L\ \mathcal{D}\bar{\Psi}_L$$

$$\begin{aligned}
&\det(\exp(-i\Gamma^a(x) T_R^a P_L))\det(\exp(i\Gamma^a(x) T_R^a P_R)) \\
&= \det(\exp(-i\Gamma^a(x) T_R^a P_L)\exp(-i\Gamma^a(x) T_R^a P_R)) \\
&= \det(\exp(-i\Gamma^a(x) T_R^a (P_L - P_R))) \\
&= \det(\exp(i\Gamma^a(x) T_R^a \gamma^5)) \\
&= \exp \mathrm{Tr}(i\Gamma^a(x) T_R^a \gamma^5)
\end{aligned}$$

Fujikawa

$$\exp \mathrm{Tr}(i\Gamma^a(x) T_R^a \gamma^5) = \lim_{M\to\infty} \exp\left\{ i\int d^4x\, \Gamma^a(x)\, \mathrm{Tr}(\gamma^5 T_R^a e^{-\not{D}_x^2/M^2} \delta^{(4)}(x-y))\right\}$$

$$= \lim_{M\to+\infty} \exp\left\{ i\int d^4x \frac{d^4k}{(2\pi)^4}\Gamma^a(x)\,\mathrm{Tr}\left[\gamma^5 T_R^a\, e^{-(\not{D}+i\not{k})^2/M^2}\right]\right\}$$

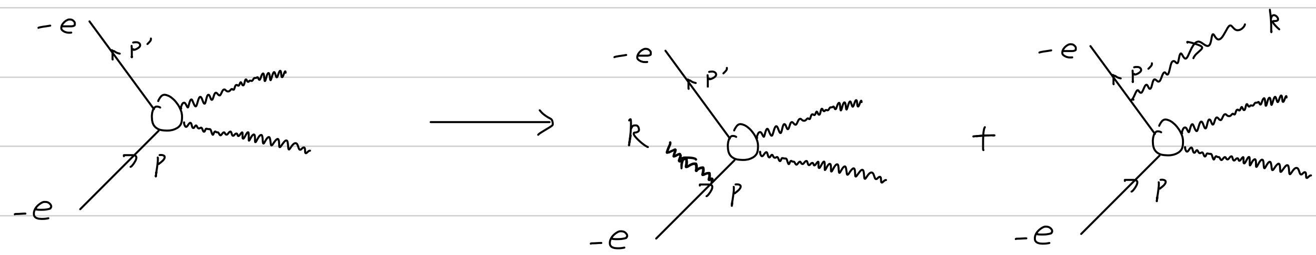
(计算): $\mathrm{Tr}\left[T_R^a\{T_R^b, T_R^c\}\right] = \frac{1}{2}A(R)d^{abc}$.

$$= \exp\left(-\frac{ig^2}{128\pi^2}A(R)\int d^4x\, \Gamma^a d^{abc}\varepsilon^{\mu\nu\rho\sigma}F^b_{\mu\nu}F^c_{\rho\sigma}\right)$$

$$\exp\left[-\frac{ig^2}{128\pi^2}\left(\sum_{\text{left}}A(R) - \sum_{\text{right}}A(R)\right)\int d^4x\,\Gamma^a d^{abc}\varepsilon_{\mu\nu\rho\sigma}F^b_{\mu\nu}F^c_{\rho\sigma}\right]$$

Infrared Divergence

○ Soft photon radiation



outgoing
incoming

Scattering amplitude of left-hand diagram

$$i\mathcal{M} = \bar{u}(p')\mathcal{M}(p',p)u(p)$$

Scattering amplitude of Right-hand Diagram

$$i\mathcal{M} = \bar{u}(p')\mathcal{M}_0(p',p-k)\frac{1}{i}\frac{\not{p}-\not{k}+m}{-(p-k)^2+m^2-i\varepsilon}(-ie\gamma^\mu)\varepsilon_\mu(k)u(p)$$

$$+\bar{u}(p')\varepsilon_\mu(k)(-ie\gamma^\mu)\frac{1}{i}\frac{\not{p}'+\not{k}+m}{-(p'+k)^2+m^2-i\varepsilon}\mathcal{M}_0(p'+k,p)u(p)$$

k is small, ignore higher order terms of k.

$$\mathcal{M}_0(p',p-k) \approx \mathcal{M}_0(p',k)$$

$$\mathcal{M}_0(p'+k,p) \approx \mathcal{M}_0(p',p)$$

$p^2=m^2,\ k^2=0$

$$-(p-k)^2+m^2-i\varepsilon = -p^2-k^2+2p\cdot k+m^2-i\varepsilon = 2p\cdot k - i\varepsilon \approx 2p\cdot k \quad -(1)$$

$$-(p'+k)^2+m^2-i\varepsilon = -(p')^2-k^2-2p'\cdot k+m^2-i\varepsilon = -2p'\cdot k-i\varepsilon \approx -2p'\cdot k \quad -(2)$$

Use Dirac Equation Simplify numerator $(\not{p}-m)u(p)=0=\bar{u}(p)(\not{p}-m)$

$$(\not{p}-\not{k}+m)\gamma^\mu u(p) \approx (\not{p}+m)\gamma^\mu u(p) = (p_\nu\gamma^\nu\gamma^\mu+m\gamma^\mu)u(p)$$

$$= \{p_\nu(\{\gamma^\nu,\gamma^\mu\}-\gamma^\mu\gamma^\nu)+m\gamma^\mu\}u(p)$$

$$= \{p_\nu(2g^{\nu\mu}-\gamma^\mu\gamma^\nu)+m\gamma^\mu\}u(p)$$

$$= \{2p^\mu-\gamma^\mu\not{p}+m\gamma^\mu\}u(p)$$

$$= \{2p^\mu-\gamma^\mu(+\not{p}-m)\}u(p)$$

$$= 2p^\mu u(p) \quad -(3)$$

$$\bar{u}(p')\gamma^\mu(\not{p}'+\not{k}+m) = \bar{u}(p')\gamma^\mu(\not{p}'+m)$$

$$= \bar{u}(p')(\gamma^\mu\gamma^\nu p'_\nu+m\gamma^\mu)$$

$$= \bar{u}(p')(\{\gamma^\mu,\gamma^\nu\}p'_\nu-\gamma^\nu\gamma^\mu p'_\nu+m\gamma^\mu)$$

$$= \bar{u}(p')(2g^{\mu\nu}p'_\nu-\not{p}'\gamma^\mu+m\gamma^\mu)$$

$$= \bar{u}(p')(2p'^\mu-(\not{p}'-m)\gamma^\mu)$$

$$= \bar{u}(p')2p'^\mu \quad -(4)$$

Combine Results (Real number modification)

$$i\mathcal{M} = \bar{u}(p')\mathcal{M}_0(p',p-k)\frac{1}{i}\frac{\not{p}-\not{k}+m}{-(p-k)^2+m^2-i\varepsilon}(-ie\gamma^\mu)\varepsilon_\mu(k)u(p)$$

$$+\bar{u}(p')\varepsilon_\mu(k)(-ie\gamma^\mu)\frac{1}{i}\frac{\not{p}'+\not{k}+m}{-(p'+k)^2+m^2-i\varepsilon}\mathcal{M}_0(p'+k,p)u(p)$$

$$= \bar{u}(p') M_0(p', p) \frac{i}{\not{}} \frac{(-ie)}{2p\cdot k} 2p^{\mu} \varepsilon_{\mu}(k) u(p)$$

$$+ \bar{u}(p') 2p'^{\mu} \varepsilon_{\mu}(k) (-ie) \frac{i}{\not{}} \frac{1}{-2p'\cdot k} M_0(p', p) u(p)$$

$$= \bar{u}(p') M_0(p', p) u(p) \times e \left\{ -\frac{p\cdot\varepsilon(k)}{p\cdot k} + \frac{p'\cdot\varepsilon(k)}{p'\cdot k} \right\}$$

Cross section with photon emission

$$\sigma(e^-(p) \to e^-(p') + \gamma(k)) = \sigma(e^-(p) \to e^-(p')) \cdot$$

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2w} \sum_{helicity} e^2 \left(\frac{p'\cdot\varepsilon(k)}{p'\cdot k} - \frac{p\cdot\varepsilon(k)}{p\cdot k} \right)^2$$

Define

$$\int d(\gamma\text{-radiation}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2w} \sum_{helicity} e^2 \left(\frac{p'\cdot\varepsilon(k)}{p'\cdot k} - \frac{p\cdot\varepsilon(k)}{p\cdot k} \right)^2$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2w} \sum_{helicity} e^2 \varepsilon_{\mu}(k) \varepsilon_{\nu}^{*}(k) \left(\frac{p'^{\mu}}{p'\cdot k} - \frac{p^{\mu}}{p\cdot k} \right) \left(\frac{p'^{\nu}}{p'\cdot k} - \frac{p^{\nu}}{p\cdot k} \right)$$

Polarization Sum

$$\sum_{helicity} \varepsilon_{\mu}(k) \varepsilon_{\nu}^{*}(k) = -g_{\mu\nu}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2w} e^2 \cdot (-g_{\mu\nu}) \left(\frac{p'^{\mu} p'^{\nu}}{(p'\cdot k)^2} + \frac{p^{\mu} p^{\nu}}{(p\cdot k)^2} - \frac{p^{\mu} p'^{\nu}}{(p\cdot k)(p'\cdot k)} - \frac{p'^{\mu} p^{\nu}}{(p\cdot k)(p'\cdot k)} \right)$$

momentum — Energy relation.

$$p^2 = m^2 \quad p'^2 = m^2$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2w} e^2 \left(\frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)} - \frac{m^2}{(p\cdot k)^2} - \frac{m^2}{(p'\cdot k)^2} \right)$$

Parametrization (Choose a reference frame, incoming & out going has Same Energy!

$p = E(1, \vec{v}) \quad p' = E(1, \vec{v}') \quad k = w(1, \hat{k}) \quad \hat{k}$ is 3-dim unit vector.

$$p\cdot p' = E^2(1 - \vec{v}\cdot\vec{v}')$$

$$p\cdot k = Ew(1 - \vec{v}\cdot\hat{k})$$

$$p'\cdot k = Ew(1 - \vec{v}'\cdot\hat{k})$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2w} e^2 \cdot \left(\frac{2E^2(1 - \vec{v}\cdot\vec{v}')}{E^2w^2(1-\vec{v}\cdot\hat{k})(1-\vec{v}'\cdot\hat{k})} - \frac{m^2}{E^2w^2(1-\vec{v}\cdot\hat{k})^2} - \frac{m^2}{E^2w^2(1-\vec{v}'\cdot\hat{k})^2} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2w} e^2 \cdot \left(\frac{2(1-\vec{v}\cdot\vec{v}')}{w^2(1-\vec{v}\cdot\hat{k})(1-\vec{v}'\cdot\hat{k})} - \frac{m^2/E^2}{w^2(1-\vec{v}\cdot\hat{k})^2} - \frac{m^2/E^2}{w^2(1-\vec{v}'\cdot\hat{k})^2} \right)$$

$$= \int w^2 dw\, d\Omega_{\hat{k}} \frac{1}{(2\pi)^3} \frac{1}{2w} e^2 \frac{1}{w^2} \left(\frac{2(1-\vec{v}\cdot\vec{v}')}{(1-\vec{v}\cdot\hat{k})(1-\vec{v}'\cdot\hat{k})} - \frac{m^2/E^2}{(1-\vec{v}\cdot\hat{k})^2} - \frac{m^2/E^2}{(1-\vec{v}'\cdot\hat{k})^2} \right)$$

$$= \int \frac{dw}{w} \frac{d\Omega_{\hat{k}}}{(2\pi)^3} \frac{e^2}{2} \cdot \left(\frac{2(1-\vec{v}\cdot\vec{v}')}{(1-\vec{v}\cdot\hat{k})(1-\vec{v}'\cdot\hat{k})} - \frac{m^2/E^2}{(1-\vec{v}\cdot\hat{k})^2} - \frac{m^2/E^2}{(1-\vec{v}'\cdot\hat{k})^2} \right)$$

$$= \int \frac{dw}{w} \left(\frac{e^2}{4\pi}\right) \frac{1}{\pi} \cdot I(\vec{v},\vec{v}')$$

$$I(\vec{v},\vec{v}') \equiv \int \frac{d\Omega_{\hat{k}}}{4\pi} \left(\frac{2(1-\vec{v}\cdot\vec{v}')}{(1-\vec{v}\cdot\hat{k})(1-\vec{v}'\cdot\hat{k})} - \frac{m^2/E^2}{(1-\vec{v}\cdot\hat{k})^2} - \frac{m^2/E^2}{(1-\vec{v}'\cdot\hat{k})^2} \right)$$

Suppose in a coordinate that $\vec{v} = (0,0,\beta)$ $\vec{v}' = -\vec{v} = (0,0,-\beta)$

$E = m\frac{1}{\sqrt{1-\beta^2}}$

$$I(\vec{v},\vec{v}') = \int \frac{2\pi d\cos\theta}{4\pi} \cdot \left(\frac{2(1+\beta^2)}{(1-\beta\cos\theta)(1+\beta\cos\theta)} - \frac{m^2/E^2}{(1-\cos\theta\beta)^2} - \frac{m^2/E^2}{(1+\cos\theta\beta)^2} \right)$$

$$= \int_{-1}^{1} \frac{1}{2} d\cos\theta \cdot \left((1+\beta^2)\left(\frac{1}{1-\beta\cos\theta} + \frac{1}{1+\beta\cos\theta}\right) - (1-\beta^2)\frac{1}{(1-\cos\theta\beta)^2} - (1-\beta^2)\frac{1}{(1+\cos\theta\beta)^2} \right)$$

$$= \frac{1}{2}\left((1+\beta^2)\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) + (1+\beta^2)\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - (1-\beta^2)\frac{1}{\beta}\left(\frac{1}{(1-\beta)} - \frac{1}{(1+\beta)}\right) - (1-\beta^2)\frac{1}{\beta}\left(\frac{1}{(1-\beta)} - \frac{1}{(1+\beta)}\right) \right)$$

$$= \frac{1}{2}\left(2(1+\beta^2)\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 2(1-\beta^2)\frac{1}{\beta}\frac{2\beta}{1-\beta^2} \right)$$

$$= (1+\beta^2)\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - (1-\beta^2)\frac{1}{\beta}\frac{2\beta}{1-\beta^2}$$

$$= (1+\beta^2)\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 2$$

$\beta \to 1$ 时有发散!

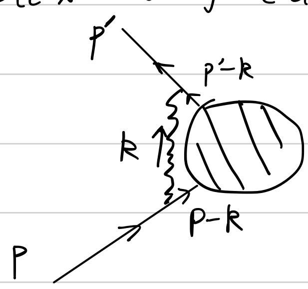
对于Soft photon emission中的第一项. (μ^2 represents mass of soft photon,

$\int \frac{dw}{w} \frac{e^2}{4\pi} \frac{1}{\pi} = \ln\left(\frac{\Lambda}{\mu^2}\right) \frac{e^2}{4\pi^2}$ $\quad \Lambda$ represents detector's sensitivity)

Combine results, Soft photon radiation.

$$\int d(\gamma - \text{radiation}) = \ln\left(\frac{\Lambda}{\mu^2}\right)\frac{e^2}{4\pi^2} \left\{ -2 + (1+\beta^2)\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) \right\}$$

- Complex modification



Original scattering amplitude

$$\mathcal{M}_0 = \bar{u}(p') M_0(p', p) u(p)$$

Modified scattering amplitude

$$\mathcal{M} = \int \frac{d^4k}{(2\pi)^4} \bar{u}(p')(-ie\gamma^\mu) \frac{1}{i} \frac{\not{p}' - \not{k} + m}{-(p'-k)^2 + m^2 - i\varepsilon} M_0(p'-k, p-k) \frac{1}{i} \frac{\not{p} - \not{k} + m}{-(p-k)^2 + m^2 - i\varepsilon}$$

$$(-ie\gamma^\nu) \frac{1}{i} \frac{g_{\mu\nu}}{k^2 + i\varepsilon} u(p)$$

Ignore lower order corrections

$$\mathcal{M} = \int \frac{d^4k}{(2\pi)^4} \bar{u}(p')(-ie\gamma^\mu) \frac{1}{i} \frac{\not{p}' + m}{2p'\cdot k - i\varepsilon} M_0(p', p) \frac{1}{i} \frac{\not{p} + m}{2p\cdot k - i\varepsilon} (-ie\gamma^\nu) \frac{1}{i} \frac{g_{\mu\nu}}{k^2 + i\varepsilon} u(p)$$

Use Dirac Equation to modify results

$$(\not{p} - m) u(p) = 0 \qquad \bar{u}(p)(\not{p} - m) = 0$$

$$\bar{u}(p')\gamma^\mu(\not{p}' + m) = \bar{u}(p')\gamma^\mu(p'_\nu\gamma^\nu + m) = \bar{u}(p')(p'_\nu\{\gamma^\mu, \gamma^\nu\} - p'_\nu\gamma^\nu\gamma^\mu + m\gamma^\mu)$$

$$= \bar{u}(p')(2p'^\mu - \not{p}'\gamma^\mu + m\gamma^\mu)$$

$$= \bar{u}(p')\, 2p'^\mu$$

$$(\not{p} + m)\gamma^\nu u(p) = (p_\alpha\gamma^\alpha + m)\gamma^\nu u(p) = \left(p_\alpha(\{\gamma^\alpha, \gamma^\nu\} - \gamma^\nu\gamma^\alpha) + m\gamma^\nu\right) u(p)$$

$$= (2p^\nu - \gamma^\nu\not{p} + \gamma^\nu m) u(p)$$

$$= (2p^\nu) u(p)$$

$$\mathcal{M} = \int \frac{d^4k}{(2\pi)^4} \bar{u}(p')(-e) \frac{2p'^\mu}{2p'\cdot k - i\varepsilon} M_0(p', p) \frac{2p^\nu}{2p\cdot k - i\varepsilon} (-e) \frac{1}{i} \frac{g_{\mu\nu}}{k^2 + i\varepsilon} u(p)$$

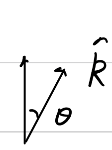
$$= \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') M_0(p', p) u(p)\, e^2 \frac{4p'\cdot p}{(2p'\cdot k - i\varepsilon)(2p\cdot k - i\varepsilon)} \frac{1}{k^2 + i\varepsilon} \left(\frac{1}{i}\right)$$

$$\equiv \mathcal{M}_0 \cdot K_V$$

K_V represents virtue modification.

$$K_V = \int \frac{d^4k}{(2\pi)^4} e^2 \frac{p'\cdot p}{(p'\cdot k - i\varepsilon)(p\cdot k - i\varepsilon)} \frac{1}{k^2 + i\varepsilon} \left(\frac{1}{i}\right)$$

Integral over k^0 first.



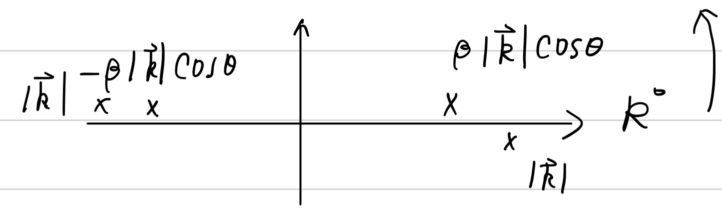
$$p \equiv E(1, \vec{v}) = E(1, 0, 0, \beta) \qquad p\cdot p' = E^2(1 + \beta^2)$$

$$p' \equiv E(1, \vec{v}') = E(1, 0, 0, -\beta)$$

$$p'\cdot k - i\varepsilon = Ek^0 - \beta E k^3 - i\varepsilon = Ek^0 - \beta E|\vec{k}|\cos\theta - i\varepsilon$$

$$p\cdot k - i\varepsilon = Ek^0 + \beta E k^3 - i\varepsilon = Ek^0 + \beta E|\vec{k}|\cos\theta - i\varepsilon$$

$$k^2 + i\varepsilon = (k^0)^2 - |\vec{k}|^2 + i\varepsilon \sim (k^0 - |\vec{k}| + i\varepsilon)(k^0 + |\vec{k}| - i\varepsilon)$$



围道取的有点奇怪.

$$K_v = \int \frac{d^4k}{(2\pi)^4} e^2 \frac{p'\cdot p}{(p'\cdot k - i\varepsilon)(p\cdot k - i\varepsilon)} \frac{1}{k^2 + i\varepsilon} \left(\frac{1}{i}\right)$$

$$= \int \frac{d^3k}{(2\pi)^4} (-ie^2)(p'\cdot p) \int dk^0 \frac{1}{Ek^0 - \beta E|\vec{k}|\cos\theta - i\varepsilon} \frac{1}{Ek^0 + \beta E|\vec{k}|\cos\theta - i\varepsilon} \frac{1}{(k^0 - |\vec{k}| + i\varepsilon)(k^0 + |\vec{k}| - i\varepsilon)}$$

$$= \int \frac{d^3k}{(2\pi)^4} (-ie^2)(p'\cdot p)(2\pi i) \Bigg(\frac{1}{-E|\vec{k}| - \beta E|\vec{k}|\cos\theta} \frac{1}{-E|\vec{k}| + \beta E|\vec{k}|\cos\theta} \frac{1}{-2|\vec{k}|}$$
$$+ \frac{1}{-2E\beta|\vec{k}|\cos\theta} \frac{1}{(-\beta|\vec{k}|\cos\theta - |\vec{k}|)(-\beta|\vec{k}|\cos\theta + |\vec{k}|)}$$
$$+ \frac{1}{2E\beta|\vec{k}|\cos\theta} \frac{1}{(\beta|\vec{k}|\cos\theta - |\vec{k}|)(\beta|\vec{k}|\cos\theta + |\vec{k}|)} \Bigg)$$

$$= \int \frac{d^3k}{(2\pi)^3} e^2 (p'\cdot p) \Bigg(\frac{1}{-2E^2|\vec{k}|^3(1+\beta\cos\theta)(1-\beta\cos\theta)} + \frac{1}{2E|\vec{k}|^3\beta\cos\theta} \frac{1}{(1+\beta\cos\theta)(1-\beta\cos\theta)}$$
$$- \frac{1}{2E|\vec{k}|^3\beta\cos\theta} \frac{1}{(1-\beta\cos\theta)} \frac{1}{(1+\beta\cos\theta)} \Bigg)$$

$$= \int \frac{d^3k}{(2\pi)^3} e^2(1+\beta^2) \frac{1}{-2\,|\vec{k}|^3(1+\beta\cos\theta)(1-\beta\cos\theta)}$$

$$= \int w^2 dw\, d\Omega \frac{1}{(2\pi)^3} e^2(1+\beta^2) \frac{1}{-2w^3} \left(\frac{1}{1-\beta\cos\theta} + \frac{1}{1+\beta\cos\theta} \right) \frac{1}{2}$$

$$= \int \frac{dw}{w} 2\pi\, d\cos\theta \frac{1}{(2\pi)^3} e^2(1+\beta^2) \frac{1}{-4} \left(\frac{1}{1-\beta\cos\theta} + \frac{1}{1+\beta\cos\theta} \right)$$

$$= \int \frac{dw}{w} \cdot \frac{1}{4\pi^2} e^2 (1+\beta^2) \frac{-1}{4} \ln\left(\frac{1+\beta}{1-\beta}\right) \times 2 \frac{1}{\beta}$$

$$= \int \frac{dw}{w} \frac{e^2}{4\pi} \frac{-1}{2\pi} (1+\beta^2) \ln\left(\frac{1+\beta}{1-\beta}\right) \frac{1}{\beta}$$

$$(1+K_v)^2 \doteq \left(1 - \ln\left(\frac{\Lambda}{\mu^2}\right) \frac{e^2}{4\pi^2} (1+\beta^2) \frac{1}{\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \right)$$

$$\left(1 + \left(\int dr \text{ radiation}\right)^2\right)(1+K_v)^2 = 1$$ (In high-energy limit, $\beta \to 1$).