约束和广义坐标

- 完整约束（若约束可表达为联系系统各坐标、时间的一组方程）

$$f_m(x_1 \cdots \quad , t) = 0 \quad m=1 \cdots k$$

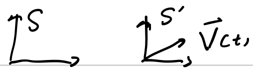
非完整约束：不属于完整约束的式子.

- 稳定约束 & 不稳定约束.

约束中不含时 or 含时.

非惯性系力学

· 非惯性系相对惯性系无转动.



S'相对于S以 $\vec V(t)$ 运动.

$$T=\frac{1}{2}m(v'+\vec V)^2=\frac{1}{2}m v'^2+\frac{1}{2}m V^2+m v'\cdot\vec V$$

$$L=T-U=\frac{1}{2}m v'^2+\frac{1}{2}m\vec V^2+m v'\cdot\vec V-U$$
$$=\frac{1}{2}m v'^2+\frac{1}{2}m\vec V^2-m r'\cdot\frac{d\vec V}{dt}-U$$

Euler-Lagrange Equation

$$\begin{cases}\frac{\partial L}{\partial r'}=-m\frac{d\vec V}{dt}-\frac{\partial U}{\partial r'}\\ \frac{\partial L}{\partial v'}=m v'\end{cases}\Rightarrow m\frac{dv'}{dt}=-m\frac{d\vec V}{dt}-\frac{\partial U}{\partial r'}$$

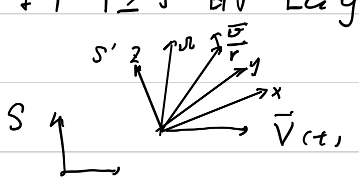
∘ S'系相对于S系有转动.

$$v'=v+\Omega\times r$$
$$L=\frac{1}{2}m v^2+\frac{1}{2}m(\Omega\times r)^2+m v\cdot(\Omega\times r)-m r\cdot\frac{d\vec V}{dt}-U$$

$$\begin{cases}\frac{\partial L}{\partial r}=m\Omega\times(\Omega\times r)+m v\times\Omega-m\frac{d\vec V}{dt}-\nabla U\\ \frac{\partial L}{\partial v}=m v+m(\Omega\times r)\end{cases}$$

$$m\frac{dv}{dt}=m\Omega\times(\Omega\times r)+2m v\times\Omega-m\frac{dV}{dt}-\nabla U-m\beta\times r$$

∘ 转动非惯性系中粒子的Lagrangian与运动方程.



都是转动系中的向量 $\left(\frac{d\vec V}{dt}\right)_{rot}=\left(\frac{d\vec V}{dt}\right)_{sta}-\Omega\times\vec V$

$$T=\frac{1}{2}m(\vec v+\Omega\times\vec r+V(t))^2$$

$$L=\frac{1}{2}m\vec v^2+\frac{1}{2}m(\Omega\times r)^2+\frac{1}{2}m(V)^2+m v\cdot(\Omega\times r)+m v\cdot V+m(\Omega\times r)\cdot V-U$$

用 $A\cdot(B\times C)=B\cdot(C\times A)$!分方向计算即可.

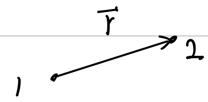
$$\frac{\partial L}{\partial r}=m\Omega\times(r\times\Omega)+m(\vec v\times\Omega)+m(\vec V\times\Omega)-\nabla U$$
$$\frac{\partial L}{\partial v}=m v+m(\Omega\times r)+m\vec V(t)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)=m\frac{d\vec v}{dt}+m(\Omega\times v)+m(\beta\times r)+m\frac{d\vec V}{dt}+m\vec V\times\Omega$$

$$m\frac{dv}{dt}=m\Omega\times(\Omega\times r)+2m v\times\Omega-m\frac{dV}{dt}-\nabla U-m\beta\times r$$

中心力场中的粒子运动

(求中心力场问题的范式).

- 中心力场之2体问题的质心系等效转化.



$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2 - V(|\vec{r}_2 - \vec{r}_1|)$$

质心系转化：

$$L = \frac{1}{2}(m_1+m_2)\dot{\vec{r}}_c^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2 - V(|\vec{r}|)$$

其中：

$$\begin{cases} \vec{r} = \vec{r}_2 - \vec{r}_1 \\ \vec{r}_c = \frac{1}{m_1+m_2}(m_1\vec{r}_1 + m_2\vec{r}_2) \end{cases}$$

相当于：

1° 质心匀速运动

2° $\frac{m_1 m_2}{m_1+m_2}$ 粒子在势场 $V(|\vec{r}|)$ 中运动.

- 中心力场问题一般解法.

由上一部分结论，中心力场等效于解 $m=(m_1m_2)/(m_1+m_2)$ 粒子在势场中运动.

$$L = \frac{1}{2} m (\dot{\vec{r}})^2 - V(|\vec{r}|)$$

极坐标系写法 (他不加证明就认为粒子在平面内运动)

↑ 不严谨口头证明用角动量守恒.

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

—— Z方向角动量守恒.(θ对应广义动量守恒)

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} = \underbrace{J}_{\text{const}}$$

—— 能量守恒.(L中不显含时间)

$$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$H = P_r\dot{r} + P_\theta\dot{\theta} - L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \underbrace{E}_{\text{const}}$$

—— 用能动量守恒得到运动方程解.

$$\begin{cases} J = m r^2 \dot{\theta} \Longrightarrow \dot{\theta} = \frac{J}{mr^2} \\ E = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \longrightarrow \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \frac{J^2}{m^2 r^4} + V(r) = E \Rightarrow \dot{r} = \sqrt{\frac{2}{m}(E - V(r)) - \frac{J^2}{m^2 r^2}} \end{cases}$$

理论上，可得 $r(t)$ & $\theta(t)$.

—— 求解运动轨道的方式.

$$\begin{cases} \dot{r} = \sqrt{\frac{2}{m}(E-V(r)) - \frac{J^2}{m^2r^2}} \\ \dot{\theta} = \frac{J}{mr^2} \end{cases}$$

$$\frac{d\theta}{dr} = \frac{J/mr^2}{\sqrt{\frac{2}{m}(E-V(r)) - \frac{J^2}{m^2r^2}}}$$

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{J/r'^2}{\sqrt{2m[E-V(r')] - \frac{J^2}{r'^2}}}\, dr'$$

—— 有效势能:

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\frac{J^2}{m^2r^4} + V(r) = E$$

$\Downarrow$ 理解为

$$V_{eff}(r) = V(r) + \frac{1}{2}\frac{J^2}{mr^2}$$

开普勒问题.

(势能是 α/r 的形式)

◦ 粒子运动的轨道

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{J/r'^2}{\sqrt{2m[E-V(r')] - \frac{J^2}{r'^2}}} dr'$$

$$= \theta_0 + \int_{r_0}^{r} \frac{J/r'^2}{\sqrt{2m(E-\frac{\alpha}{r'}) - \frac{J^2}{r'^2}}} dr'$$

$$= \theta_0 + \int_{r_0}^{r} \frac{J\frac{1}{r'^2}}{\sqrt{2mE - 2m\frac{\alpha}{r'} - \frac{J^2}{r'^2}}} dr'$$

$$= \theta_0 + \int_{\frac{1}{r_0}}^{\frac{1}{r}} -J \frac{1}{\sqrt{2mE + \frac{m^2\alpha^2}{J^2} - (\frac{J}{r'} + \frac{m\alpha}{J})^2}} d(\frac{1}{r'})$$

$$= \theta_0 + \int_{\frac{1}{r_0}}^{\frac{1}{r}} - \frac{1}{\sqrt{1 - \left(\frac{\frac{J}{r'} + \frac{m\alpha}{J}}{\sqrt{2mE + \frac{m^2\alpha^2}{J^2}}}\right)^2}} \frac{J}{\sqrt{2mE + \frac{m^2\alpha^2}{J^2}}} d(\frac{1}{r'})$$

$$\left\{\begin{aligned} &\frac{\frac{J}{r'} + \frac{m\alpha}{J}}{\sqrt{2mE + \frac{m^2\alpha^2}{J^2}}} = \cos\alpha \\ &-\frac{1}{\sqrt{1-(\sim)^2}} d(\sim) = +\frac{\sin\alpha}{\sin\alpha} \cdot d\alpha = d\alpha \end{aligned}\right.$$

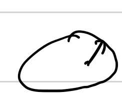
$$= \arccos\left(\frac{\frac{J}{r} + \frac{m\alpha}{J}}{\sqrt{2mE + \frac{m^2\alpha^2}{J^2}}}\right) + \theta'$$

$$\frac{\frac{J}{r} + \frac{m\alpha}{J}}{\sqrt{2mE + \frac{m^2\alpha^2}{J^2}}} = \cos(\theta - \theta')$$

$$\frac{J}{r} + \frac{m\alpha}{J} = \sqrt{2mE + \frac{m^2\alpha^2}{J^2}} \cos(\theta - \theta')$$

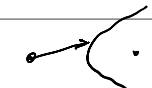
$$r = \frac{J}{\sqrt{2mE + \frac{m^2\alpha^2}{J^2}} \cos\theta - \frac{m\alpha}{J}}$$

$$= \frac{\frac{-J^2}{m\alpha}}{1 + \sqrt{\frac{2EJ^2}{m\alpha^2} + 1} \cos\theta} \quad \alpha < 0$$



椭圆.

$$= \frac{J^2/m\alpha}{-1 + \sqrt{1 + 2EJ^2/m\alpha^2} \cos\theta} \quad \alpha > 0$$



双曲

龙格——楞次矢量.

Laplace — Runge — Lenz — vector.

∘ 证明龙格——楞石兹矢量守恒.

上面解决的开普勒问题 ⟶ 轨道闭合 ⟶ 有更高的对称性(动力学对称性)

$$\vec{M} = \vec{p}\times\vec{L} + m\alpha\frac{\vec{r}}{|\vec{r}|}$$

证明其守恒性

$$\frac{d\vec{M}}{dt} = \frac{d\vec{p}}{dt}\times\vec{L} + m\alpha\frac{d}{dt}\left(\frac{\vec{r}}{|\vec{r}|}\right)$$

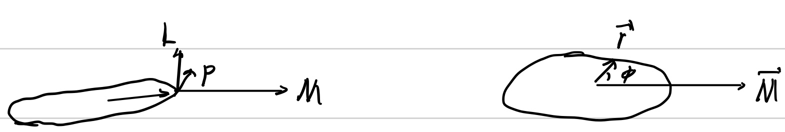
$$= +\frac{\alpha}{|\vec{r}|^3}\vec{r}\times\vec{L} + m\alpha\left(\frac{1}{|\vec{r}|}\left(\frac{d\vec{r}}{dt}\right) - \frac{1}{|\vec{r}|^2}\frac{d|\vec{r}|}{dt}\vec{r}\right)$$

$$= + m\frac{\alpha}{|\vec{r}|^3}\vec{r}\times\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) + m\alpha\left(\frac{1}{|\vec{r}|}\left(\frac{d\vec{r}}{dt}\right) - \frac{1}{|\vec{r}|^2}\frac{d|\vec{r}|}{dt}\vec{r}\right)$$

$$= m\alpha\left(-\frac{1}{|\vec{r}|^3}\left[\frac{d\vec{r}}{dt}|\vec{r}|^2 - \vec{r}\underbrace{\left(\frac{d\vec{r}}{dt}\cdot\vec{r}\right)}_{=|\vec{r}|\frac{d|\vec{r}|}{dt}}\right] + \frac{1}{|\vec{r}|}\left(\frac{d\vec{r}}{dt}\right) - \frac{1}{|\vec{r}|^2}\frac{d|\vec{r}|}{dt}\vec{r}\right)$$

$$= 0$$

o 用龙格——楞石兹矢量得到运动的解.



$$\vec{r}\cdot\vec{M} = |\vec{r}|\cdot|\vec{M}|\cdot\cos\phi$$

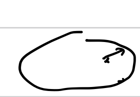
$$= \vec{r}\cdot\left(\vec{p}\times\vec{L} + m\alpha\frac{\vec{r}}{|\vec{r}|}\right)$$

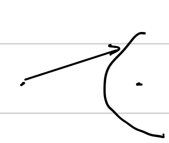
$$= \vec{r}\cdot(\vec{p}\times\vec{L}) + m\alpha|\vec{r}|$$

$$= J^2 + m\alpha|\vec{r}|$$

$$|\vec{r}|\cdot\left(-m\alpha + |\vec{M}|\cos\phi\right) = J^2$$

$$|\vec{r}| = \begin{cases} \dfrac{-J^2/m\alpha}{1+\dfrac{|\vec{M}|}{-m\alpha}\cos\phi} & \alpha<0 \\[2ex] \dfrac{J^2/m\alpha}{-1+\dfrac{|\vec{M}|}{m\alpha}\cos\phi} & \alpha>0 \end{cases}$$

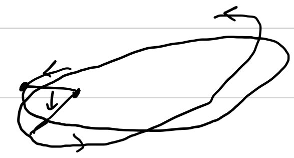




近日点进动

◦ 中心势偏离进动推导范式.

$$V(r) = -\frac{\alpha}{r} + \delta V(r) \qquad (吸引力).$$



中心力场问题:

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{J/r'^2}{\sqrt{2m\left[E - V(r')\right] - \frac{J^2}{r'^2}}} dr'$$

从 $r_{min} \to r_{max} \to r_{min}$，转过的角度

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{J/r'^2}{\sqrt{2m[E - V(r')] - \frac{J^2}{r'^2}}} dr'$$

$$= 2\int_{r_{min}}^{r_{max}} \frac{\frac{J}{r'^2}}{\sqrt{2m\left(E + \frac{\alpha}{r'} - \delta V(r')\right) - \frac{J^2}{r'^2}}} dr'$$

r_{min} & r_{max} 代入都有圆结果

$$= 2\int_{r_{min}}^{r_{max}} \frac{1}{2} \frac{\frac{J}{r'^2} \cdot 2m\,\delta V(r')}{\left(2m\left(E + \frac{\alpha}{r'}\right) - \frac{J^2}{r'^2}\right)^{3/2}} dr' + 2\pi$$

↑ 但 r_{min} & r_{max} 的变化对其无影响? 积分区间.

$$= 2\int_{r_{min}}^{r_{max}} m\,\delta V(r') \frac{\partial}{\partial J}\left(\frac{1}{\sqrt{2m\left(E + \frac{\alpha}{r'}\right) - \frac{J^2}{r'^2}}}\right) dr'$$

$$= \frac{\partial}{\partial J}\left(2m\int_{r_{min}}^{r_{max}} \frac{\delta V(r')}{\sqrt{2m\left(E + \frac{\alpha}{r'}\right) - \frac{J^2}{r'^2}}} dr'\right)$$

$$d\phi = dr' \cdot \frac{J/r'^2}{\sqrt{2m\left[E - V(r')\right] - \frac{J^2}{r'^2}}}$$

用 ϕ 表示积分.

$$= \frac{\partial}{\partial J}\left(\int_0^{\pi} \frac{2m}{J} r'^2 V(r')\, d\phi\right)$$

扌震动问题

单粒子/一维振子振动

- 自由弹簧

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x} \\ \frac{\partial L}{\partial x} = -kx \end{cases}$$

$$m\ddot{x} + kx = 0$$

$$x = A\cos(\omega_0 t) + B\sin(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

- 受迫振动

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + xF(t)$$

$$\begin{cases} \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ \frac{\partial L}{\partial x} = -kx + F(t) \end{cases}$$

$$m\ddot{x} + m\omega_0^2 x - F(t) = 0$$

若:

$$F(t) = f\cos(\Omega t + \beta)$$

则: 猜解:

1° $\omega_0 \neq \omega$

$$x = \underbrace{A\cos(\omega_0 t + \alpha)}_{\text{系数待定}} + B\cdot\cos(\Omega t + \beta)$$

$$B(-\omega^2 + \omega_0^2)m = f$$

$$B = \frac{f}{m(\omega_0^2 - \omega^2)} \qquad (\omega_0 \neq \omega)$$

2° $\omega_0 = \omega$

$$x = A\cos(\omega_0 t + \alpha) + B t\cdot\sin(\Omega t + \beta)$$

$$\begin{cases} \frac{d}{dt}(Bt\sin(\omega t + \beta)) = B\sin(\omega t + \beta) + \omega B t\cos(\omega t + \beta) \\ \frac{d^2}{dt^2}(Bt\sin(\omega t + \beta)) = B\omega\cos(\omega t + \beta) + \omega B\cos(\omega t + \beta) - \omega^2 B t\sin(\omega t + \beta) \end{cases}$$

$$m\left[2B\omega\cos(\omega_0 t + \beta) - \omega_0^2 B^2 t^2\sin(\omega_0 t + \beta) + \omega_0^2 B^2 t^2 \sin(\omega_0 t + \beta)\right] = f\cos(\omega t + \beta)$$

$$B = \frac{f}{2\omega m}$$

$$x = A\cos(\omega_0 t + \alpha) + \frac{f}{2\omega m} t\sin(\omega_0 t + \beta)$$

o阻尼振动.

$$m\ddot{x} = -kx - 2m\gamma\dot{x}$$ 似乎是有个什么耗散项之类的.

$$\ddot{x} = -\omega_0^2 x - 2\gamma\dot{x}$$

猜指数衰减解.

$$x = e^{-\gamma t}(A\cos\omega t + B\sin\omega t)$$

$$\begin{cases} \frac{dx}{dt} = -\gamma e^{-\gamma t}(A\cos\omega t + B\sin\omega t) + \omega e^{-\gamma t}(-A\sin\omega t + B\cos\omega t) \\ \frac{d^2x}{dt^2} = +\gamma^2 e^{-\gamma t}(A\cos\omega t + B\sin\omega t) - \omega\gamma e^{-\gamma t}(-A\sin\omega t + B\cos\omega t) \\ \qquad - \omega\gamma e^{-\gamma t}(-A\sin\omega t + B\cos\omega t) \\ \qquad + \omega^2 e^{-\gamma t}(-A\cos\omega t - B\sin\omega t) \end{cases}$$

代入 运动微分方程.

$$\gamma^2 e^{-\gamma t}(A\cos\omega t + B\sin\omega t) \cancel{- 2\omega\gamma e^{-\gamma t}(-A\sin\omega t + B\cos\omega t)}$$
$$+ \omega^2 e^{-\gamma t}(-A\cos\omega t - B\sin\omega t) - 2\gamma^2 e^{-\gamma t}(A\cos\omega t + B\sin\omega t)$$
$$\cancel{+ 2\gamma\omega e^{-\gamma t}(-A\sin\omega t + B\cos\omega t)} + \omega_0^2 e^{-\gamma t}(A\cos\omega t + B\sin\omega t) = 0$$

$$-\omega^2 - \gamma^2 + \omega_0^2 = 0$$
$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

1° $\gamma < \omega_0$, $e^{-\gamma t} \times$(振动项)

2° $\gamma = \omega_0$ $x = e^{-\gamma t} \cdot (At + B)$

$$\begin{cases} \dot{x} = -\gamma \cdot x + A \cdot e^{-\gamma t} \\ \ddot{x} = \gamma^2 \cdot x - 2\gamma A e^{-\gamma t} \\ \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \end{cases}$$

$$\gamma^2 x - 2\gamma A e^{-\gamma t} - 2\gamma^2 x + 2\gamma A e^{-\gamma t} + \omega_0^2 x = 0$$
$$\gamma^2 - 2\gamma^2 + \omega_0^2 = 0$$ (恒成立)

3° $\gamma < \omega_0$, 纯指数衰减.

多个粒子简谐振动

- 多个粒子振动模式与振动频率

多个粒子振动 Lagrangian 写为:

$$L = \frac{1}{2} m_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} k_{ij} x_i x_j \quad (m \text{ 和 } k \text{ is symmetry matrix})$$

运动方程是:

$$\begin{cases} \frac{\partial L}{\partial x_i} = -k_{ij} x_j & (k \text{ 为对称 matrix}) \\ \frac{\partial L}{\partial \dot{x}_i} = m_{ij} \dot{x}_j & (m \text{ 为对称 matrix}) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \end{cases}$$

$$m_{ij} \ddot{x}_j = -k_{ij} x_j$$

$$m_{ij} \ddot{x}_j + k_{ij} x_j = 0$$

猜解:

$$x_i = \eta_i e^{-i\omega t}$$

代入得.

$$(-\omega^2 m_{ij} + k_{ij}) \eta_j = 0 \Longrightarrow \text{矩阵形式.} [k - \omega^2 m][\eta] = 0$$

线性方程有解:

$$\det(k - \omega^2 m) = 0 \longrightarrow \text{解为 frequency } \omega_1^2 \cdots \omega_n^2.$$

$\downarrow$ 对于解 $\omega_{(i)}^2$

$$(k - \omega_{(i)}^2 m) \eta^{(i)} = 0 \quad (\text{可得特征矢量})$$

$\downarrow$ 特征矢量相互正交

$$\begin{cases} (k_{ij} - \omega_{(a)}^2 m_{ij}) \eta_j^{(a)} = 0 \longrightarrow \eta_i^{(b)} (k_{ij} - \omega_{(a)}^2 m_{ij}) \eta_j^{(a)} = 0 \\ (k_{ij} - \omega_{(b)}^2 m_{ij}) \eta_j^{(b)} = 0 \longrightarrow \eta_i^{(a)} (k_{ij} - \omega_{(b)}^2 m_{ij}) \eta_j^{(b)} = 0 \end{cases} \quad m, k \text{ are symmetry!}$$

$$\downarrow$$

$$\eta_i^{(a)} (\omega_{(a)}^2 - \omega_{(b)}^2) m_{ij} \eta_j^{(b)} = 0$$

(正交性)

$$\begin{cases} \text{取} \quad \tilde{\eta}_i^{(a)} = \eta_i^{(a)} \cdot \frac{1}{\sqrt{\eta^{T(a)} m \eta^{(a)}}} \\ \text{则有归一化:} \quad \tilde{\eta}^{T(a)} \cdot m \cdot \tilde{\eta}^{(b)} = \delta_{ab} \end{cases}$$

(归一化操作可用来求简振坐标)

已知振动 frequency $\omega_{(a)}$ 以及模式 $\eta^{(a)}$ 后, 可将振动 x_i 写为形式如下.

$$\begin{bmatrix} x \end{bmatrix}_{n\times 1} = \begin{bmatrix} \eta^{(1)} \end{bmatrix} (A_{(1)} \cos \omega_{(1)} t + B_{(1)} \sin \omega_{(1)} t) + \begin{bmatrix} \eta^{(2)} \end{bmatrix} (\sim) + \cdots$$

$$= \begin{bmatrix} \eta^{(1)} & \eta^{(2)} & \cdots & \eta^{(n)} \end{bmatrix}_{n\times n} \cdot \begin{bmatrix} A_{(1)} \cos \omega_{(1)} t + B_{(1)} \sin \omega_{(1)} t \\ A_{(2)} \cos \omega_{(2)} t + B_{(2)} \sin \omega_{(2)} t \\ \vdots \\ A_{(n)} \cos \omega_{(n)} t + B_{(n)} \sin \omega_{(n)} t \end{bmatrix}_{n\times 1}$$

求解系数的方式:(下面详细说)

$$(\eta^{(1)} \cdots \eta^{(n)}) \cdot \begin{pmatrix} A_{(1)} \\ \vdots \\ A_{(n)} \end{pmatrix} = \begin{pmatrix} x(t=0) \end{pmatrix}$$

$$(\eta^{(1)} \cdots\cdots \eta^{(n)}) \begin{pmatrix} -\omega_{(1)} B_{(1)} \\ \vdots \\ -\omega_{(n)} B_{(n)} \end{pmatrix} = \begin{pmatrix} \dot{x}(t=0) \end{pmatrix}$$

◦ 简振坐标,模态矩阵

—— 定义 模态矩阵

$$A_{ij} = \left[\tilde{\eta}^{(1)} \cdots \tilde{\eta}^{(n)} \right]_{ij}$$

—— 模态矩阵引入简振坐标

定义

$$x_i = A_{ij} Q_j$$

则

$$L = \frac{1}{2} m_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} k_{ij} x_i x_j = \frac{1}{2} Q^T A^T m \cdot A Q - \frac{1}{2} Q^T A^T k A Q$$

对角化性质! $A^T m A = \begin{bmatrix} \tilde{\eta}^{(1)T} \\ \vdots \\ \tilde{\eta}^{(n)T} \end{bmatrix} m \left[\tilde{\eta}^{(1)} \cdots \tilde{\eta}^{(n)} \right]$

$$\left\{ \tilde{\eta}^{T(a)} \cdot m \cdot \tilde{\eta}^{(b)} = \delta_{ab} \right.$$

$$= I_{n\times n}$$

$$A^T k A = \begin{bmatrix} \tilde{\eta}^{(1)T} \\ \vdots \\ \tilde{\eta}^{(n)T} \end{bmatrix} k \cdot \left[\tilde{\eta}^{(1)} \cdots \tilde{\eta}^{(n)} \right]$$

$$(\omega_{(b)}^2 m - k) \tilde{\eta}^{(b)} = 0$$

$$\tilde{\eta}^{(a)T} (\omega_{(b)}^2 m - k) \tilde{\eta}^{(b)} = 0$$

$$\tilde{\eta}^{(a)T} m \tilde{\eta}^{(b)} = \delta_{ab}$$

$$\tilde{\eta}^{(a)T} k \tilde{\eta}^{(b)} = \omega_{(b)}^2 \delta_{ab}$$

$$= \begin{bmatrix} \omega_{(1)}^2 & & \\ & \ddots & \\ & & \omega_{(n)}^2 \end{bmatrix}$$

则,Q的每个坐标相互独立振动! $[x] = [A] \cdot [Q] = [A] \cdot \begin{bmatrix} A_{(1)} \cos\omega_{(1)} t + B_{(1)} \sin\omega_{(1)} t \\ \vdots \\ A_{(n)} \sim + B_{(n)} \sim \end{bmatrix}$

简振坐标 $Q = A^{-1} x$

A^{-1} 的求法

由于 $A^{-1}A = I$

$A^T m A = I$

则： $A^{-1} = A^T m$

① 求振动问题的范式

(1) $\det(w^2 m - k) = 0$

求 $w^2_{(i)}$

(2) 求 $\eta^{(a)}$，振动模式

$(w^2_{(a)} m - k)\eta^{(a)} = 0$

(3) 归一化

$$\tilde{\eta}^{(a)} = \frac{\eta^{(a)}}{\sqrt{\eta^{(a)T} m \eta^{(a)}}}$$

(4) 模态 matrix

$$A = [\tilde{\eta}^{(1)} \cdots \tilde{\eta}^{(n)}]$$

$$A^T m A = I$$

$$A^T k A = \mathrm{Diag}(w^2_{(1)} \cdots w^2_{(n)})$$

(5) 解的一般形式

$$A \begin{bmatrix} A_{(1)}\cos(w_1 t) + B_{(1)}\sin(w_{(1)} t) \\ \vdots \\ A_{(n)}\cos(w_n t) + B_{(n)}\sin(w_n t) \end{bmatrix} = \begin{bmatrix} x(t) \end{bmatrix}$$

(6) 解：

$$\begin{bmatrix} A_{(1)} \\ \vdots \\ A_{(n)} \end{bmatrix} = A^T m \begin{bmatrix} x(0) \end{bmatrix}$$

$$\begin{bmatrix} w_{(1)} B_{(1)} \\ \vdots \\ w_{(n)} B_{(n)} \end{bmatrix} = A^T m \begin{bmatrix} \dot{x}(0) \end{bmatrix}$$

刚体运动学&动力学

- 刚体的角动量&动能

刚体的动能（引入惯量张量）（取质心为原点）

$$T = \frac{1}{2}\sum m_i(v_c + \vec{r}\times\vec{\omega})^2$$

$$= \frac{1}{2}\sum_i m_i v_c^2 + \frac{1}{2}\sum_i m_i(\vec{r}\times\vec{\omega})^2 + \sum_i m_i v_c\cdot(\vec{r}\times\vec{\omega})$$

$$= \frac{1}{2}M v_c^2 + \frac{1}{2}\sum_i m_i \underline{(\vec{r}\times\vec{\omega})\cdot(\vec{r}\times\vec{\omega})} + \underline{\left(\sum_i m_i \vec{r}\right)}\cdot(\vec{\omega}\times\vec{v_c})$$

$\sum_i m_i \vec{r} = 0$ ：质心条件

$(\vec{r}\times\vec{\omega})\cdot(\vec{r}\times\vec{\omega}) = \vec{r}\cdot[\vec{\omega}\times(\vec{r}\times\vec{\omega})] = \vec{r}[\vec{r}\omega^2 - \vec{\omega}(\vec{r}\cdot\vec{\omega})] = r^2\omega^2 - (\vec{r}\cdot\vec{\omega})^2$

$A\cdot(B\times C) = C\cdot(A\times B)$ $\quad A\times(B\times C) = B\cdot(A\cdot C) - C(A\cdot B)$

$$= \frac{1}{2}M v_c^2 + \frac{1}{2}\sum_i m_i\left(r_i^2\omega^2 - (r_i\cdot\omega)^2\right)$$

$$= \frac{1}{2}M v_c^2 + \frac{1}{2}\left(\sum_i m_i r_i^2\right)\cdot\delta_{\alpha\beta}\,\omega_\alpha\cdot\omega_\beta - \frac{1}{2}\left(\sum_i m_i r_{i\alpha} r_{i\beta}\right)\omega_\alpha\,\omega_\beta$$

$$= \frac{1}{2}M v_c^2 + \frac{1}{2}I_{\alpha\beta}\,\omega_\alpha\omega_\beta$$

$$I_{\alpha\beta} = \sum_i m_i r_i^2\delta_{\alpha\beta} - \sum_i m_i r_{i\alpha} r_{i\beta}$$

刚体的角动量（取质心为原点）

$$\vec{L} = \sum_i m_i \vec{r_i}\times(\omega\times\vec{r_i}) = \sum_i m_i\,\vec{\omega}(r_i^2) - \sum_i m_i \vec{r_i}(\vec{\omega}\cdot\vec{r_i})$$

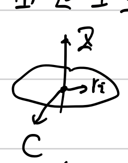
$$L_\alpha = \sum_i m_i r_i^2\omega_\alpha - \sum_i m_i r_{i\alpha} r_{i\beta}\,\omega_\beta$$

$$= I_{\alpha\beta}\,\omega_\beta$$

惯量主轴：

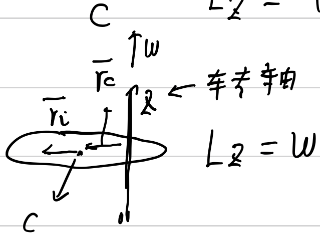
$I_{\alpha\beta} = \sum_i m_i r_i^2\delta_{\alpha\beta} - \sum_i m_i r_{i\alpha} r_{i\beta}$ 仅有对角元非0.

- 平行轴定理



只有绕C转动时.

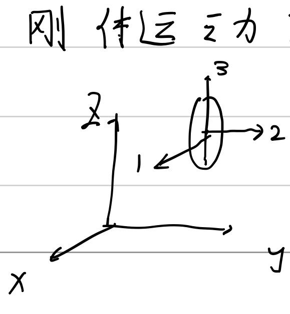
$$L_z = \omega_z\cdot I_z = \omega_z\cdot\sum_i m_i r_i^2$$



$$L_z = \omega_z\cdot I_z + m\omega^2|\vec{r_c}|$$

$$I_z' = I_z + m\omega^2|\vec{r_c}|$$

- 刚体运动 Euler Equation



主轴即随刚体固定

$$\vec{\Omega} = \Omega_1\hat{i}_1 + \Omega_2\hat{i}_2 + \Omega_3\hat{i}_3$$

$$\vec{L} = L_1\Omega_1\hat{i}_1 + L_2\Omega_2\hat{i}_2 + L_3\Omega_3\hat{i}_3$$

地面/转动系 矢量变换关系.

$$\left(\frac{dL}{dt}\right)_{ground} = \left(\frac{dL}{dt}\right)_{rot} + \Omega \times L$$

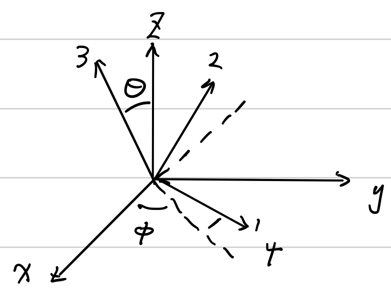
Euler 运动方程.

$$\frac{dL_1}{dt} = \frac{d\Omega_1}{dt} I_1 + \Omega_2\Omega_3(I_3 - I_2) = N_1$$

$$\frac{dL_2}{dt} = \frac{d\Omega_2}{dt} I_2 + \Omega_1\Omega_3(I_1 - I_3) = N_2$$

$$\frac{dL_3}{dt} = \frac{d\Omega_3}{dt} I_3 + \Omega_1\Omega_2(I_2 - I_1) = N_3$$

∘ Euler Angle.



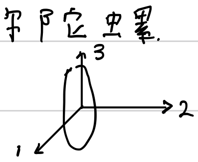
1,2,3 是惯量主轴.

$$\Omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\Omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\Omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$$

∘ 自由不对称陀螺.



自由指无外力，不对称指 $I_1 \neq I_2 \neq I_3$. 一般认为 $I_1 < I_2 < I_3$

运动方程.

$$I_1\frac{d\Omega_1}{dt} = (I_2 - I_3)\Omega_2\Omega_3$$

$$I_2\frac{d\Omega_2}{dt} = (I_3 - I_1)\Omega_1\Omega_3$$

$$I_3\frac{d\Omega_3}{dt} = (I_1 - I_2)\Omega_1\Omega_2$$

由运动方程得到守恒量.

$$\frac{d}{dt}\left[\frac{1}{2}I_1\Omega_1^2 + \frac{1}{2}I_2\Omega_2^2 + I_3\Omega_3^2\right] = \Omega_1\Omega_2\Omega_3(I_2 - I_3 + I_3 - I_1 + I_1 - I_2) = 0$$

↳ $\frac{L_1^2}{2I_1E} + \frac{L_2^2}{2I_2E} + \frac{L_3^2}{2I_3E} = 1$ ⟶ 椭球

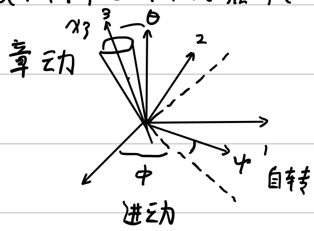
$$\frac{d}{dt}\left[\frac{1}{2}I_1^2\Omega_1^2 + \frac{1}{2}I_2^2\Omega_2^2 + \frac{1}{2}I_3^2\Omega_3^2\right] = \Omega_1\Omega_2\Omega_3\left[I_1(I_2 - I_3) + I_2(I_3 - I_1) + I_3(I_1 - I_2)\right] = 0$$

↳ $\frac{L_1^2}{L^2} + \frac{L_2^2}{L^2} + \frac{L_3^2}{L^2} = 1$ ⟶ 三个轴都相等的椭球.

⟶ 轨道是2球交线!

3轴长度: $\sqrt{2I_1E} < \sqrt{2I_2E} < \sqrt{2I_3E}$

○ 对称陀螺定点转动.



$$I_1' = I_2' = I_1 + ml^2$$

$$L = \frac{1}{2}(I_1 + ml^2)\,\Omega_1^2 + \frac{1}{2}(I_1 + ml^2)\,\Omega_2^2 + \frac{1}{2} I_3 \Omega_3^2 - mgl\cos\theta$$

$$\begin{cases} \Omega_1 = \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi \\ \Omega_2 = -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi \\ \Omega_3 = \dot\psi + \dot\phi\cos\theta \end{cases}$$

$$= \frac{1}{2}(I_1 + ml^2)(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 - mgl\cos\theta$$

Euler – Lagrange Equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}$$

ψ, ϕ 不在 L 中显含，p_ψ, p_ϕ 守恒.

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = L_3$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = (I_1'\sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\dot\psi\cos\theta = L_z$$

$$\Rightarrow \dot\phi = \dot\phi(\theta), \quad \dot\psi = \dot\psi(\theta)$$

能量也是守恒的(无证明,直接写出.

$$E = \frac{1}{2} I_1'(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{I_3}{2}(\dot\psi + \dot\phi\cos\theta)^2 + mgl\cos\theta$$

$$\dot\phi = \frac{L_z - L_3\cos\theta}{I_1'\sin^2\theta} \qquad \dot\psi = \frac{L_3}{I_3} - \frac{L_z - L_3\cos\theta}{I_1'\sin^2\theta}\cos\theta$$

$$\Downarrow$$

$$E = E(\dot\theta, \theta) \Rightarrow \dot\theta = f(\theta) \Rightarrow \text{积分得 } \theta = \theta(t).$$

正则变换 canonical Transformation

- Hamiltonian 引入

$$L = L(q, \dot{q}, t)$$

EL equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

动量定义:

$$p = \frac{\partial L}{\partial \dot{q}} \Rightarrow \dot{q} = \dot{q}(q, p, t)$$

定义:

$$H = p\dot{q} - L$$

$$dH = d(p\dot{q}) - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial t}dt$$

$$= p\,d\dot{q} + (dp)\dot{q} - \frac{d}{dt}(p)\,dq - p\,d\dot{q} - \frac{\partial L}{\partial t}dt$$

$$= \dot{q}\,dp - \dot{p}\,dq - \frac{\partial L}{\partial t}dt$$

Heisenberg equation

$$\left(\frac{\partial H}{\partial p}\right) = \dot{q}$$

$$\left(\frac{\partial H}{\partial q}\right) = -\dot{p}$$

$$\frac{\partial H}{\partial t} = -\partial L/\partial t$$

- 泊松括号.

A, B 为 (q, p) 的 function

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

—— 意义: $A = A(q, p, t)$

$$\{A, H\} = \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q}$$

$$= \frac{\partial A}{\partial q}\dot{q} + \frac{\partial A}{\partial p}\dot{p}$$

$$= \frac{dA}{dt} - \frac{\partial A}{\partial t}$$

- 正则变换 & 辛矩阵/辛条件.

$$H = H(q, p) \qquad \text{坐标变换: } P = P(q, p) \quad Q = Q(q, p)$$

$$H' = H'(P, Q)$$

希望 P, Q 满足方程.

$$\begin{cases} \dfrac{dQ}{dt} = \dfrac{\partial H'}{\partial P} \\ \dfrac{dP}{dt} = -\dfrac{\partial H'}{\partial Q} \end{cases} \Longrightarrow \frac{d}{dt}\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \partial H'/\partial Q \\ \partial H'/\partial P \end{pmatrix} \quad —(1)$$

用坐标变化表示新的 Q, P 变化率.

$$\begin{cases} \dfrac{dQ}{dt} = \dfrac{\partial Q}{\partial q}\dfrac{dq}{dt} + \dfrac{\partial Q}{\partial p}\dfrac{dp}{dt} = \dfrac{\partial Q}{\partial q}\dfrac{\partial H}{\partial p} - \dfrac{\partial Q}{\partial p}\dfrac{\partial H}{\partial q} \\ \dfrac{dP}{dt} = \dfrac{\partial P}{\partial q}\dfrac{\partial H}{\partial p} - \dfrac{\partial P}{\partial p}\dfrac{\partial H}{\partial q} \end{cases} \Rightarrow \frac{d}{dt}\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} \quad —(2)$$

由于:

$$\frac{\partial H}{\partial q} = \frac{\partial H'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H'}{\partial P}\frac{\partial P}{\partial q}$$
$$\frac{\partial H}{\partial p} = \frac{\partial H'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H'}{\partial P}\frac{\partial P}{\partial p}$$
$$\Rightarrow \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial H'}{\partial Q} \\ \frac{\partial H'}{\partial P} \end{pmatrix} \qquad -(3)$$

结合 (1) (2) (3):

$$\underbrace{\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{(2)} \cdot \underbrace{\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix}}_{(3)} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{(1)}$$

高维:(辛条件)

行指标 → 列指标 →
$$\frac{\partial(Q,P)}{\partial(q,p)} \cdot \Omega \cdot \frac{\partial(Q,P)}{\partial(q,p)}^T = \Omega \qquad \Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

↑ 辛矩阵.

。生成函数.

$$\delta \int_{t_0}^{t} \left(\sum_\alpha p_\alpha \dot{q}_\alpha - H \right) dt = 0 \Longrightarrow$$ 可用变分法得 Heisenberg 方程.
$$\delta \int_{t_0}^{t} \left(\sum_\alpha P_\alpha \dot{Q}_\alpha - H' \right) dt = 0$$

$$\sum_\alpha p_\alpha \dot{q}_\alpha - \overset{H(q,p)}{H} = \sum_\alpha P_\alpha \dot{Q}_\alpha - \overset{H'(Q,P)}{H'} + \frac{dF}{dt}$$

$$F = F(q, p, Q, P, t)$$

1° $F = F_1(q, Q, t)$

$$\frac{dF}{dt} = \frac{\partial F_1}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial F_1}{\partial Q_\alpha}\dot{Q}_\alpha + \frac{\partial F_1}{\partial t}$$

$$p_\alpha = \frac{\partial F_1}{\partial q_\alpha} \qquad P_\alpha = -\frac{\partial F_1}{\partial Q_\alpha} \qquad H' = H + \frac{\partial F_1}{\partial t}$$

2° $F = F_2(q, P, t) - \sum_\alpha P_\alpha Q_\alpha$

$$\frac{dF}{dt} = \frac{\partial F_2}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial F_2}{\partial P_\alpha}\dot{P}_\alpha - Q_\alpha \dot{P}_\alpha - P_\alpha \dot{Q}_\alpha + \frac{\partial F_2}{\partial t}$$

$$p_\alpha = \frac{\partial F_2}{\partial q_\alpha} \qquad Q_\alpha = \frac{\partial F_2}{\partial P_\alpha} \qquad H' = H + \frac{\partial F_2}{\partial t}$$

3° $F = F_3(p, Q, t) + \sum_\alpha q_\alpha p_\alpha$

$$q_\alpha = -\frac{\partial F_3}{\partial p_\alpha} \qquad P_\alpha = -\frac{\partial F_3}{\partial Q_\alpha} \qquad H' = H + \frac{\partial F_3}{\partial t}$$

4° $F = F_4(p, P, t) - \sum_\alpha P_\alpha Q_\alpha + \sum_\alpha p_\alpha q_\alpha$

$$Q_\alpha = \frac{\partial F_4}{\partial P_\alpha} \qquad q_\alpha = -\frac{\partial F_4}{\partial p_\alpha} \qquad H' = H + \frac{\partial F_4}{\partial t}.$$

。泊松括号条件.

证明： $\{Q_\alpha, P_\beta\} = \delta_{\alpha\beta}$

$$\{Q_\alpha, P_\beta\} = \sum_i \left(\frac{\partial Q_\alpha}{\partial q_i}\frac{\partial P_\beta}{\partial p_i} - \frac{\partial Q_\alpha}{\partial p_i}\frac{\partial P_\beta}{\partial q_i} \right)$$

$$= \left(\frac{\partial Q_\alpha}{\partial q_1} \cdots \frac{\partial Q_\alpha}{\partial q_n} \right) \cdot \begin{pmatrix} \frac{\partial P_\beta}{\partial p_1} \\ \vdots \\ \frac{\partial P_\beta}{\partial p_n} \end{pmatrix} - \left(\frac{\partial Q_\alpha}{\partial p_1} \cdots \frac{\partial Q_\alpha}{\partial p_n} \right) \cdot \begin{pmatrix} \frac{\partial P_\beta}{\partial q_1} \\ \vdots \\ \frac{\partial P_\beta}{\partial q_n} \end{pmatrix}$$

若将 $\{Q_\alpha, P_\beta\}$ 理解为矩阵.

$$\{Q, P\} = \underset{\downarrow}{J_Q(q)} \cdot J_P(p) - J_Q(p) \cdot J_P(q)$$

$$\begin{pmatrix} \frac{\partial Q_1}{\partial q_1} & \cdots & \frac{\partial Q_1}{\partial q_n} \\ \vdots & & \\ \frac{\partial Q_n}{\partial q_1} & \cdots & \frac{\partial Q_n}{\partial q_n} \end{pmatrix}$$

也就是，要证明

$$J_Q(q) J_P(p) - J_Q(p) J_P(q) = \mathbb{I}$$

辛条件，

$$\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} J_Q(q) & J_Q(p) \\ J_P(q) & J_P(p) \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} J_Q(q) & J_P(q) \\ J_Q(p) & J_P(p) \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} J_Q(q) & J_Q(p) \\ J_P(q) & J_P(p) \end{pmatrix} \begin{pmatrix} J_Q(p) & J_P(p) \\ -J_Q(q) & -J_P(q) \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

$$\Downarrow$$

$$J_Q(q) J_P(p) - J_Q(p) J_P(q) = \mathbb{I}$$

证毕!

▫ 作用量S随端点的变化

$$S=\int_{t_0}^{t_1} L(q(t),\dot{q}(t),t)\,dt$$

若无端点变化 $\delta S=\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial \dot q}\frac{d}{dt}(\delta q)\right]dt$

端点变化： $(t_0,q_0)\underset{\text{解是 }q_1(t)}{\overset{\Downarrow}{\text{to}}}(t_1,q_1)\rightarrow(t_0,q_0)\underset{\text{解是 }q_2(t)}{\overset{\Downarrow}{\text{to}}}(t_1+dt_1,q_1+dq_1)$

$$\delta S=\int_{t_0}^{t_1+dt_1}L(q_2(t),\dot q_2(t),t)\,dt-\int_{t_0}^{t_1}L(q_1(t),\dot q_1(t),t)\,dt$$

$$=\int_{t_0}^{t_1}\{L(q_2(t),\dot q_2(t),t)-L(q_1(t),\dot q_1(t),t)\}dt+L(q_2(t_1),\dot q_2(t_1),t_1)dt_1$$

$$=\int_{t_0}^{t_1}\frac{\partial L}{\partial q}(q_2(t)-q_1(t))dt+\int_{t_0}^{t_1}\frac{\partial L}{\partial \dot q}(\dot q_2(t)-\dot q_1(t))dt+L(q_1(t_1),\dot q_1(t_1),t_1)dt_1$$

$$=\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right](q_2(t)-q_1(t))+\Delta\left[\frac{\partial L}{\partial \dot q}(q_2(t)-q_1(t))\right]\Big|_{t_0}^{t_1}$$
$$+L(q_1(t_1),\dot q_1(t_1),t_1)dt_1$$

$$=\frac{\partial L}{\partial \dot q}\Big|_{t_1}(q_2(t_1)-q_1(t_1))+L(t_1)dt_1$$

$$\Big\downarrow\leftarrow\Big\{\ q_2(t_1)=q_1+dq_1-\dot q_2(t_1)\cdot dt_1\approx q_1+dq_1-\dot q_1(t_1)dt_1$$

$$=\frac{\partial L}{\partial \dot q}\Big|_{t_1}\cdot(dq_1-\dot q_1(t_1)dt_1)+L(t_1)dt_1$$

$$=P(t_1)dq_1+(L-P\dot q)dt_1$$

$$S=S(q_0,t_0,q_1,t_1)$$

$$\underbrace{\frac{\partial S}{\partial q_1}=P(t_1)\qquad\frac{\partial S}{\partial t_1}=L-P\dot q}=-H$$

$$\Downarrow$$

$H(q,\frac{\partial S}{\partial q},t)+\frac{\partial S}{\partial t}=0$. ⟶ Hamilton — Jacobi Equation.

↳ 有 n+1 个积分const，1个以 S+S₀ 形式存在.

↓

H 中不显含时间时 → $S=S(q)-Et$

↑ 有n-1个积分常数.　↑ 积分常数