连续媒介质中的Maxwell方程组

- 真空中的 Maxwell 方程组.

$$\nabla\cdot E(\vec r,t)=\frac{1}{\varepsilon_0}\rho(\vec r,t) \qquad \nabla\times\vec E(\vec r,t)=-\frac{\partial\vec B(\vec r,t)}{\partial t}$$

$$\nabla\cdot B(\vec r,t)=0 \qquad \nabla\times\vec B(\vec r,t)=\mu_0\vec j(\vec r,t)+\mu_0\varepsilon_0\frac{\partial\vec E(\vec r,t)}{\partial t}$$

能量密度 $u=\frac{1}{2}(E\cdot D+H\cdot B)$

能流 ; $S=E\times H$

—— 在媒介质中, 分为自由电荷/电流, 和束缚电荷/电流.

$$\rho=\rho_b+\rho_f$$

$$j=j_b+j_f$$

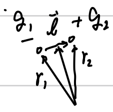
- 束缚电荷density与电极化密度 矢量 的关系:

电极化密度矢量的定义:

$$\vec P=\lim_{V\to0}\frac{\overbrace{\sum_i\vec p_i(t)}^{}}{V}=\lim_{V\to0}\frac{\sum_i q_i\vec r_i}{V}$$

分子内每一个电荷量

单个分子的电偶极矩矢量.



$q_1\vec r_1+q_2\vec r_2=-q\vec r_1+q\vec r_2=q\cdot\vec l=\vec p$ ← 单个分子电偶极矩.

—— 束缚电荷性质之总合为0!

$$\iiint dV\ \rho_b(\vec r,t)=0$$

认为: $\rho_b=\nabla\cdot G(\vec r,t)$

$$\iiint dV\ \nabla\cdot G(\vec r,t)=\oiint d\vec S\cdot G(\vec r,t)=0$$

boundary of 介质!

—— 总偶极矩为:

$$\mathcal{P}=\iiint dV\ \rho_b(\vec r,t)\vec r=\iiint dV\cdot\nabla\cdot\vec G(\vec r,t)\vec r=\iiint dxdydz\cdot\left(\frac{\partial G_x}{\partial x}+\frac{\partial G_y}{\partial y}+\frac{\partial G_z}{\partial z}\right)(x\hat i+y\hat j+z\hat k)$$

$$=\iiint dxdydz\left[\left(x\frac{\partial G_x}{\partial x}+x\frac{\partial G_y}{\partial y}+x\frac{\partial G_z}{\partial z}\right)\hat i+\cdots\right]$$

$$=\iint dydz\cdot\left(\Delta(x\,G_x)\Big|_{x-}^{x+}-\int dx\,G_x\right)\hat i+\iint dxdz\cdot\Delta[x\,G_y]\Big|_{y-}^{y+}\hat i+\iint dxdy\ \Delta[xG_z]\Big|_{z-}^{z+}\hat i$$

$$+\cdots$$

$$=\iiint dxdydz\cdot(-\vec G)$$

另外, 由于: $\mathcal{P}=\iiint dxdydz\ \vec P=\iiint dxdydz(-\nabla\cdot\vec P)\vec r$

In all:

$$\vec G=-\vec P$$

$$\rho_b=-\nabla\cdot\vec P$$

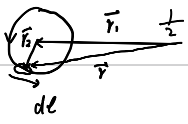
- 束缚电流与磁偶极矩密度的关系!

磁化密度矢量的定义:

$$\vec M=\lim_{V\to0}\frac{\sum_i M_i}{V}=\lim_{V\to0}\frac{\int dV\frac{1}{2}\vec r\times\vec j}{V}$$

分子磁偶极矩

分子内电流



$\frac{1}{2}\oint\underbrace{(\vec r_1+\vec r_2)\times d\vec l}_{积分后是0}\ I=\vec m$

—— 束缚电流分类:

由束缚电荷引起

$J_b=J_b^P+J_b^M$ ← 由分子磁矩引起.

$$J_b^P=\lim_{V\to0}\frac{\sum_i q_iV_i}{V}=\frac{\partial}{\partial t}\left(\lim_{V\to0}\frac{\sum_i q_i r_i}{V}\right)=\frac{\partial}{\partial t}\vec P(\vec r,t)=\frac{\partial}{\partial t}\vec P(\vec r,t)$$

—— 由分子电流引起的束缚电流满足性质引出束缚分子电流表示

$$\oiint J_b^M \cdot dS = 0$$ （任何封闭面通过总分子电流是0）

$$\iiint \nabla \cdot J_b^M \, dx\,dy\,dz = 0$$

$$\downarrow$$

$$\nabla \cdot J_b^M = 0$$

Suppose： $J_b^M = \nabla \times K(\vec{r}, t)$

总磁偶极矩：

$$\mathcal{M}_{tot} = \frac{1}{2} \iiint \vec{r} \times J_b^M(\vec{r}, t) \cdot d^3 r$$

$$= \frac{1}{2} \iiint \vec{r} \times (\nabla \times K(\vec{r}, t))\, d^3 r$$

$$\nabla(\vec{K} \cdot \vec{r}) = (K \cdot \nabla)\vec{r} + (\vec{r} \cdot \nabla)\vec{K} + \vec{r} \times (\nabla \times \vec{K}) + \vec{K} \times (\nabla \times \vec{r})$$

$$\Downarrow$$

$$r \times (\nabla \times K) = \nabla(K \cdot r) - (K \cdot \nabla) r - (r \cdot \nabla) K - K \times (\nabla \times r)$$

$$= \nabla(K \cdot r) - (K \cdot \nabla) r - (r \cdot \nabla) K$$

$$= \nabla(K \cdot r) - K - (r \cdot \nabla) K$$

$$= \frac{1}{2} \iiint \vec{r} \times (\nabla \times K)\, d^3 r$$

$$= \frac{1}{2} \iiint \nabla(K \cdot r)\, d^3 r - \frac{1}{2} \iiint K\, d^3 r - \frac{1}{2} \iiint (r \cdot \nabla) K\, d^3 r$$

$$= -\frac{1}{2} \iiint K\, dV - \frac{1}{2} \iiint \left(x \frac{\partial K}{\partial x} + y \frac{\partial K}{\partial y} + z \frac{\partial K}{\partial z}\right) d^3 r$$

$$= -\frac{1}{2} \iiint K\, dV + \frac{1}{2} \iiint (K + K + K)\, dx\,dy\,dz$$

$$= \iiint K\, dx\,dy\,dz$$

$\longrightarrow \iiint \nabla(K \cdot r)\, d^3 r = 0$ 证明

仅 consider x direction!

$$\nabla(K \cdot r)\big|_x = \frac{\partial}{\partial x}(K \cdot r)$$

$$\iiint \frac{\partial}{\partial x}(K \cdot r)\, dx\,dy\,dz$$

$$= \iint \Delta(K \cdot r)\Big|_{x-}^{x+} dy\,dz$$

$$= 0$$

由于： $\mathcal{M}_{tot} = \iiint M\, dx\,dy\,dz = \iiint \frac{1}{2} \vec{r} \times (\nabla \times M)\, d^3 r$

M：磁化密度矢量.

对比得：（束缚分子电流表示为）：

$$K = M$$

$$J_b^M = \nabla \times M$$

- 实验发现：

—— 电场

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$$

$\vec{P}$：极化密度矢量. χ_e：媒介的电导率. $\vec{E}$：电场强度矢量 ε_r：相对介电场数.

$\downarrow$ 电位移矢量.

$$\vec{D} = \varepsilon_0 E + \vec{P} = \varepsilon_0 \varepsilon_r \vec{E}$$

—— 磁场.

定义： $H = \frac{1}{\mu_0} B - M$

H：磁场强度矢量 B：磁感应强度矢量. M：磁化密度矢量

实验发现： $M = \chi_m H = (\mu_r - 1) H$

则： $B = \mu_0 (H + M) = \mu_0 (M + \frac{1}{\mu_r - 1} M) = \frac{\mu_0 \mu_r}{\mu_r - 1} M \Rightarrow M = \frac{1}{\mu_0}(1 - \frac{1}{\mu_r}) B \Rightarrow H = \frac{1}{\mu_0 \mu_r} B$

—— 欧姆定律：

$$\vec{j}_f = \overset{\text{某介质的电导率}}{\sigma} \vec{E}$$

○ 改写 Maxwell Equation 为 ρ_f 和 j_f 对应的方程

Original maxwell Eq

$$\nabla \cdot E = \frac{1}{\varepsilon_0}(\rho_f + \rho_b) \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0(j_f + j_b^P + j_b^M) + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

将此结论带入：

$$\nabla \cdot E = \frac{1}{\varepsilon_0}(\rho_f - \nabla \cdot P) \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0\left(j_f + \frac{\partial \vec{P}}{\partial t} + \nabla \times M\right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

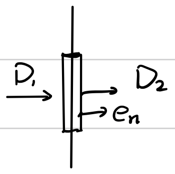
$$\Downarrow$$

$$\begin{cases} \varepsilon_0 \vec{E} + \vec{P} = \vec{D} \\ \frac{1}{\mu_0} \vec{B} - \vec{M} = \vec{H} \end{cases}$$

$$\begin{cases} \nabla \cdot \vec{D} = \rho_f & \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} = 0 & \nabla \times \vec{H} = j_f + \frac{\partial \vec{P}}{\partial t} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \\ & \quad = j_f + \frac{\partial \vec{D}}{\partial t} \end{cases}$$

○ 电磁场的边值关系.

—— 电位移矢量与自由电荷面密度.



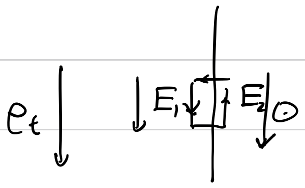
$$\nabla \cdot D = \rho_f \Rightarrow \iiint d^3r \, \nabla \cdot D = \iiint d^3r \, \rho_f$$

$$\oint D \cdot dS = S(D_2 - D_1) = S \sigma_f$$

$$D_2 - D_1 = \sigma_f$$

$$e_n \cdot (D_2 - D_1) = \sigma_f$$

—— 电场强度矢量切向守恒.



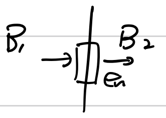
$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\iint \nabla \times E \cdot ds = -\iint \frac{\partial B}{\partial t} \cdot ds = 0$$

$$\oint E \cdot dl = (E_1 - E_2) l = 0$$

$$e_n \times (E_2 - E_1) = 0$$

—— 磁感应强度矢量法向守恒.



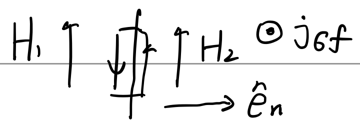
$$\nabla \cdot B = 0$$

$$\Downarrow$$

$$e_n \cdot (B_2 - B_1) = 0$$

—— 磁场强度矢量切向关系

$$\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$



$$\nabla \times H = j_f + \frac{\partial \vec{D}}{\partial t} \Rightarrow \oint dl \cdot H = \iint ds \cdot \left(j_f + \frac{\partial \vec{D}}{\partial t}\right) = l \cdot j_{\sigma f}$$

$$e_n \times (H_2 - H_1) = J_{\sigma f}(r, t)$$

电磁场的Lorentz变换

- $E'_{\parallel}(x',y',z',t') = E_{\parallel}(x,y,z,t)$

  $B'_{\parallel}(x',y',z',t') = B_{\parallel}(x,y,z,t)$

  $E'_{\perp}(x',y',z',t') = \gamma[E_{\perp}(x,y,z,t) + \vec{V} \times \vec{B}(x,y,z,t)]$

  $B'_{\perp}(x',y',z',t') = \gamma[B_{\perp}(x,y,z,t) - \frac{\vec{v}}{c^2} \times \vec{E}(x,y,z,t)]$

- Green Function 法是为了求解静电学问题(无磁场)

由 Maxwell Equation.

$$\nabla\times E = -\frac{\partial B}{\partial t} = 0$$

电场可由势函数的散度表示：

$$E = -\nabla\Phi.$$

电位移矢量 $D=\epsilon E$，由 Maxwell Eq，

$$\nabla\cdot D = \rho_f$$

$$-\nabla\cdot(\nabla\Phi) = \frac{\rho_f}{\epsilon}$$

$$\nabla^2\Phi = -\frac{\rho_f}{\epsilon}$$

—— 边值问题的分类：

1类边值问题：空间区域 Ω 内电荷分布 ρ_f 与边界 $\partial\Omega$ 上电势 Φ 已给定.

2类边值问题：空间 Ω 内 电荷分布 ρ_f 与边界 $\partial\Omega$ 上电场法向 $\frac{\partial\Phi}{\partial n}$ 给定.

—— Green Function 的定义：

$$\nabla_r^2 G(r,r') = -\frac{1}{\varepsilon_0}\delta(r-r')$$ 理解为：r' 处的电荷 1 产生的影响.

Green Function 性质①：

$$\nabla_{r'}^2 G(r',r) = -\frac{1}{\varepsilon_0}\delta(r'-r) = -\frac{1}{\varepsilon}\delta(r-r')$$

$$= \nabla_r^2 G(r,r')$$

Green Function 性质②：

在一类边界条件下. $G(r,r') = G(r',r)$ 成立.

—— Green Function 的作用.(Φ函数的解)

已知：

$$\begin{cases}\nabla_{r'}^2\Phi(r') = -\frac{1}{\varepsilon}\rho(r') \\ \nabla_r^2 G(r,r') = -\frac{1}{\varepsilon_0}\delta(r-r')\end{cases}$$

$$\Downarrow$$

$$G(r,r')\nabla_{r'}^2\Phi(r') - \Phi(r')\nabla_r^2 G(r,r') = -\frac{1}{\varepsilon}\rho_f(r')G(r,r') + \frac{1}{\varepsilon_0}\delta(r-r')\Phi(r')$$

$$\begin{cases}G(r,r') = G(r',r) \\ \nabla_r^2 G(r,r') = \nabla_{r'}^2 G(r',r)\end{cases}$$ ⇐ 一类边界条件时的性质.

↑为什么在二类边界也可用此性质?

$$G(r',r)\nabla_{r'}^2\Phi(r') - \Phi(r')\nabla_{r'}^2 G(r',r) = -\frac{1}{\varepsilon}\rho_f(r')G(r,r') + \frac{1}{\varepsilon_0}\delta(r-r')\Phi(r')$$

$$\iiint d^3r'\left[G(r',r)\nabla_{r'}^2\Phi(r') - \Phi(r')\nabla_{r'}^2 G(r',r)\right] = \iiint d^3r'\cdot\left[-\frac{1}{\varepsilon}\rho_f(r')G(r,r') + \frac{1}{\varepsilon_0}\delta(r-r')\Phi(r')\right]$$

$$\iiint d^3r'\,\nabla_{r'}\cdot\left[G(r',r)\nabla_{r'}\Phi(r') - \Phi(r')\nabla_{r'}G(r',r)\right] = -\frac{1}{\varepsilon}\iiint d^3r'\,\rho_f(r')G(r,r') + \frac{1}{\varepsilon_0}\Phi(r)$$

$$\oiint dS'\cdot\left[G(r',r)\nabla_{r'}\Phi(r') - \Phi(r')\nabla_{r'}G(r',r)\right] = -\frac{1}{\varepsilon}\iiint d^3r'\,\rho_f(r')G(r,r') + \frac{1}{\varepsilon_0}\Phi(r)$$

$$\Phi(r) = \frac{\varepsilon_0}{\varepsilon}\iiint d^3r'\,\rho_f(r')G(r,r') + \varepsilon_0\oiint_{\partial\Omega} dS'\left[G(r',r)\nabla_{r'}\Phi(r') - \Phi(r')\nabla_{r'}G(r',r)\right]$$

↓理解为单幅时边界势为0.

一类边值问题：$G(r,r')\big|_{\partial\Omega} = 0$，Φ已知。

$$\Phi(r) = \frac{\varepsilon_0}{\varepsilon}\iiint d^3r'\,\rho_f(r')G(r,r') - \varepsilon_0\oiint_{\partial\Omega} dS'\cdot\Phi(r')\frac{\partial G(r',r)}{\partial n'}$$

—— 常见的 $G(r,r')$ 解 ① 无界空间中的 Green Function，无穷远是边界/单电荷，边界势能为0.

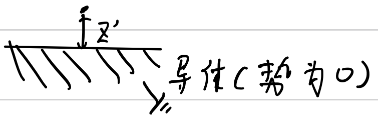
$$\nabla_r G(r,r') = -\frac{1}{\varepsilon_0}\delta(r-r')$$

$$G_0(r,r') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|r-r'|}$$

$$\downarrow \leftarrow \left\{ \nabla_r^2 \frac{1}{|r-r'|} = -4\pi\delta(r-r') \right.$$

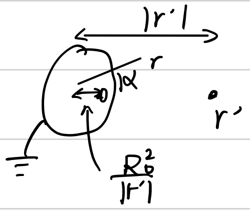
$$\nabla_r^2 G_0(r,r') = -\frac{1}{\varepsilon_0}\delta(r-r')$$

—— ② 上半平面格林函数/单电荷，边界势为0.



$$G_1(r,r') = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right)$$

—— ③ 球外空间格林函数.



$$G_2(r,r') = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\alpha}} - \frac{R_0/r'}{\sqrt{r^2+b^2-2rb\cos\alpha}}\right)$$

↓

电像法!

电偶极矩与电四极矩.

- function Tylor 展开.

$$f(r-r') \approx f(r) + \sum_{i=1}^{3} \frac{\partial f(r)}{\partial x_i}(-x_i') + \frac{1}{2!}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 f(r)}{\partial x_i \partial x_j}(-x_i')(-x_j')$$

$$= f(r) - r'\cdot\nabla_r f(r) + \frac{1}{2!}(r'\cdot\nabla_r)^2 f(r)$$

- 介质中的电势.

$$\Phi(r) \approx \frac{1}{4\pi\varepsilon}\iiint dr' \frac{\rho_f(r')}{|r-r'|}$$

$$= \frac{1}{4\pi\varepsilon}\iiint dr' \rho_f(r')\left[\frac{1}{r} - r'\cdot\nabla_r\frac{1}{r} + \frac{1}{2!}\sum_{ij} x_i' x_j' \frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{r}\right]$$

$$= \frac{1}{4\pi\varepsilon}\left[\frac{\iiint dr' \rho_f(r')}{r} - \underbrace{\left(\iiint dr' \rho_f(r') r'\right)}_{\vec{P}}\cdot\nabla_r\frac{1}{r} + \frac{1}{6}\left(\sum_{ij}\underbrace{\iiint d^3r' \cdot 3x_i' x_j' \cdot \rho_f(r')}_{D_{ij}}\right)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{1}{r}\right]$$

用球谐函数的方法求解

○ 球坐标下的 Laplace 算符以及它的求解.

球坐标下的 Laplace Equation

$$\nabla^2 f = \left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right\} \cdot f(r,\theta,\varphi) = 0$$

分离变量法求解:

$$f(r,\theta,\varphi) = R(r)\cdot Y_L^M(\theta,\varphi)$$

分别满足方程.

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\varphi^2} \right) Y_L^M(\theta,\varphi) = -L(L+1)\,Y_L^M(\theta,\varphi)$$

$$\left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L(L+1)}{r^2} \right\} R_L(r) = 0$$

它的解是: $L = 0, 1, \cdots$ $\quad M = -L, -L+1, -L+2, \cdots +L.$

$$P_L^M(x) = \begin{cases} (-1)^M (1-x^2)^{M/2} \frac{d^M}{dx^M} P_L(x) & M \geq 0 \\ (-1)^{|M|} \frac{(L-|M|)!}{(L+|M|)!} P_L^{|M|}(x) & M < 0 \end{cases} \qquad \left\} \begin{matrix} P_0^0 \sim 1 \\ P_1^0 \sim \cos\theta \end{matrix} \right.$$

记住!

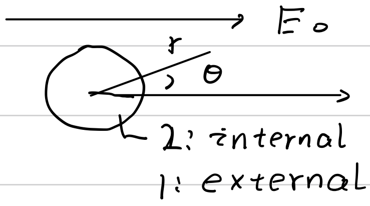
$$P_L(x) = \frac{1}{2^L L!}\frac{d^L}{dx^L}(x^2-1)^L$$

$$R_L(x) = \left(C_L r^L + D_L \frac{1}{r^{L+1}} \right)$$

总之.

$$f(r,\theta,\varphi) = \sum_{L=0}^{+\infty}\sum_{M=-L}^{L}\left(C_L r^L + D_L \frac{1}{r^{L+1}} \right)\cdot Y_L^M(\theta,\varphi)$$

○ 真空中有电场 $\vec{E}_0$. 且有一个介质球, ε, 不带自由电荷. 问, 真空中的电场分布.



$\Phi = -E_0 r\cos\theta$

关于 φ 对称, $M=0$.

$$\Phi_1(r,\theta,\varphi) = \sum_L \left(a_L r^L + b_L \frac{1}{r^{L+1}} \right) P_L(\cos\theta) \qquad \text{external}$$

$$\Phi_2(r,\theta,\varphi) = \sum_L \left(C_L r^L + d_L \frac{1}{r^{L+1}} \right) P_L(\cos\theta) \qquad \text{internal}$$

$\longleftarrow d_L = 0$ (不发散)

external 的边界条件:

$$\sum_L \left(a_L r^L + b_L \frac{1}{r^{L+1}} \right) P_L(\cos\theta) \Big|_{r\to\infty} = -E_0 r\cos\theta$$

得到.

$$a_0 + a_1 \cdot r \cdot \cos\theta = -E_0 r \cos\theta$$

$$a_1 = -E_0, \quad others = 0$$

$$\begin{cases} \Phi_1(r,\theta,\varphi) = -E_0 r \cdot \cos\theta + \sum_L b_L \frac{1}{r^{L+1}} P_L(\cos\theta) \\ \Phi_2(r,\theta,\varphi) = \sum_{L=0}^{+\infty} C_L r^L \cdot P_L(\cos\theta) \end{cases}$$

内外 Φ 连续, $\frac{\partial\Phi}{\partial n} \cdot \varepsilon$ 连续.

$$-E_0 R\cos\theta + \sum_{L=0}^{+\infty} \underset{\sim}{b_L} \frac{1}{R^{L+1}} \cdot P_L(\cos\theta) = \sum_{L=0}^{+\infty} (\underset{\sim}{C_L} R^L) \cdot P_L(\cos\theta)$$

$$\varepsilon_0 \left\{ -E_0 \cos\theta - \sum_{L=0}^{+\infty} (L+1) \underset{\sim}{b_L} \frac{1}{R^{L+2}} P_L(\cos\theta) \right\} = \varepsilon \left\{ \sum_{L=0}^{+\infty} (\underset{\sim}{L C_L} \cdot R^{L-1}) P_L(\cos\theta) \right\}$$

1° 对比 $L=0$ 的 coefficient:

$$\begin{aligned} b_0 \frac{1}{R} &= C_0 \\ -b_0 \frac{1}{R^2} \varepsilon_0 &= 0 \end{aligned} \quad \longrightarrow \quad b_0 = C_0 = 0$$

2° 对比 $L=1$ 的 coefficient

$$-E_0 R + b_1 \frac{1}{R^2} = C_1 R$$

$$\varepsilon_0 \left\{ -E_0 - 2 b_1 \frac{1}{R^3} \right\} = \varepsilon C_1$$

$$\downarrow$$

$$\longleftarrow \quad C_1 = -\frac{\varepsilon_0}{\varepsilon} E_0 - 2 \frac{\varepsilon_0}{\varepsilon} \frac{1}{R^3} b_1$$

$$-E_0 R + b_1 \frac{1}{R^2} = -\frac{\varepsilon_0}{\varepsilon} E_0 R - 2 \frac{\varepsilon_0}{\varepsilon} \frac{1}{R^2} b_1$$

$$b_1 = R^2 \frac{E_0 R - \frac{\varepsilon_0}{\varepsilon} E_0 R}{1 + 2 \frac{\varepsilon_0}{\varepsilon}}$$

$$= E_0 R^3 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}$$

$$C_1 = -E_0 \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0}$$

3° 对比 2 … 以上阶 coefficient.

$$\begin{aligned} b_L \frac{1}{R^{L+1}} &= C_L \cdot R^L \\ \varepsilon_0 \left\{ -b_L \frac{L+1}{R^{L+2}} \right\} &= \varepsilon \left\{ C_L \, L \, R^{L-1} \right\} \end{aligned} \quad \longrightarrow \quad C_L = b_L = 0$$

In all

$$\Phi_1(r,\theta,\varphi) = -E_0 r\cos\theta + E_0 R^3 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \frac{1}{r^2} \cos\theta$$

$$\Phi_2(r,\theta,\varphi) = -E_0 \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} r \cos\theta$$

静磁学

○ Maxwell 方程

$$\nabla\times H = j_f \qquad \nabla\cdot B = 0$$

假设：

$$B = \nabla\times A \Rightarrow A \to A + \nabla\Lambda \text{ 不改变 } B \Rightarrow \text{columb gauge} \quad \nabla\cdot A = 0$$

↙ 默认了 H, B 比例关系

$$\nabla\times H = \nabla\times\left(\frac{1}{\mu}B\right) = \frac{1}{\mu}\nabla\times(\nabla\times A) = j_f$$

↓ ← $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$

$$\frac{1}{\mu}\nabla(\nabla\cdot A) - \frac{1}{\mu}\nabla^2 A = j_f$$

↓ ← $\nabla\cdot A = 0$

$$\nabla^2 A = -\mu j_f$$

类比静电势满足方程. $\nabla^2\Phi = -\frac{1}{\varepsilon}\rho_f$

$$A = \frac{\mu}{4\pi}\iiint\frac{j_f(r')}{|r-r'|}dx'dy'dz'$$

$$B(r) = \nabla_r\times A(r) = \nabla_r\times\left(\frac{\mu}{4\pi}\iiint\frac{j_f(r')}{|r-r'|}dx'dy'dz'\right)$$

$$= \frac{\mu}{4\pi}\iiint\left(\nabla_r\frac{1}{|r-r'|}\right)\times j_f(r')\,dx'dy'dz'$$

$$= \frac{\mu}{4\pi}\iiint\left(-\frac{r-r'}{|r-r'|^3}\right)\times j_f(r')\,d^3r'$$

$$= \frac{\mu}{4\pi}\iiint j_f(r')\times\frac{r-r'}{|r-r'|^3}d^3r'$$

在根导线上时 $= \frac{\mu}{4\pi}\oint_c I\,dl\times\frac{r-r'}{|r-r'|^3}$

磁标势法

- Maxwell 方程(引出磁标势)

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 j + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

$$\Downarrow$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 (j_f + \underbrace{\nabla \times M}_{j_b^M} + \underbrace{\frac{\partial P}{\partial t}}_{j_b^P}) + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

$$\Downarrow \leftarrow \begin{cases} H = \frac{1}{\mu_0} B - M \\ D = \varepsilon_0 E + P \end{cases} \Rightarrow \begin{cases} \text{一般情况. } D = \varepsilon_0 \varepsilon_r E \quad H = \frac{1}{\mu_0 \mu_r} B. \\ \text{但磁标势法一般用于铁磁物质不符合} \end{cases}$$

$$\nabla \cdot B = 0 \qquad \nabla \times H = j_f + \frac{\partial D}{\partial t}$$

$$\Downarrow \begin{cases} \text{无 } j_f \text{, 无 } D \end{cases}$$

$$\nabla \cdot B = 0 \qquad \nabla \times H = 0$$

$$\Downarrow$$

$$H = - \nabla \Phi_M$$

- 用类比束缚电荷的方法引入磁荷.

$$H = \frac{1}{\mu_0} B - M$$

$$B = \mu_0 H + \mu_0 M$$

$$\nabla \cdot B = \mu_0 \nabla \cdot H + \mu_0 \nabla \cdot M = 0$$

$$\nabla \cdot P = - \rho_b \overset{\text{类比}}{\Longleftrightarrow} \mu_0 \nabla \cdot M = - \rho_M \leftarrow \text{磁荷.}$$

$$\Downarrow$$

$$\mu_0 \nabla \cdot H = \rho_M$$

$$\nabla \cdot H = \frac{1}{\mu_0} \rho_M$$

$$\Downarrow \leftarrow \} \; H = - \nabla \Phi_M$$

$$\nabla^2 \Phi_M = - \frac{1}{\mu_0} \rho_M$$

磁多极矩.

○ A 的表达式:

$$A(r) = \frac{\mu_0}{4\pi} \iiint_\Omega \frac{j_f(r')}{|r-r'|}\, dx'dy'dz'$$

$$\downarrow \leftarrow \left\{ \frac{1}{|r-r'|} = \frac{1}{r} - (r'\cdot\nabla_r)\frac{1}{r} \right.$$

$$= \frac{\mu_0}{4\pi} \iiint \frac{j_f(r')}{r}\, dx'dy'dz' - \frac{\mu_0}{4\pi} \iiint j_f(r')\left(r'\cdot\nabla_r \frac{1}{r}\right) dx'dy'dz'$$

$$= \frac{\mu_0}{4\pi r} \iiint_\Omega j_f(r')\, dx'dy'dz' + \frac{\mu_0}{4\pi} \iiint j_f(r')\left(\sum_i \frac{x_i' x_i}{r^3}\right) dx'dy'dz'$$

自由电流，别的都是任意spacial Function.

$$\iiint_\Omega \left(f(r')[j_f(r')\cdot\nabla_{r'}]g(r') + \underbrace{g(r')[j_f(r')\cdot\nabla_{r'}]f(r')}\right) dx'dy'dz' = 0$$

proof:

$$\iiint_\Omega g(r')[j_f(r')\cdot\nabla_{r'}]f(r')\, dx'dy'dz'$$

$$= \iiint_\Omega g(r')\left[j_{fx}(r')\frac{\partial f(r')}{\partial x'} + j_{fy}\frac{\partial f}{\partial y'} + j_{fz}\frac{\partial f}{\partial z'}\right] dx'dy'dz'$$

$$= \iint_{S_x} g(r')\, j_{fx}(r')\, f(r')\Big|_{x'=-\infty}^{x'=+\infty} dy'dz' - \iiint dx'dy'dz'\, f(r')\cdot \frac{\partial}{\partial x'}[g(r')\, j_{fx}(r')]$$

$$+ \cdots$$

$$\downarrow \leftarrow \left\{ j_{fx}\big|_{x=-\infty} = j_{fx}\big|_{x=+\infty} = 0 \right.$$ ！，电流边界条件！

$$= -\iiint dx'dy'dz'\, f(r')\frac{\partial}{\partial x'}(g(r')\, j_{fx}(r'))$$

$$- \iiint dx'dy'dz'\, f(r')\frac{\partial}{\partial y'}(g(r')\, j_{fy}(r'))$$

$$- \iiint dx'dy'dz'\, f(r')\frac{\partial}{\partial z'}(g(r')\, j_{fz}(r'))$$

$$= -\iiint dx'dy'dz'\, f(r')[\nabla_{r'}\cdot j_f(r')]\, g(r')$$

$$- \iiint dx'dy'dz'\, f(r')[j_f(r')\cdot\nabla_{r'}]\, g(r')$$

$$\downarrow \leftarrow \left\{ \nabla_{r'}\cdot j_f(r') = 0 \right.$$ ，电流稳恒条件.

$$= -\iiint dx'dy'dz'\, f(r')[j_f(r')\cdot\nabla_{r'}]\, g(r')$$

1° $f(r') = 1 \quad g(r') = x_i'$

$$\iiint_\Omega dx'dy'dz' [j_f(r')\cdot\nabla_{r'}]x_i' = 0$$

$$\downarrow$$

$$\iiint_\Omega dx'dy'dz'\, j_f(r') = 0$$

2° $f(r') = x_i' \quad f(r') = x_k$

$$\iiint_\Omega dx'dy'dz' \left(x_k'[j_f(r')\cdot\nabla_{r'}]x_i' + x_i'[j_f(r')\cdot\nabla_{r'}]x_k'\right) = 0$$

$$\iiint_\Omega dx'dy'dz' \left(x_k'\, j_{fi}(r') + x_i'\, j_{fk}(r') \right) = 0$$

$$\Downarrow$$

$\Downarrow$

$$\sum_{i=1}^{3}\iiint dx'dy'dz'\ j_{fk}(r')\,x_i'x_i = \sum_{i=1}^{3} x_i\iiint dx'dy'dz'\ j_{fk}(r')\,x_i'$$

↑第k分量!

$$= \sum_{i=1}^{3} x_i\left(\frac{1}{2}\iiint_\Omega j_{fk}(r')\,x_i'\ dx'dy'dz' + \frac{1}{2}\iiint j_{fk}(r')\,x_i'\ dx'dy'dz'\right)$$

$$= \sum_{i=1}^{3} x_i\left(-\frac{1}{2}\iiint j_{fi}(r')\,x_k'\,dx'dy'dz' + \frac{1}{2}\iiint j_{fk}(r')\,x_i'\ dx'dy'dz'\right)$$

$$= -\frac{1}{2}\sum_{i=1}^{3} \underset{a}{x_i} \iiint_\Omega \left(\underset{c}{j_{fi}(r')}\,\underset{b}{x_k'} - j_{fk}(r')\,x_i'\right) dx'dy'dz'$$

$$a\times(b\times c) = (a\cdot c)\,b - (a\cdot b)\,c$$

$$= -\frac{1}{2}\left(r\times\iiint_\Omega (r'\times j_f(r'))\,dx'dy'dz'\right)_k$$

$$= \frac{\mu_0}{4\pi r}\iiint_\Omega j_f(r')\,dx'dy'dz' + \frac{\mu_0}{4\pi}\iiint j_f(r')\left(\sum_i \frac{x_i'x_i}{r^3}\right)dx'dy'dz'$$

$$\iiint j_f(r')\,d^3r' = 0$$

$$\iiint j_f(r')\sum_i x_i'x_i\ d^3r' = -\frac{1}{2}\left(r\times\iiint (r'\times j_f(r'))\,d^3r'\right) = \underbrace{m}_{\downarrow}\times r$$

$$m = \frac{1}{2}\iiint (r'\times j_f(r'))\,d^3r'$$

$$= \frac{\mu_0}{4\pi}(m\times r)\cdot\frac{1}{r^3}$$

∘ B 的表达式.

$$B = \nabla\times A = \frac{\mu_0}{4\pi}\nabla\times\left(\frac{m\times r}{r^3}\right)$$

$$a\times(b\times c) = (a\cdot c)\,b - (a\cdot b)\,c$$

$$= \frac{\mu_0}{4\pi}\left(\left(\frac{\vec{r}}{r^3}\cdot\nabla\right)m + \left(\nabla\cdot\frac{\vec{r}}{r^3}\right)m - (m\cdot\nabla)\frac{\vec{r}}{r^3} - (\nabla\cdot m)\frac{\vec{r}}{r^3}\right)$$

$$= \frac{\mu_0}{4\pi}\left(\left(\nabla\cdot\frac{\vec{r}}{r^3}\right)m - (m\cdot\nabla)\frac{\vec{r}}{r^3}\right)$$

$$\nabla\cdot\frac{\vec{r}}{r^3} = \nabla\cdot\left(-\nabla\frac{1}{r}\right) = -\nabla^2\frac{1}{r} = -(-4\pi\delta(r)) = 4\pi\delta(r)$$

$$= -\frac{\mu_0}{4\pi}(m\cdot\nabla)\frac{\vec{r}}{r^3} = \frac{\mu_0}{4\pi}\,\frac{3(m\cdot e_r)e_r - m}{r^3}$$

静磁场的能量.

- 静磁场能量的 $A - j_f$ 表达形式：

$$W = \iiint \frac{1}{2} B\cdot H \, d^3r$$

$$B = \nabla \times A$$

$$\nabla \times H = j_r$$

$$\nabla\cdot(a\times b) = (\nabla\times a)\cdot b - a\cdot(\nabla\times b)$$

$$= \iiint \frac{1}{2}(\nabla\times A)\cdot H \, d^3r$$

$$= \iiint \frac{1}{2}\nabla\cdot(A\times H)\, d^3r + \frac{1}{2}\iiint A\cdot(\nabla\times H)\, d^3r$$

$$= \frac{1}{2}\oiint (A\times H)\cdot dS + \frac{1}{2}\iiint A\cdot j_f \, d^3r$$

$$= \frac{1}{2}\iiint A\cdot j_f(r)\, d^3r$$

- 场分布（j_f → 引起 A）（j_e → 引起 A_e）.

$$W = \frac{1}{2}\iiint (A + A_e)\cdot(j_f + j_e)\, d^3r$$

$$= W + W_e + W_{int}$$

$$= \frac{1}{2}\iiint A\, j_f \, d^3r + \frac{1}{2}\iiint A_e\cdot j_e \, d^3r + \frac{1}{2}\iiint (A\, j_e + A_e\cdot j_f)\, d^3r$$

$$W_{int} = \frac{1}{2}\iiint (A\cdot j_e + A_e\cdot j_f)\, d^3r.$$

$$\iiint_{R^3} A_e\cdot j_f \, d^3r$$

$$= \frac{\mu}{4\pi}\iiint_{R^3} dx\,dy\,dz \iiint \frac{j_f(r)\cdot j_e(r')}{|r - r'|}\, dx'dy'dz'$$

$$= \iiint_{R^3} A\cdot j_e \, d^3r$$

$$= \iiint_{R^3} A\cdot j_e(r)\, d^3r$$

电磁波的传播.

◦ 从 Maxwell 方程到 电磁波动方程.

Maxwell 方程:

$$\nabla\cdot D = \rho_f \qquad \nabla\times E = -\frac{\partial B}{\partial t}$$

$$\nabla\cdot B = 0 \qquad \nabla\times H = j_f + \frac{\partial D}{\partial t}$$

$\Big\downarrow$ $\rho_f = 0 \quad j_f = 0$; 无源条件.

$D = \varepsilon E \quad H = \frac{1}{\mu_0\mu_r}B = \frac{1}{\mu}B$ 线性条件.

$$\nabla\cdot E = 0 \qquad \nabla\times E = -\mu\frac{\partial H}{\partial t}$$

$$\nabla\cdot H = 0 \qquad \nabla\times H = \varepsilon\frac{\partial E}{\partial t}$$

$\Big\downarrow$ $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$

$$\nabla\times(\nabla\times E) = \nabla(\nabla\cdot E) - \nabla^2 E = -\nabla^2 E \quad \Rightarrow \quad \nabla^2 E - \mu\varepsilon\frac{\partial^2 E}{\partial t^2} = 0$$

$$= -\mu\frac{\partial}{\partial t}(\nabla\times H) = -\mu\varepsilon\frac{\partial^2 E}{\partial t^2}$$

$$\nabla\times(\nabla\times H) = \nabla(\nabla\cdot H) - \nabla^2 H = -\nabla^2 H \quad \Rightarrow \quad \nabla^2 H - \mu\varepsilon\frac{\partial^2 H}{\partial t^2} = 0$$

$$= \varepsilon\frac{\partial}{\partial t}(\nabla\times E) \quad = -\mu\varepsilon\frac{\partial^2 H}{\partial t^2}$$

◦ 电场的平面波解:

$$E = E_0 \exp(i\omega t - i\vec{k}\cdot\vec{x})$$

$\Big\downarrow$ $\nabla^2 E - \mu\varepsilon\frac{\partial^2 E}{\partial t^2} = 0$

$$-|\vec{k}|^2 - \mu\varepsilon\cdot(-\omega^2) = 0$$

$$\mu\varepsilon\omega^2 - |\vec{k}|^2 = 0$$

$$\omega = \frac{1}{\sqrt{\mu\varepsilon}}|\vec{k}|$$ Equation of state → 波速 $v = \frac{1}{\sqrt{\mu\varepsilon}}$

$\Big\downarrow$ $\nabla\cdot E = 0$

$$\vec{k}\cdot\vec{E}_0 = 0$$ Equation of 横波条件.

◦ 电场平面波解与磁场平面波解的关系.

$$B = B_0 \exp(i\omega t - i\vec{k}\cdot\vec{x})$$

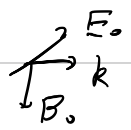
$$E = E_0 \exp(i\omega t - i\vec{k}\cdot\vec{x})$$

$\Big\downarrow$ $\nabla\times E = -\frac{\partial B}{\partial t}$ $\qquad \omega = \frac{1}{\sqrt{\mu\varepsilon}}|\vec{k}|$

$$ik\times E_0 = i\omega B_0$$

$$e_k\times E_0 = \frac{\omega}{|\vec{k}|}B_0 = \frac{1}{\sqrt{\mu\varepsilon}}B_0$$

$$B_0 = \sqrt{\mu\varepsilon}\; e_k\times E_0$$



◦注. 这种解称为 TEM 波(Transverse − E,M) 指电磁场都 ⊥ $\vec{k}$ (是横波).

° 能量密度和能流密度.

$$U = \frac{1}{2} E\cdot D + \frac{1}{2} B\cdot H = \frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu} B^2 = \frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu}\mu\varepsilon E^2 = \varepsilon E^2$$

$$S = E\times H = E\times\left(\frac{1}{\mu}\sqrt{\mu\varepsilon}\, e_k\times E\right) = \sqrt{\frac{\varepsilon}{\mu}}\cdot E\times(e_k\times E)$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\, E^2 \hat{e}_k$$

$$= \sqrt{\frac{1}{\mu\varepsilon}}\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\varepsilon E^2\right)$$

$$= \sqrt{\frac{1}{\mu\varepsilon}}\cdot\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu} B^2\right)$$

$$= \underset{\uparrow\text{波速}}{\sqrt{\frac{1}{\mu\varepsilon}}}\cdot U$$

↑波速.

趋肤效应

导体内电磁波.

- 良导体条件（导体内无自由电荷） ← 线性介质.

$$\begin{cases} \nabla \cdot E = \frac{1}{\varepsilon} \rho_f(r,t) & -(1) \\ \sigma E = j & -(2) \\ \nabla \cdot j + \frac{\partial \rho_f}{\partial t} = 0 & -(3) \end{cases}$$

则：(3) → (2)

$$\sigma \nabla \cdot E = \nabla \cdot j = -\frac{\partial \rho_f}{\partial t} \quad -(4)$$

(1), (4) ⇓

$$-\frac{1}{\sigma}\frac{\partial \rho_f}{\partial t} = \frac{1}{\varepsilon}\rho_f$$

$$\frac{\partial \rho}{\partial t} = -\frac{\sigma}{\varepsilon}\rho$$

$$\rho_f(r,t) = \rho_f(r,0) \cdot \exp(-\frac{\sigma}{\varepsilon}t)$$

良导体条件.（自由电荷快速变为0）

$$\frac{\varepsilon}{\sigma} << T$$

- 导体内的波动方程（线性介质，复电容率条件）

无自由电荷/线性介质 ⟶

$$\nabla \cdot E = 0 \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times H = \sigma E + \frac{\partial D}{\partial t} = \sigma E + \varepsilon \frac{\partial E}{\partial t}$$

解：

$$\begin{cases} E = E(\vec{r}) \cdot \exp(-i\omega t) \\ B = B(\vec{r}) \cdot \exp(-i\omega t) \end{cases}$$

则

$$\nabla \cdot E = 0 \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu(-i\omega\varepsilon + \sigma)E = (-i\omega)(\mu)(\varepsilon + \frac{i}{\omega}\sigma)E.$$

复电容率：$\varepsilon' = \varepsilon + i\frac{\sigma}{\omega}$

波动Eq：$(\nabla^2 + \mu\varepsilon'\frac{\partial^2}{\partial t^2})(\omega) = 0.$

解：$|k| = \sqrt{\mu\varepsilon'} \cdot \omega = \sqrt{\mu(\varepsilon + \frac{i}{\omega}\sigma)} \quad \omega$

开根：$|k| = \alpha + i\beta = \sqrt{\mu\varepsilon + \frac{i\sigma}{\omega}\mu} \cdot \omega$

$$\alpha^2 - \beta^2 + i2\alpha\beta = (\mu\varepsilon + i\frac{\sigma\mu}{\omega})\omega^2$$

$$\begin{cases} \alpha^2 - \beta^2 = \mu\varepsilon\omega^2 \\ 2\alpha\beta = \sigma\mu\omega \end{cases} \longrightarrow \quad \alpha = \frac{\sigma\mu\omega}{2}\frac{1}{\beta} \longrightarrow \quad (\frac{\sigma\mu\omega}{2})^2\frac{1}{\beta^2} - \beta^2 = \mu\varepsilon\omega^2$$

$$\Big\downarrow \beta^2 = x$$

$$x^2 + \omega^2\mu\varepsilon x - (\frac{\sigma\mu\omega}{2})^2 = 0$$

$$\beta = \sqrt{\frac{\omega^2\mu\varepsilon \pm \sqrt{\omega^4\mu^2\varepsilon^2 + (\sigma\mu\omega)^2}}{2}}$$

$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{1}{2}(1 + \sqrt{(\frac{\sigma}{\omega\varepsilon})^2} - 1)}$$

↓ ← 良导体条件：$\frac{\varepsilon}{\sigma} << T \Rightarrow \frac{\sigma}{\omega\varepsilon} >> 1$

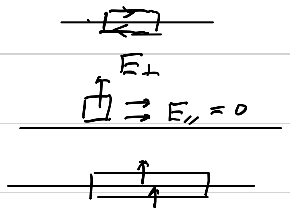
$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{2\omega\varepsilon}} = \frac{1}{Z_c} \leftarrow \text{传透深度!}$$

▫ 导体边界条件：

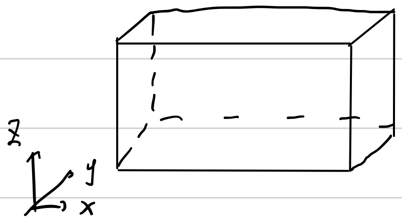
$\nabla\times E = 0 \Rightarrow E_{/\!/} = 0$ （导体内无电场）

$\nabla\cdot E = 0 \Rightarrow \frac{\partial E_\perp}{\partial \perp} = 0$ $\quad \nabla\cdot E = \frac{\partial E_\perp}{\partial \perp}$

$\nabla\cdot B = 0 \Rightarrow B_\perp = 0$



▫ 谐振腔



电场波动方程.

$$\nabla^2 E - \mu\varepsilon\frac{\partial^2 E}{\partial t^2} = 0$$

分量解

$$\nabla^2 u - \mu\varepsilon\frac{\partial^2 u}{\partial t^2} = 0$$

猜解：

$$u = (C_1\cos k_x x + D_1\sin k_x x)(C_2\cos k_y y + D_2\sin k_y y)(C_3\cos k_z z + D_3\sin k_z z)\cdot\exp(-i\omega t)$$

边界条件

$E_{/\!/} = 0 \quad \frac{\partial E_\perp}{\partial \perp} = 0$

解：

$$E_x = A_1\cos k_x x\ \sin k_y y\ \sin k_z z\ e^{-i\omega t}$$
$$E_y = A_2\sin k_y y\ \cos k_y y\ \sin k_z z\ e^{-i\omega t}$$
$$E_z = A_3\sin k_x x\ \sin k_y y\ \cos k_z z\ e^{-i\omega t}$$

$k_x = n_x\frac{\pi}{L_1} \quad k_y = n_y\frac{\pi}{L_2} \quad k_z = n_z\frac{\pi}{L_3} \qquad n_{x,y,z} = 0, 1, \cdots$

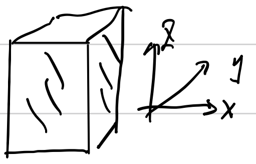
$$\omega = \frac{\pi}{\sqrt{\mu\varepsilon}}\cdot\sqrt{\left(\frac{n_x}{L_1}\right)^2+\left(\frac{n_y}{L_2}\right)^2+\left(\frac{n_z}{L_3}\right)^2}$$

横场条件 $\nabla\cdot E = 0 \Rightarrow k_x A_1 + k_y A_2 + k_z A_3 = 0$

ω 最小（波长最长）的波（$L_1 > L_2 > L_3$），$\{n_x, n_y, n_z\}$ 中不可有 2 个 $= 0$.

$$\omega_{min} = \frac{\pi}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{L_1}\right)^2+\left(\frac{1}{L_2}\right)^2}$$

▫ 电磁波导. （考虑 boundary condition $E_{/\!/} = 0$ $\partial E_\perp/\partial\perp = 0$ ）



解：
$$E_x = A_1\cdot\cos k_x x\cdot\sin k_y y\ e^{-ik_z z}\ e^{-i\omega t}$$
$$E_y = A_2\sin k_x x\ \cos k_y y\ e^{ik_z z}\ e^{-i\omega t}$$
$$E_z = A_3\sin k_x x\ \sin k_y y\ e^{-ik_z z}\ e^{-i\omega t}$$

$k_x = \frac{\pi}{L_1}n_1 \quad k_y = \frac{\pi}{L_2}n_2$

$$\omega = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\pi}{L_1}n_1\right)^2+\left(\frac{\pi}{L_2}n_2\right)^2+k_z^2}$$

横场条件 $\nabla\cdot E = 0 \quad k_x A_1 + k_y A_2 - ik_z A_3 = 0$

— TE 波. $A_3 = 0$ (横波条件) (TE 指电磁波是横波)

$$n_1 A_1 + n_2 A_2 = 0$$

— TE_{10} 波.

由于 $w = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{(\frac{\pi}{L_1} n_1)^2 + (\frac{\pi}{L_2} n_2)^2 + k_z^2}$ (n_1, n_2 不同时 $= 0$) $L_1 > L_2$

$$\Downarrow$$

则 $w_{min} = W_{10} = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{k_z^2 + (\frac{\pi}{L_1})^2} \geq \frac{1}{\sqrt{\mu\varepsilon}} \cdot (\frac{\pi}{L_1})$

$$w = \frac{2\pi}{T} = \frac{2\pi}{\lambda} \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\lambda_{max} \leq 2 L_1$$

对于 $w = w_{10}$, 考虑 TE 波 TE_{10}

$$E_x = 0$$

$$E_y = A_2 \cdot \sin k_x x \cdot \cos k_y y \cdot e^{i k_z z} e^{-i w t} = A_2 \sin(\frac{\pi}{L_1} x) e^{i k_z z} e^{-i w t}.$$

$$E_z = 0$$

求解磁场.

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{suppose: } B = B(\vec{r}) \cdot e^{-i w t}$$

$$\nabla \times E = i w B$$

$$B = -i \frac{1}{w} \nabla \times E$$

$$= -i \frac{1}{w} \cdot \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & A_2 \sin(\frac{\pi}{L_1} x) e^{i k_z z} & 0 \end{bmatrix} e^{-i w t}$$

$$= -i \frac{1}{w} \cdot \left(-i k_z A_2 \sin(\frac{\pi}{L_1} x) e^{i k_z z} \hat{x} + \frac{\pi}{L_1} A_2 \cos(\frac{\pi}{L_1} x) e^{i k_z z} \hat{z} \right) e^{-i w t}$$

线性介质 $H = \frac{1}{\mu} B = -i \frac{1}{\mu w} \left(-i k_z A_2 \sin(\frac{\pi}{L_1} x) e^{i k_z z} \cdot \hat{x} + \frac{\pi}{L_1} A_2 \cos(\frac{\pi}{L_1} x) e^{i k_z z} \hat{z} \right) e^{-i w t}$

$$w = w_{10} = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{(\frac{\pi}{L_1})^2 + k_z^2}$$

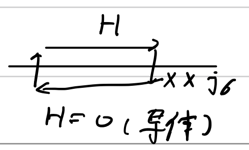
边界电流: $x = 0$, $x = L$ 的 $y-z$ 面. $H_{//} = 0$

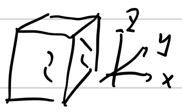
$y = 0$ / $y = L$ 的 $x-z$ 面

Maxwell eq: $\nabla \times H = j_f + \frac{\partial D}{\partial t}$

$$\downarrow$$

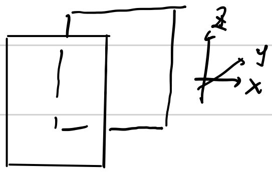
$$H l = j_\sigma \cdot l \leftarrow$$





$x = \frac{L_1}{2}$ 时, $H_z = 0 \rightarrow j_{\sigma x} = 0 \rightarrow$ 沿 z 方向裂缝无影响!

- 两个介质面内只能传播一种横波.



z方向传播. $\vec{E} = \vec{E}_0 e^{ikz - i\omega t}$

$\vec{E}_0 = E_0 \hat{y}$

$\vec{E}_0$ 不可有 $\hat{x}$ 分量 $\longrightarrow$ 不然 $E_{\parallel} = 0$ at the boundary.

等离子体

- 等离子体内电子振荡.

认为正电荷不移动，电子有分布场 $n(r,t)$、速度场 $v(r,t)$

电荷体 density

$$\delta\rho(r,t) = (-e)(n(r,t) - n_0)$$

Maxwell eq (真空)

$$\nabla\cdot E = \frac{1}{\varepsilon_0}\delta\rho(r,t) \qquad —(1)$$

电子运动方程.

$$m\frac{dv}{dt} = m\left(\frac{\partial v}{\partial t} + (v\cdot\nabla)v\right) = -eE \quad - (2)$$

电荷守恒

$$\frac{\partial}{\partial t}n(r,t) + \nabla\cdot(nv(r,t)) = 0 \qquad —(3)$$

(1), (2), (3) 的微扰条件.

$$\nabla\cdot E = \frac{1}{\varepsilon_0}\delta\rho(r,t) \qquad —(4)$$

$$m\frac{\partial v}{\partial t} = -eE \qquad —(5)$$

$$\frac{\partial}{\partial t}(\delta\rho) - en_0\nabla\cdot v = 0 \qquad —(6)$$

$\nabla\cdot(5)$ 以

$$m\frac{\partial}{\partial t}\nabla\cdot v = -e\nabla\cdot E \quad \downarrow 代入(4),(6)$$

$$m\frac{\partial}{\partial t}\left(\frac{1}{en_0}\frac{\partial}{\partial t}\delta\rho\right) = -e\cdot\frac{1}{\varepsilon_0}\delta\rho$$

$$\frac{\partial^2}{\partial t^2}\delta\rho = -\frac{e^2n_0}{m\varepsilon_0}\delta\rho$$

$$w_p = \sqrt{\frac{e^2n_0}{m\varepsilon_0}}$$

↑电子振动的本征频率.

- 电磁波在等离子体内的传播.

$$E = E(\vec{r})e^{-iwt}$$

$$m\frac{dv}{dt} = E\cdot(-e)$$

$$v \sim -\frac{e}{m}\int Edt \sim -\frac{ie}{mw}E$$

$$j \sim v\cdot n_0\cdot(-e) \sim \frac{-ie^2n_0}{mw}E = i\frac{w_p^2\varepsilon_0}{w}E$$

则: Maxwell eq: (真空)

$$\nabla\cdot E = 0 \qquad \nabla\times E = -\frac{\partial B}{\partial t}$$

$$\nabla\cdot B = 0 \qquad \nabla\times B = \mu_0\left(i\frac{w_p^2\varepsilon_0}{w}\right)E + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$$

$$\sim \left(i\frac{w_p^2}{w}\mu_0\varepsilon_0 - iw\mu_0\varepsilon_0\right)E \Rightarrow \mu_0\varepsilon' \sim \mu_0\left(1 - \frac{w_p^2}{w^2}\right)\varepsilon_0$$

则: $\nabla^2(\sim) - \mu_0\varepsilon_0\left(1 - \frac{w_p^2}{w^2}\right)\frac{\partial^2}{\partial t^2}(\sim) = 0 \Rightarrow k = \sqrt{\mu_0\varepsilon_0\left(1-\frac{w_p^2}{w^2}\right)}\ w$, $n = \sqrt{1 - w_p^2/w^2}$!

$w > w_p$ 短波可反射

$w < w_p$ 不反射!

电磁波的产生.

- Lorentz 规范, Maxwell 方程 for 电势/磁矢势.

Maxwell 方程

$$\nabla\cdot E = \frac{\rho}{\varepsilon_0} \qquad \nabla\times E = -\frac{\partial B}{\partial t} \longrightarrow E = -\nabla\Phi - \frac{\partial A}{\partial t}$$

$$\nabla\cdot B = 0 \qquad \nabla\times B = \mu_0 j + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$$

$$\Downarrow$$

$$B = \nabla\times A$$

规范不确定性:

$$\begin{cases} A' = A + \nabla\Lambda \\ \Phi' = \Phi - \frac{\partial\Lambda}{\partial t} \end{cases}$$

$$\begin{cases} B' = \nabla\times A' = \nabla\times A = B \\ E' = -\nabla\Phi + \nabla\frac{\partial\Lambda}{\partial t} - \frac{\partial A}{\partial t} - \nabla\frac{\partial\Lambda}{\partial t} = E \end{cases}$$

(B, E 不变)

Lorentz 规范

$$\nabla\cdot A + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$$

$$\nabla\cdot A + \mu_0\varepsilon_0\frac{\partial\Phi}{\partial t} = 0$$

—— Lorentz 规范下 Maxwell 方程.(真空条件)

$$\nabla\cdot E = \frac{1}{\varepsilon_0}\rho_f \qquad \nabla\times E = -\frac{\partial B}{\partial t}$$

$$\nabla\cdot B = 0 \qquad \nabla\times B = \mu_0 j_f + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$$

$$\begin{cases} B = \nabla\times A \\ E = -\nabla\Phi - \frac{\partial A}{\partial t} \end{cases}$$

$$\nabla\times(\nabla\times A) = \mu_0 j_f + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial A}{\partial t}\right)$$

$$\nabla\times(\nabla\times A) = -\nabla^2 A + \nabla(\nabla\cdot A)$$

$$-\nabla^2 A + \nabla(\nabla\cdot A) = \mu_0 j_f - \mu_0\varepsilon_0\frac{\partial}{\partial t}(\nabla\Phi) - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2}$$

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla\left(\nabla\cdot A + \mu_0\varepsilon_0\frac{\partial}{\partial t}\Phi\right) = -\mu_0 j_f$$

$$\nabla\cdot\left(-\nabla\Phi - \frac{\partial A}{\partial t}\right) = -\nabla^2\Phi - \frac{\partial}{\partial t}(\nabla\cdot A) = \frac{1}{\varepsilon_0}\rho_f$$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial}{\partial t}(\nabla\cdot A) = -\frac{1}{\varepsilon_0}\rho_f$$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\nabla A + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) = -\frac{1}{\varepsilon_0}\rho_f$$

Lorentz Gauge

$$\nabla\cdot A + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$$

D'Alembert Func.

$$\begin{cases} \nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 j_f \\ \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{1}{\varepsilon_0}\rho_f \end{cases}$$

直接给出D'Alembert方程的解

$$A(r,t) = \frac{\mu_0}{4\pi}\iiint d^3r' \frac{j_f\left(r', t - \frac{|r-r'|}{c}\right)}{|r-r'|}$$

$$\Phi(r,t) = \frac{1}{4\pi\varepsilon_0}\iiint d^3r' \frac{\rho_f\left(r', t - \frac{|r-r'|}{c}\right)}{|r-r'|}$$

proof:

$$\nabla^2 \Phi(r,t) - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = \frac{1}{4\pi\varepsilon_0}\iiint d^3r'\Big[\left(-4\pi\delta^{(3)}(r-r')\right)\rho_f\left(r', t-\frac{|r-r'|}{c}\right)$$

$$+\frac{1}{|r-r'|}\underline{\nabla^2\rho_f\left(r', t-\frac{|r-r'|}{c}\right)} + 2\nabla\frac{1}{|r-r'|}\nabla\rho_f\left(r', t-\frac{|r-r'|}{c}\right)$$

$$+\frac{1}{|r-r'|}\underline{\frac{\partial^2\rho_f}{\partial t^2}\left(-\frac{1}{c^2}\right)}\Big]$$

$$\nabla\frac{1}{|r-r'|}\cdot\nabla\rho_f\left(r', t-\frac{|r-r'|}{c}\right) = -\frac{(r-r')}{|r-r'|^3}\cdot\left(-\frac{1}{c}\right)\cdot\frac{(r-r')}{|r-r'|}\frac{\partial}{\partial t}\rho_f\circ\left(r', t-\frac{|r-r'|}{c}\right)$$

$$\nabla^2\rho_f\left(r', t-\frac{|r-r'|}{c}\right) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\rho_f\left(r', t-\frac{|r-r'|}{c}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\rho_f\left(r', t-\frac{|r-r'|}{c}\right)\right) + \cdots$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\rho_f}{\partial t}\circ\left(r', t-\frac{|r-r'|}{c}\right)\cdot\left(-\frac{1}{c}\right)\cdot\frac{x-x'}{|r-r'|}\right) + \cdots$$

$$= \frac{\partial^2\rho_f}{\partial t^2}\circ\left(r', t-\frac{|r-r'|}{c}\right)\cdot\frac{1}{c^2}\frac{(x-x')^2}{|r-r'|^2} + \cdots$$

$$+\frac{\partial\rho_f}{\partial t}\circ\left(r', t-\frac{|r-r'|}{c}\right)\left(-\frac{1}{c}\right)\frac{1}{|r-r'|} + \cdots$$

$$+\frac{\partial\rho_f}{\partial t}\circ\left(r', t-\frac{|r-r'|}{c}\right)\left(+\frac{1}{c}\right)\frac{(x-x')^2}{|r-r'|^3}$$

$$= \frac{\partial^2\rho_f}{\partial t^2}\circ\left(r', t-\frac{|r-r'|}{c}\right)\cdot\frac{1}{c^2}$$

$$+\left(-\frac{1}{c}\right)\frac{2}{|r-r'|}\frac{\partial\rho_f}{\partial t}\circ\left(r', t-\frac{|r-r'|}{c}\right)$$

$$-\frac{1}{c^2}\frac{\partial^2\rho_f\left(r', t-\frac{|r-r'|}{c}\right)}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2\rho_f}{\partial t^2}\circ\left(r', t-\frac{|r-r'|}{c}\right)$$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -\frac{1}{\varepsilon_0}\rho_f$$

电偶极辐射.

∘ 交变电流引起的磁矢势

由矢势的产生方程:

$$\mathbf{j}(r';t') = \mathbf{j}(r')\exp(-i\omega t')$$

$$A(r,t) = \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{j}(r', t - \frac{|r-r'|}{c})}{|r-r'|} dx'dy'dz'$$

$$= \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{j}(r')}{|r-r'|}\exp(-i\omega t + i\frac{\omega}{c}|r-r'|)\, dx'dy'dz'$$

$$= \underbrace{\left[\frac{\mu_0}{4\pi}\iiint \frac{\mathbf{j}(r')}{|r-r'|}\exp(-ik|r-r'|)\, dx'dy'dz'\right]}_{A(r)} \exp(-i\omega t) \qquad k = \frac{\omega}{c}$$

$$= A(r)\cdot\exp(-i\omega t)$$

∘ 用磁矢势 A 得到 E、B.

$$B(r,t) = (\nabla\times A(r))e^{-i\omega t}$$

suppose: $E(r,t) = E(r)\, e^{-i\omega t}$

$$\nabla\times B = \mu_0\varepsilon_0\frac{\partial E}{\partial t}$$

$$\nabla\times B(r)\ \exp(-i\omega t) = \mu_0\varepsilon_0(-i\omega)E(r)\cdot\exp(-i\omega t)$$

$$E(r) = \frac{1}{\mu_0\varepsilon_0}\frac{i}{\omega}\cdot\nabla\times B(r)$$

$$= \frac{ic}{k}(\nabla\times B(r))\cdot e^{-i\omega t}$$

∘ 电偶极辐射

$$A(r) = \iiint \frac{\mu_0}{4\pi}\frac{\mathbf{j}(r')}{|r-r'|}\cdot\exp(i\frac{\omega}{c}|r-r'|)\ d^3r$$

$\frac{\omega}{c} = \frac{2\pi}{\lambda}$

$\lambda \ll |r-r'|$ (远区)

$l \ll \lambda \qquad l \ll r \qquad$ (要求)

↑ j的范围

$$\doteq \frac{\mu_0}{4\pi}\iiint d^3r'\ \frac{\mathbf{j}(r')}{r}(1+\frac{1}{r}r'\cdot\hat{e}_r)\exp(ikr)\cdot(1-ikr'\cdot\hat{e}_r)$$

$$\approx \frac{\mu_0}{4\pi}\iiint d^3r'\ \frac{\mathbf{j}(r')\exp(ikr)}{r}\left(1+\frac{1}{r}r'\cdot\hat{e}_r - ikr'\cdot\hat{e}_r\right)$$

$(\frac{2\pi}{\lambda} \gg \frac{1}{r})$, 远区条件.

$$= \frac{\mu_0}{4\pi}\iiint d^3r'\ \frac{\mathbf{j}(r')\exp(ikr)}{r}(\underset{A^{(1)}}{1} - \underset{A^{(2)}}{ikr'\cdot\hat{e}_r})$$

电偶极辐射 　电4极辐射.

电偶极辐射:

大r不是向量

$$A^{(1)} = \frac{\mu_0}{4\pi}\frac{\exp(ikr)}{r}\iiint d^3r'\ \mathbf{j}(r')$$

↓

$$\iiint \dot{j}(r')\, d^3r' = -\iiint_{\Omega} r' \cdot (\nabla_{r'} \cdot \dot{j}(r'))\, d^3r'$$

$$\frac{\partial \rho}{\partial t}(r', t) + \nabla_{r'} \cdot \dot{j}(r', t) = 0$$

$$\rho(r', t) = \rho(r') \exp(-i\omega t) \qquad \dot{j}(r', t) = \dot{j}(r') \exp(-i\omega t)$$

$$-i\omega \rho(r') + \nabla_{r'} \cdot \dot{j}(r') = 0$$

$$A^{(1)}(r) = \frac{\mu_0}{4\pi r} e^{ikr} \iiint \left[-r' (\nabla_{r'} \cdot \dot{j}(r')) \right] d^3r'$$

$$= \frac{\mu_0}{4\pi r} e^{ikr} \iiint d^3r' \left[-r' \cdot i\omega\, \rho(r') \right] d^3r'$$

$$\vec{p} = \iiint \rho(r') \vec{r}'\, d^3r'$$

$$= -\frac{i\omega\mu_0}{4\pi r} e^{ikr} \vec{p}$$

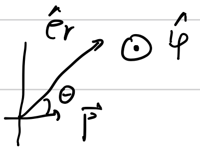
—— 磁感应强度矢量

$$B^{(1)}(r) = \nabla \times A^{(1)}(r) = \nabla \times \left(-\frac{i\omega\mu_0}{4\pi r} e^{ikr} \vec{p} \right)$$

$$= -\frac{i\mu_0\omega}{4\pi} \frac{e^{ikr}}{r} \left(ik\, \hat{e}_r \times \vec{p} - \frac{1}{r} \hat{e}_r \times \vec{p} \right)$$

$k = \frac{2\pi}{\lambda} >> \frac{1}{r}$ 远场条件

$$\doteq \frac{\mu_0 \omega k}{4\pi} \frac{e^{ikr}}{r} \hat{e}_r \times \vec{p}$$



$$= \frac{\omega^2}{4\pi\varepsilon_0 c^3} \frac{e^{ikr}}{r} \hat{e}_r \times \vec{p} = -\frac{e^{ikr}}{4\pi\varepsilon_0 c^3 r} e_r \times \ddot{p} = -\frac{\omega^2 e^{ikr}}{4\pi\varepsilon_0 c^3 r} |p| \sin\theta\, \hat{e}_\varphi$$

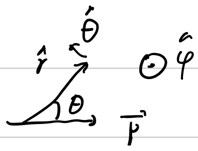
注 $B^{(1)}(r,t) = B^{(1)}(r) \exp(-i\omega t)$

—— 电场强度矢量

$$E^{(1)}(r) = \frac{ic}{k} \nabla_r \times B^{(1)}(r)$$

$$= \frac{ic}{k} \nabla_r \times \left(\frac{\omega^2}{4\pi\varepsilon_0 c^3} \frac{e^{ikr}}{r} e_r \times \vec{p} \right)$$

$$= -\frac{\omega^2 e^{ikr}}{4\pi\varepsilon_0 c^3 r} e_r \times (e_r \times \vec{p}) = -\frac{\omega^2 e^{ikr}}{4\pi\varepsilon_0 c^2 r} |\vec{r}| \sin\theta\, \hat{e}_\theta$$



$$= \frac{e^{ikr}}{4\pi\varepsilon_0 c^2 r} e_r \times (e_r \times \ddot{\vec{p}})$$

Lienard - Wiechert Potential

- 求带电粒子产生的 potential Φ 和 A.

$$A(r,t) = \iiint \frac{\mu_0}{4\pi} \frac{j_f(r', t - \frac{|r-r'|}{c})}{|r-r'|} dx'dy'dz'$$

$$\Phi(r,t) = \iiint \frac{1}{4\pi\varepsilon_0} \frac{\rho(r', t - \frac{|r-r'|}{c})}{|r-r'|} dx'dy'dz'$$

带电粒子形成电流密度/电荷密度

$$j_f(r',t') = q\cdot\delta(r' - r_0(t'))\cdot v(t') \qquad (\text{只有 } j_f(r_0(t'),t') \neq 0)$$

$$\rho_f(r',t') = q\cdot\delta(r' - r_0(t')) \qquad (\text{只有 } \rho_f(r_0(t'),t') \neq 0)$$

Lorentz 变换: 为了求 (r, t) 处的电磁势.

↓

它的产生元是 $(r_0(t'), t')$ 约束关系: $t' = t - \frac{|r - r_0(t')|}{c}$

↓

S' 系以速度 $v(t')$ 运动相对 S。

↓

$v'(\tilde{t}') = 0$, 此时 S' 系中坐标: $(\tilde{r}_0(\tilde{t}'), \tilde{t}')$ $\qquad \tilde{r}_0(\tilde{t}') = \widetilde{r_0(t')}$

$r_0(t')$ Lorentz Trans 后.

由光速不变原理: $\tilde{t} = \tilde{t}' + \frac{1}{c}|\tilde{r} - \tilde{r}_0(\tilde{t}')|$

↓

相当于 $(\tilde{r}, \tilde{t})$ 处的 $\tilde{\Phi}, \tilde{A}$ 由 $(\tilde{r}_0(\tilde{t}'), \tilde{t}')$ 处电荷产生.

$$\tilde{\Phi}(\tilde{r}, \tilde{t}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\tilde{r} - \widetilde{r_0(t')}|}$$

$$\tilde{A}(\tilde{r}, \tilde{t}) = 0$$

$(\frac{\Phi}{c}, \vec{A})$ 构成 Lorentz 矢量.

$$A(r,t) = \frac{\tilde{A}(\tilde{r},\tilde{t}) + \frac{\vec{v}(t')}{c^2}\cdot\tilde{\Phi}(\tilde{r},\tilde{t})}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-v^2/c^2}} \frac{q\vec{v}}{4\pi\varepsilon_0 c^2|\tilde{r} - \tilde{r}'|}$$

$$\Phi(r,t) = \frac{\tilde{\Phi}(\tilde{r},\tilde{t}) + v\cdot\tilde{A}(\tilde{r},\tilde{t})}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-v^2/c^2}} \frac{q}{4\pi\varepsilon_0|\tilde{r} - \tilde{r}'|}$$

$$\left\{ |\tilde{r} - \tilde{r}'| = c(\tilde{t} - \tilde{t}') = c\frac{|t - t'| - \frac{v(t')}{c^2}\cdot(r - r')}{\sqrt{\sim}} \right.$$

$$= \frac{|r - r_0| - \frac{v(t')}{c^2}\cdot(r - r')}{\sqrt{\sim}}$$

$$\left\{ \begin{aligned} A(r,t) &= \frac{qv(t')}{4\pi\varepsilon_0 c^2 \left(|\vec{r} - r_0(t')| - \frac{v(t')}{c}\cdot(r - r'(t'))\right)} \\ \Phi(r,t) &= \frac{q}{4\pi\varepsilon_0\left(|\vec{r} - \vec{r}_0(t')| - \frac{v(t')}{c}\cdot(\vec{r} - r_0(t'))\right)} \end{aligned} \right. \qquad \left(t' = t - \frac{|r - r_0(t')|}{c}\right)$$