Action of point particle

- Poincare invariant , parametrization invariant action

$$S_{PP} = -m \int d\tau \left(-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}\right)^{1/2}$$

- Another useful form Action.

$$S'_{PP} = \frac{1}{2}\int d\tau \left(\eta^{-1}\frac{dX_\mu}{d\tau}\frac{dX^\mu}{d\tau} - \eta m^2\right)$$

—— Reparametrization invariance

$$X'^\mu(\tau'(\tau)) = X^\mu(\tau)$$

$$\eta'(\tau')d\tau' = \eta(\tau)d\tau$$

$$d\tau\left(\eta^{-1}\frac{dX_\mu}{d\tau}\frac{dX^\mu}{d\tau} - \eta m^2\right) = d\tau\left(\frac{1}{\eta}\frac{dX_\mu}{d\tau}\frac{dX^\mu}{d\tau} - \eta m^2\right)$$

$$= d\tau'\left(\frac{1}{\eta'}\frac{dX'_\mu}{d\tau'}\frac{dX'^\mu}{d\tau'} - \eta' m^2\right)$$

—— Equation of motion varying tetrad "η".

$$S'_{PP} = \frac{1}{2}\int d\tau \left(\eta^{-1}\frac{dX_\mu}{d\tau}\frac{dX^\mu}{d\tau} - \eta m^2\right)$$

$$\delta S'_{PP} = \frac{1}{2}\int d\tau \left(-\frac{\delta\eta}{\eta^2}\frac{dX_\mu}{d\tau}\frac{dX^\mu}{d\tau} - \delta\eta\, m^2\right)$$

$$= -\frac{1}{2}\int d\tau \left(\frac{1}{\eta^2}\frac{dX_\mu}{d\tau}\frac{dX^\mu}{d\tau} + m^2\right)\delta\eta$$

Variational principle

$$\eta^2 = -\frac{1}{m^2}\dot{X}_\mu \dot{X}^\mu$$

insert obtained tetrad into action results in original action

$$S'_{PP} = \frac{1}{2}\int d\tau \left(\eta^{-1}\frac{dX_\mu}{d\tau}\frac{dX^\mu}{d\tau} - \eta m^2\right)$$

$$= \frac{1}{2}\int d\tau \left(\frac{1}{\sqrt{-\frac{1}{m^2}\dot{X}_\mu\dot{X}^\mu}}\dot{X}_\mu\dot{X}^\mu - \sqrt{-\frac{1}{m^2}\dot{X}_\mu\dot{X}^\mu}\, m^2\right)$$

$$= \frac{1}{2}\int d\tau \cdot \frac{1}{\sqrt{-\frac{1}{m^2}\dot{X}_\mu\dot{X}^\mu}}\left(\dot{X}_\mu\dot{X}^\mu - m^2\left(-\frac{1}{m^2}\dot{X}_\mu\dot{X}^\mu\right)\right)$$

$$= \frac{1}{2}\int d\tau \frac{1}{\sqrt{-\frac{1}{m^2}\dot{X}_\mu\dot{X}^\mu}}\, 2\dot{X}_\mu\dot{X}^\mu$$

$$= \int d\tau \frac{1}{\sqrt{-\frac{1}{m^2}\dot{X}_\mu\dot{X}^\mu}}\dot{X}_\mu\dot{X}^\mu$$

$$= -m\int d\tau \left(-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}\right)^{1/2} = S_{PP}$$

Action of String.

- Namb – Goto action. $X^\mu = X^\mu(\tau, \sigma)$ $\sigma^a = (\tau, \sigma)$
  - —— induced matrix h_{ab}, define induced matrix as

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu$$

  - —— Namb – Goto Action defined as

$$S_{NG} = \int_M d\tau d\sigma \left(-\frac{1}{2\pi\alpha'}\right)(-\det h_{ab})^{1/2}$$

  - —— Reparametrization invoriant

$$X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma)$$

- Brink – Di Vecchia – Howe – Deser – Zumino action or Polyakov action.

$$S_P[X, \gamma] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\,(-\gamma)^{1/2}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu \qquad \gamma = \det \gamma_{ab}$$

  - —— Variction of determiant

$$\ln \det M = \mathrm{tr} \ln M$$

$$\frac{1}{\det M} d(\det M) = \mathrm{tr}\, M^{-1} dM$$

$$\frac{1}{\det M} d(\det M) = (M^{-1})_{ab}\, dM_{ba}$$

$$\frac{1}{\gamma}\delta\gamma = \gamma^{ab}\,\delta\gamma_{ba}$$

$$\delta\gamma = \gamma\,\gamma^{ab}\,\delta\gamma_{ba}$$

Considering

$$\gamma^{ab}\gamma_{bc} = \delta_{ac} \quad ; \quad \gamma^{ab}\delta\gamma_{ba} + \delta\gamma^{ab}\cdot\gamma_{ba} = 0 \qquad \gamma^{ab}\delta\gamma_{ba} = -\gamma_{ba}\,\delta\gamma^{ab}$$

$$\delta\gamma = -\gamma\cdot\gamma_{ba}\,\delta\gamma^{ab}$$

  - —— Variational principle respect to γ matrix.

$$S_P[X, \gamma] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\,(-\gamma)^{1/2}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu$$

$$= -\frac{1}{4\pi\alpha'}\int_M d\tau\, d\sigma\cdot(-\gamma)^{-1/2}\cdot\gamma^{ab}\partial_a X^\mu \partial_b X^\mu\ \frac{1}{2}(-1)\cdot d\gamma$$
$$-\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\,(-\gamma)^{1/2}\ \partial_a X^\mu \partial_b X_\mu\ \delta\gamma^{ab}$$

$$= -\frac{1}{4\pi\alpha'}\int_M d\tau\, d\sigma\cdot(-\gamma)^{-1/2}\cdot\gamma^{cd}\partial_c X^\mu \partial_d X^\mu\ \frac{1}{2}\gamma\cdot\gamma_{ba}\,\delta\gamma^{ab}$$
$$-\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\,(-\gamma)^{1/2}\ \partial_a X^\mu \partial_b X_\mu\ \delta\gamma^{ab}$$

$$= -\frac{1}{4\pi\alpha'}\int_M d\tau\, d\sigma\cdot(-\gamma)^{-1/2}\cdot\gamma^{cd}\, h_{cd}\ \frac{1}{2}\gamma\cdot\gamma_{ba}\,\delta\gamma^{ab}$$
$$-\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\,(-\gamma)^{1/2}\, h_{ab}\ \delta\gamma^{ab}$$

$$= -\frac{1}{4\pi\alpha'}\int_M d\tau\, d\sigma\cdot(-\gamma)^{1/2}\left(h_{ab}\,\delta\gamma^{ab} - \frac{1}{2}\gamma^{cd}h_{cd}\,\gamma_{ba}\,\delta\gamma^{ab}\right)$$

$$= -\frac{1}{4\pi\alpha'}\int_M d\tau\, d\sigma\,(-\gamma)^{1/2}\left(h_{ab} - \frac{1}{2}\gamma^{cd}h_{cd}\cdot\gamma_{ba}\right)\delta\gamma^{ab}$$

Variation invariant.

$$h_{ab} - \frac{1}{2}\gamma^{cd}h_{cd}\cdot\gamma_{ba} = 0$$

$$\gamma_{ba} \sim h_{ab} = \partial_a X^\mu \partial_b X_\mu = h_{ba}$$

by insertion

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\,(-\gamma)^{1/2}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu$$

$$\sim -\frac{1}{2\pi\alpha'}\int_M d\tau d\sigma\,(-h)^{1/2} \sim S_{NG}[X,\gamma]$$

Parametrization invariant of polyakov action

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\,(-\gamma)^{1/2}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu$$

$$= -\frac{1}{4\pi\alpha'}\int_M d\tau\, d\sigma\cdot(-\det\gamma)^{1/2}\,(\gamma^{-1})_{ab}\,\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X_\mu}{\partial\sigma^b}$$

reparametrization of γ matrix

$$\frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau',\sigma') = \gamma_{ab}(\tau,\sigma)$$

$$\frac{\partial\sigma'^c}{\partial\sigma^a}\gamma'_{cd}(\tau',\sigma')\frac{\partial\sigma'^d}{\partial\sigma^b} = \gamma_{cd}(\tau',\sigma')$$

$$\Lambda^T\gamma'\Lambda = \gamma \qquad \Lambda_{ab} \equiv \frac{\partial\sigma'^a}{\partial\sigma^b}$$

inverse $\Lambda^{-1}{}_{ab} \equiv \dfrac{\partial\sigma^a}{\partial\sigma'^b}$ check : $\Lambda^{-1}{}_{ab}\cdot\Lambda_{bc} = \dfrac{\partial\sigma^a}{\partial\sigma'^b}\dfrac{\partial\sigma'^b}{\partial\sigma^c} = \delta_{ac}$

$$\gamma' = (\Lambda^T)^{-1}\gamma\Lambda^{-1}$$

$$= (\Lambda^{-1})^T\gamma\Lambda^{-1}$$

reparametrization of determiant of γ matrix.

$$\det(\gamma') = \det((\Lambda^{-1})^T)\,\det(\gamma)\,\det(\Lambda^{-1})$$

$$= \frac{1}{\det^2(\Lambda)}\cdot\det(\gamma)$$

reparametrization of γ inverse.

$$(\gamma')^{-1} = \Lambda\gamma^{-1}\Lambda^T$$

Reparametrization of h matrix.

$$h'_{ab} = \frac{\partial X'^\mu}{\partial\sigma'^a}\frac{\partial X'_\mu}{\partial\sigma'^b}$$

$$= \frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial X^\mu}{\partial\sigma^c}\frac{\partial X_\mu}{\partial\sigma^d}\frac{\partial\sigma^d}{\partial\sigma'^b}$$

$$= (\Lambda^{-1})^T\, h\,\Lambda^{-1}$$

reparametrization of polyakov action

original action

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\, (-\gamma)^{1/2}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu$$

$$= -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\, (-\gamma)^{1/2}\, tr(\gamma^{-1}h)$$

reparametrized action

$$S'_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d^2\sigma'\, (-\gamma')^{1/2}\, tr(\gamma'^{-1}h')$$

$$= -\frac{1}{4\pi\alpha'}\int_M d^2\sigma \cdot det\left(\frac{\partial\sigma'}{\partial\sigma}\right)\cdot\left[-\frac{1}{det^2(\Lambda)}\cdot det(\gamma)\right]^{1/2}$$

$$tr\left(\Lambda\gamma^{-1}\Lambda^T(\Lambda^{-1})^T h\Lambda^{-1}\right)$$

$$= -\frac{1}{4\pi\alpha'}\int_M d^2\sigma \cdot det(\Lambda)\cdot(-det(\gamma))^{1/2}\cdot det^{-1}(\Lambda)\cdot$$
$$tr(\gamma^{-1}h)$$

$$= -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\, (-det(\gamma))^{1/2}\, tr(\gamma^{-1}h)$$

$$= S_P[X,\gamma]$$

Reparametrizational invariant !

Two dimensional weyl invariance.

$$X'^\mu(\tau,\sigma) = X^\mu(\tau,\sigma)$$
$$\gamma'(\tau,\sigma) = exp(2w(\tau,\sigma))\,\gamma(\tau,\sigma)$$

Weyl transformation of γ inverse matrix

$$\gamma'^{-1} = exp(-2w(\tau,\sigma))\,\gamma^{-1}(\tau,\sigma)$$

Weyl transformation of γ determinant.

$$det(\gamma') = exp(4w(\tau,\sigma))\, det(\gamma(\tau,\sigma))$$

Weyl transformation of polyakov action.

$$S'_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\, (-\gamma')^{1/2}\, tr(\gamma'^{-1}h)$$

$$= -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\, (-\gamma)^{1/2}\cdot exp(2w(\tau,\sigma))\, exp(-2w(\tau,\sigma))$$
$$tr(\gamma^{-1}h)$$

$$= -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\, (-\gamma)^{1/2}\cdot tr(\gamma^{-1}h)$$

From Action to correlator

- Generation func

$$\langle x(\tau_1)\cdots x(\tau_n)\rangle = \frac{\int \mathcal{D}x\; x(\tau_1)\cdots x(\tau_n)\, e^{-S_E[x(\tau)]}}{\int \mathcal{D}x\, e^{-S_E[x(\tau)]}}$$ 下面不写E与τ, 写为t与S

$$Z[j] = \int \mathcal{D}x\; \exp -\{S[x(t)] - \int dt\, j(t)\, x(t)\}$$

$$= \int \mathcal{D}x\; \exp\left(\int dt\, j(t)\, x(t)\right) \exp\left(-S[x(t)]\right)$$

$$= \int \mathcal{D}x \sum_n \frac{1}{n!} \int dt_{i_1}\cdots dt_{i_n}\, j(t_{i_1})\cdots j(t_{i_n})\, x(t_{i_1})\cdots x(t_{i_n})\, \exp\left(-S[x(t)]\right)$$

$$= \sum_n \frac{1}{n!} \int dt_{i_1}\cdots dt_{i_n}\cdot j(t_{i_1})\cdots j(t_{i_n})\, \langle x(t_{i_1})\cdots x(t_{i_n})\rangle \cdot Z[0]$$

$$\langle x(t_1)\cdots x(t_n)\rangle = Z^{-1}[0]\, \frac{\delta}{\delta j(t_1)} \cdots \frac{\delta}{\delta j(t_n)}\, Z[j]\Big|_{j=0}$$

- Free Boson

$$S = \frac{1}{2} g \int d^d x\, \{\partial_\mu\varphi\, \partial^\mu\varphi + m^2\varphi^2\}$$

表示为 Gauss 积分形式.

作用于 $(x-y)$! 作用在 y上

$$\int d^dx\, d^dy\; \varphi(x)\{-\partial_\mu\partial^\mu\, \delta(x-y)\}\varphi(y) = \int d^dx\, d^dy\; \varphi(x)\{+\partial_{x_\mu}\partial_y^\mu\delta(x-y)\}\varphi(y)$$

$$= \int d^dx\, d^dy\; \partial_{x_\mu}\varphi(x)\{-\partial_y^\mu\delta(x-y)\}\varphi(y)$$

$$= \int d^dx\, d^dy\; \partial_{x_\mu}\varphi(x)\, \delta(x-y)\, \partial_y^\mu\varphi(y)$$

$$= \int d^dx\; \partial_{x_\mu}\varphi(x)\, \partial_x^\mu\varphi(x)$$

$$\int d^dx\, d^dy\, \varphi(x)\left(-\delta(x-y)\,\partial_y^2\right)\varphi(y) = \int d^dx -\varphi(x)\,\partial_x^2\varphi(x) = \int d^dx\, \partial_\mu\varphi\,\partial^\mu\varphi$$

有两种表示 kernel 的方式.

$$A(x,y) = \begin{cases} 1^\circ & g\{-\partial^2 + m^2\}\delta(x-y) \\ 2^\circ & g\,\delta(x-y)\{-\partial^2+m^2\} \end{cases}$$

1° partial 作用在 δ function

2° partial 作用在之后的 function 上.

$$S = \frac{1}{2}\int d^dx\, d^dy\; \varphi(x)\, A(x,y)\, \varphi(y)$$

Generating function

$$Z[j] = \int \mathcal{D}\varphi\; \exp\{-S[\varphi] + \int d^dx\, J(x)\varphi(x)\}$$

$$= \int \mathcal{D}\varphi\; \exp\{-\frac{1}{2}\int d^dx\, d^dy\; \varphi(x) A(x,y)\varphi(y) + \int d^dx\, J(x)\varphi(x)\}$$

Gauss integral

$$\int d^D v\, \exp\left(-\frac{1}{2} v^T A v + \rho^T v\right) = (2\pi)^{D/2} \exp\left(-\frac{1}{2}\mathrm{Tr}\ln A\right) \exp\left(\frac{1}{2}\rho^T A^{-1}\rho\right)$$

因此, 求出 A 的逆比较重要.

—— Use fourier transformation get A^{-1}

$$A(x,y) = A(x-y) = g\{-\partial^2 + m^2\}\delta(x-y)$$

$$A(\vec r) = g(-\partial^2 + m^2)\,\delta(\vec r)$$

$$= g(-\partial^2 + m^2)\int \frac{d^dk}{(2\pi)^d}\, e^{i\vec k\cdot\vec r}$$

$$= g\int \frac{d^dk}{(2\pi)^d}(k^2+m^2)\, e^{i\vec{k}\cdot\vec{r}}$$

$$A^{-1}(\vec{r}) = \frac{1}{g}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\, e^{i\vec{k}\cdot\vec{r}}$$

in two dimensions

$$A^{-1}(\vec{r}) = \frac{1}{g}\int \frac{dk^1\,dk^2}{(2\pi)^2}\frac{1}{k^2+m^2}\, e^{i\vec{k}\cdot\vec{r}}$$

$$= \frac{1}{g}\int \frac{k\,d\theta\,dk}{(2\pi)^2}\frac{1}{k^2+m^2}\, e^{ikr\cos\theta}$$

$$= \frac{1}{g}\int \frac{k\,dk}{(2\pi)^2}\frac{1}{k^2+m^2}\,d\theta\, e^{ikr\cos\theta}$$

$$= \frac{1}{g}\int \frac{k\,dk}{(2\pi)^2}\frac{1}{k^2+m^2}\,2\pi J_0(kr)$$

$$\approx \frac{1}{g}\frac{1}{2\pi r}e^{-mr} \quad \text{for } r\to+\infty$$

——— Use Green function evaluate $A^{-1}(r) = K(r)$

$$\int d^dy\; A(x,y)K(y,z) = \delta(x-z)$$

$$\int d^dy\; g(-\partial^2+m^2)\delta(x-y)\;K(y,z) = \delta(x-z)$$

$$\int d^dy\; g\{(-\partial_y^2+m^2)\delta(x-y)\}\;K(y,z) = \delta(x-z)$$

$$\int d^dy\; g\;\delta(x-y)(-\partial_y^2+m^2)K(y,z) = \delta(x-z)$$

$$g(-\partial_x^2+m^2)K(x,y) = \delta(x-y)$$

in two dimensions. $\vec{x}-\vec{y}\equiv\vec{r}$

$$g(-\partial^2+m^2)K(\vec{r}) = \delta(\vec{r})$$

$$g\left(-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)K(\vec{r},\theta) = \delta(\vec{r})$$

integration over (r,θ)

$$2\pi g\int_0^r d\rho\,\rho\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho K'(\rho))+m^2K(\rho)\right]=1$$

$$2\pi g\left[-rK'(r)+m^2\int_0^r d\rho\,\rho K(\rho)\right]=1$$

$m=0$ solution

$$2\pi g[-rK'(r)] = 1$$

$$K(r) = -\frac{1}{2\pi g}\ln r + \text{const}$$

$$= -\frac{1}{4\pi g}\ln r^2 + \text{const.}$$

$m\neq 0$ solution

$$\frac{d}{dr}\left\{-rK'(r)+m^2\int_0^r d\rho\,\rho K(\rho)\right\} = 0$$

$$-K'(r)-rK''(r)+m^2rK(r) = 0$$

$$K''(r)+\frac{1}{r}K'(r)-m^2K(r) = 0$$

$$\Downarrow$$

modified Bessel function

$$K(r) = \frac{1}{2\pi g}K_0(mr) \qquad K_0(x) = \int_0^\infty dt\,\frac{\cos(xt)}{\sqrt{t^2+1}}$$

$$K(r)\sim e^{-mr}$$

——— Generating function

$$Z[J] \sim \exp\left(\frac{1}{2}\int d^d x\, d^d y \;\; \varphi(x)\, K(x,y)\, \varphi(y)\right)$$

$$\langle \varphi(x)\, \varphi(y) \rangle \sim \left(\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} Z[J]\right) \frac{1}{Z[0]} = K(x,y) = K(\vec{r})$$

conformal field theory

Lorentz symmetry

- Generator of transformation

—— coordinate transformation & field trans & action trans.

coor: $x'^{\mu} = x^{\mu} + w_a \frac{\delta x^{\mu}}{\delta w_a}$

field: $\Phi'(x') = \Phi(x) + w_a \frac{\delta F}{\delta w_a}(x) = F(\Phi(x))$

generator:

$$\Phi'(x) - \Phi(x) = \Phi\left(x - w_a \frac{\delta x^{\mu}}{\delta w_a}\right) + w_a \frac{\delta F}{\delta w_a}(x) - \Phi(x)$$

$$= -w_a \frac{\delta x^{\mu}}{\delta w_a} \partial_{\mu} \Phi(x) + w_a \frac{\delta F}{\delta w_a}(x) \qquad (\emptyset)$$

$$= -i w_a \left(i \frac{\delta F}{\delta w_a}(x) - i \frac{\delta x^{\mu}}{\delta w_a} \partial_{\mu} \Phi \right)$$

$$\equiv -i w_a G_a$$

—— Generator of translation

$$x'^{\mu} = x^{\mu} + w^{\mu}$$
$$= x^{\mu} + \delta^{\mu}_{\nu} w^{\mu}$$

$$\frac{\delta x^{\mu}}{\delta w^{\nu}} \partial_{\mu} \Phi = \delta^{\mu}_{\nu} \partial_{\mu} \Phi = \partial_{\nu} \Phi$$

$$\Phi'(x) - \Phi(x) \sim -i w^{\nu} \left(-i \frac{\delta x^{\mu}}{\delta w^{\nu}} \partial_{\mu} \Phi\right) \sim -i w^{\nu} \left(-i \delta^{\mu}_{\nu} \partial_{\mu} \Phi\right)$$
$$\sim -i w^{\mu} (-i \partial_{\mu} \Phi)$$

$$P_{\mu} \equiv -i \partial_{\mu}$$

—— Generator of Lorentz trans

$$x'^{\mu} = (\delta^{\mu}_{\nu} + w^{\mu}{}_{\nu}) x^{\nu}$$

$$\delta x^{\mu} = x^{\mu} + w^{\mu}{}_{\nu} x^{\nu}$$

$$\Phi'(x') = F(\Phi(x)) = \Phi(x) - \frac{i}{2} w_{\mu\nu} S^{\mu\nu} \Phi(x)$$

$$w_a \frac{\delta F}{\delta w_a} = -\frac{i}{2} w_{\mu\nu} S^{\mu\nu} \Phi(x)$$

$$w_a \frac{\delta x^{\mu}}{\delta w_a} \partial_{\mu} = w^{\mu}{}_{\nu} x^{\nu} \partial_{\mu} = w_{\mu\nu} x^{\nu} \cdot \partial^{\mu} = \frac{1}{2} w_{\mu\nu} (x^{\nu} \partial^{\mu} - x^{\mu} \partial^{\nu})$$

$$\Phi'(x) - \Phi(x) = \left(-\frac{1}{2} w_{\mu\nu} (x^{\nu} \partial^{\mu} - x^{\mu} \partial^{\nu}) - \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right) \Phi(x)$$

$$= -\frac{i}{2} w_{\mu\nu} \left(i (x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu}) + S^{\mu\nu} \right) \Phi(x)$$

$$L^{\mu\nu} \equiv i (x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu}) + S^{\mu\nu}$$

Conserved current and ward identity

∘ Conserved current.

$$S = \int d^d x \ \mathcal{L}(\phi(x), \partial_\mu \phi(x))$$

$$S' = \int d^d x' \ \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'))$$

— change of field ϕ and coordinate

$$\phi'(x') = \phi(x) + w_a \frac{\delta F}{\delta w_a}(x)$$

$$x'^\mu = x^\mu + w_a \frac{\delta x^\mu}{\delta w_a}$$

— determinant of integrant

$$d^d x' = d^d x \cdot \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)$$

$$= d^d x \left| \delta^\mu_\nu + \partial_\nu \left(w_a \frac{\delta x^\mu}{\delta w_a}\right)\right|$$

$$= d^d x \left(1 + \partial_\mu \left(w_a \frac{\delta x^\mu}{\delta w_a}\right)\right)$$

— change of derivative $x^\nu = x'^\nu - w_a \frac{\delta x^\nu}{\delta w_a}$

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \left(\delta^\nu_\mu - \partial_\mu \left(w_a \frac{\delta x^\nu}{\delta w_a}\right)\right)\partial_\nu$$

— Change of action

$$S' = \int d^d x \left(1 + \partial_\mu \left(w_a \frac{\delta x^\mu}{\delta w_a}\right)\right) \mathcal{L}\left(1 + w_a \frac{\delta F}{\delta w_a}, \left(\delta^\nu_\mu - \frac{\partial}{\partial x^\mu}\left(w_a \frac{\delta x^\nu}{\delta w_a}\right)\right) \partial_\nu \left(\phi(x) + w_a \frac{\delta F}{\delta w_a}(x)\right)\right)$$

$$= S + \int d^d x \ \partial_\mu \left(w_a \frac{\delta x^\mu}{\delta w_a}\right) \mathcal{L} + \int d^d x \ \frac{\partial \mathcal{L}}{\partial(\phi(x))} w_a \frac{\delta F}{\delta w_a}$$

$$+ \int d^d x \ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\left(\partial_\mu \left(w_a \frac{\delta F}{\delta w_a}\right) - \partial_\mu (\delta x^\nu)\, \partial_\nu \phi(x)\right)$$

$$= S + \int d^d x \left(\frac{\partial \mathcal{L}}{\partial(\phi(x))} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right)\right) w_a \frac{\delta F}{\delta w_a} + \int d^d x \ \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} w_a \frac{\delta F}{\delta w_a}\right)$$

$$+\int d^d x\ \partial_\mu \left(W_a \frac{\delta X^\nu}{\delta W_a}\right)\left(\delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\partial_\nu \phi(x)\right)$$

$$\Delta S = \int d^d x \left(\frac{\partial \mathcal{L}}{\partial(\phi(x))} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right)\right) W_a \frac{\delta F}{\delta W_a} + \int d^d x\ \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} W_a \frac{\delta F}{\delta W_a}\right)$$

$$+\int d^d x\ \partial_\mu \left\{\left(W_a \frac{\delta X^\nu}{\delta W_a}\right)\left(\delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\partial_\nu \phi(x)\right)\right\}$$

$$-\int d^d x\ W_a \frac{\delta X^\nu}{\delta W_a}\left(\frac{\partial \mathcal{L}}{\partial \phi}\partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu \partial_\nu \phi - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\partial_\nu \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\partial_\mu \partial_\nu \phi(x)\right)$$

$$= \int d^d x \left(\frac{\partial \mathcal{L}}{\partial(\phi(x))} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right)\right) W_a \frac{\delta F}{\delta W_a} + \int d^d x\ \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} W_a \frac{\delta F}{\delta W_a}\right)$$

$$+\int d^d x\ \partial_\mu \left\{\left(W_a \frac{\delta X^\nu}{\delta W_a}\right)\left(\delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\partial_\nu \phi(x)\right)\right\}$$

$$-\int d^d x\ W_a \frac{\delta X^\nu}{\delta W_a}\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right)\partial_\nu \phi$$

$$= \int d^d x\ \partial_\mu j^\mu \qquad j^a_\mu \quad \delta x^\nu \quad \delta \phi$$

$$j^\mu = \left(W_a \frac{\delta X^\nu}{\delta W_a}\right)\left(\delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\partial_\nu \phi(x)\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} W_a \frac{\delta F}{\delta W_a}$$

$$= W_a\, j^\mu_a(x)$$

Question，可否认为 $\partial_\mu j^\mu = 0$， $\partial_\mu j^\mu_a = 0$

—— conserved current.

$$S' - S = \int d^d x\, \partial_\mu (W_a j^\mu_a(x))$$

$$= \int d^d x\ (\partial_\mu W_a) j^\mu_a(x) + \int d^d x\ W_a\, \partial_\mu j^\mu_a(x)$$

contains part with no derivative of W $\Rightarrow$ rigid transformation.

if S is invariant under rigid transformation

$$\Delta S = \int d^d x\ (\partial_\mu W_a)\, j^\mu_a(x)$$

$$= \boxed{-} \int d^d x\ W_a\, \partial_\mu j^\mu_a(x) \qquad \text{注意这里有负号!}$$

if field satisfies EOM $\Rightarrow$ S is minimal $\Rightarrow$ $\Delta S = 0$ for all $W_a(x)$!

$$\partial_\mu j^\mu_a = 0.$$

- Transformation of coorlation function

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{Z}\int D\phi\ \phi(x_1)\cdots\phi(x_n)\, e^{-S[\phi]}$$

$$\langle \phi(x_1')\cdots\phi(x_n')\rangle = \frac{1}{Z}\int D\phi'\ \phi'(x_1')\cdots\phi'(x_n')\, e^{-S[\phi']}$$

$$= \frac{1}{Z}\int D\phi\ F[\phi(x_1)]\cdots F[\phi(x_n)]\, e^{-S[\phi]}$$

invariance of action & invariance of integrand.

$$= \langle F[\phi(x_1)]\cdots F[\phi(x_n)]\rangle$$

- Ward identity.

$$\langle \phi(x_1)\,\phi(x_2)\cdots\phi(x_n)\rangle = \frac{1}{Z}\int D\phi\ \phi(x_1)\cdots\phi(x_n)\, e^{-S[\phi]}$$

$$= \frac{1}{Z}\int D\phi'\ \phi'(x_1)\cdots\phi'(x_n)\, e^{-S[\phi']}$$

$$= \frac{1}{Z}\int D\phi\ \phi'(x_1)\cdots\phi'(x_n)\, e^{-S[\phi]-\int d^dx\,\partial_\mu(j^\mu_a w_a)}$$

$$= \frac{1}{Z}\int D\phi\,\big(\phi(x_1)-i w_a G_a\phi(x_1)\big)\cdots\big(\phi(x_n)-i w_a G_a\phi(x_n)\big)\, e^{-S[\phi]}\quad \Big(1-\int d^dx\,\partial_\mu(j^\mu_a w_a)\Big)$$

$$= -\int d^dx\,\partial_\mu\langle j^\mu_a(x)\,\phi(x_1)\cdots\phi(x_n)\rangle w_a$$

$$- i\sum_{i=1}^{n}\langle\phi(x_1)\cdots G_a\phi(x_i)\cdots\phi(x_n)\rangle w_a$$

$$+\langle\phi(x_1)\cdots\phi(x_n)\rangle$$

$$\int d^dx\,\partial_\mu\langle j^\mu_a(x)\,\phi(x_1)\cdots\phi(x_n)\rangle w_a = -i\sum_{i=1}^{n}\langle\phi(x_1)\cdots G_a\phi(x_i)\cdots\phi(x_n)\rangle w_a$$

$$= -i\int d^dx\ w_a\sum_{i=1}^{n}\langle\phi(x_1)\cdots G_a\phi(x_i)\cdots\phi(x_n)\rangle\,\delta(x-x_i)$$

Question $\partial_\mu j^\mu(x)=0$ 吗? 那这个式子说什么意了?

$$\partial_\mu\langle j^\mu_a(x)\,\phi(x_1)\cdots\phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle\phi(x_1)\cdots G_a\phi(x_i)\cdots\phi(x_n)\rangle$$

From ward identity obtain commutation relation of conserved charge and field.

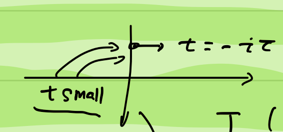
Ward identity

$$\partial_\mu\langle j^\mu_a(x)\,\phi(x_1)\cdots\phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle\phi(x_1)\cdots G_a\phi(x_i)\cdots\phi(x_n)\rangle$$

integrate from $x^0=x_1^0-$ to $x^0=x_1^0+$ $\qquad Q_a(t)=\int d^{d-1}x\ j^0(x)$

Suppose all other times x_i^0 are greater than x_1^0

Wick rotation, T(big t ... small t)



$t=-i\tau \quad \tau\to-\infty$

T (big τ ... small τ)

$\tau\to+\infty.$

$$\int_{x_1^0-}^{x_1^0+}\int d^{d-1}x\ \partial_\mu\langle j^\mu_a(x)\,\phi(x_1)\,\phi(x_2)\cdots\phi(x_n)\rangle = \langle Q_a(t_+)\,\phi(x_1)\,Y\rangle - \langle\phi(x_1)\,Q_a(t_-)\,Y\rangle$$

$$= -i\,\langle G_a\phi(x_1)\ \phi(x_2)\cdots\rangle$$

$$\langle 0|\,[Q_a,\phi(x_1)]\,Y\,|0\rangle = -i\,\langle 0|\,G_a\phi(x_1)\,Y\,|0\rangle$$

$$[Q_a,\phi(x_1)] = -iG_a\,\phi(x_1)$$

Wick rotation 前后守恒荷定义的区别。

$$\frac{dQ}{d\tau} = \frac{dQ}{i\,dt} = \frac{d(-iQ)}{dt}$$

Quantum Statistics

- Density operator and its property.

$$\rho \equiv \exp(-\beta H)$$

—— Partition function

$$Z = \sum_n \langle n| e^{-\beta H} |n\rangle = Tr(\rho)$$

—— Expectation of operator A

$$\langle A\rangle = \frac{\sum_n \langle n|e^{-\beta H} A|n\rangle}{\sum_n \langle n| e^{-\beta H}|n\rangle} = \frac{Tr(\rho A)}{Tr(\rho)}$$

- Use path integral evaluate partition function and expectation value

—— partition function evaluated by density operator in coordinate basis ($d-1$ dim)

$$\begin{aligned}
Z &= \sum_n \langle n| e^{-\beta H} |n\rangle \\
&= \sum_n \langle n|n\rangle \langle n|e^{-\beta H}|n\rangle \\
&= \int d^{d-1}x \sum_n \langle n|\vec{x}\rangle\langle \vec{x}|n\rangle \quad \langle n|e^{-\beta H}|n\rangle \\
&= \int d^{d-1}x \sum_n \langle \vec{x}|n\rangle\langle n| e^{-\beta H} |n\rangle\langle n|\vec{x}\rangle \\
&= \int d^{d-1}x \sum_{n,m} \langle \vec{x}|n\rangle\langle n| e^{-\beta H} |m\rangle\langle m|\vec{x}\rangle \\
&= \int d^{d-1}x \langle \vec{x}| e^{-\beta H} |\vec{x}\rangle
\end{aligned}$$

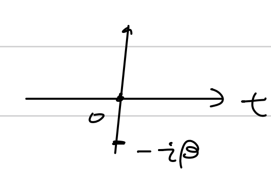
—— kernel of density operator evaluated by path integral.

Define kernel of density operator as

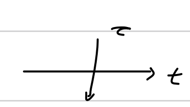
$$\begin{aligned}
\rho(\vec{x}_f, \vec{x}_i) &= \langle \vec{x}_f| e^{-\beta H} |\vec{x}_i\rangle \\
&= \langle \vec{x}_f| e^{-iH(-i\beta)} |\vec{x}_i\rangle
\end{aligned}$$

Compare with QM

$$\begin{aligned}
\rho(\vec{x}_f, \vec{x}_i) &= \int_{\vec{x}(0)=\vec{x}_i}^{\vec{x}(-i\beta)=\vec{x}_f} D\vec{x} \quad \exp\left(i\int_0^{-i\beta} dt\, L(\vec{x}, \dot{\vec{x}})\right) \\
&= \int_{\vec{x}(0)=\vec{x}_i}^{\vec{x}(-i\beta)=\vec{x}_f} D\vec{x}\ \exp(iS[\vec{x}])
\end{aligned}$$



Wick rotation, $iS_E[\vec{x}(\tau)] = S[\vec{x}(t)] \quad -i\tau = t$



$$\rho(\vec{x}_f, \vec{x}_i) = \int_{\vec{x}(0)=\vec{x}_i}^{\vec{x}(\beta)=\vec{x}_f} D\vec{x}\quad \exp(-S_E[\vec{x}])$$

—— Expectation of operator A.

$$\begin{aligned}
\langle A\rangle &= \frac{1}{Z}\int dx\ \langle x|\rho A|x\rangle \\
&= \frac{1}{Z}\int dx\, dy\ \langle x|\rho|y\rangle\langle y|A|x\rangle \\
&= \frac{1}{Z}\int dx\, dy \int_{(x,0)}^{(y,\beta)} Dx\ e^{-S_E[x(\tau)]}\ A(x)\delta(y-x) \\
&= \frac{1}{Z}\int dx \int_{(x,0)}^{(x,\beta)} Dx\ e^{-S_E[x(\tau)]}\ A(x(0))
\end{aligned}$$

$$= \frac{1}{Z} \int_{(x,0)=(x,\beta)} \mathcal{D}x \; e^{-S_E[x(\tau)]} A(x(0))$$

partition function

$$Z = \int dx \langle x | \rho | x \rangle$$

$$= \int_{(x,0)=(x,\beta)} \mathcal{D}x \; e^{-S_E[x(\tau)]} A(x(0))$$

Conformal group

- a Conformal transformation of coordinate is an invertible mapping $x \to x'$ which leaves the metric tensor invariant up to a scale. (g 和 η 是一个意思，但以后用 η 表示 Euclidian, g 表示 Minkowski

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x)$$

—— coordinate transformation generates metric transformation.

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)$$

Length under different coordinate system.

$$g'_{\mu\nu} dx'^{\mu} dx'^{\nu} = dx^{\alpha} dx^{\beta} g_{\alpha\beta}$$

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$$

$$dx'^{\mu} = dx^{\mu} + \partial_\nu \varepsilon^{\mu} dx^{\nu}$$

$$dx^{\alpha} = dx'^{\alpha} - \partial_\nu \varepsilon^{\alpha} dx^{\nu}$$

$$\simeq dx'^{\alpha} - \partial_\mu \varepsilon^{\alpha} dx'^{\mu}$$

$$= \delta^{\alpha}_{\mu} dx'^{\mu} - \partial_\mu \varepsilon^{\alpha} dx'^{\mu}$$

$$g'_{\mu\nu} = (\delta^{\alpha}_{\mu} - \partial_\mu \varepsilon^{\alpha})(\delta^{\beta}_{\nu} - \partial_\nu \varepsilon^{\beta}) g_{\alpha\beta}$$

$$= g_{\mu\nu} - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)$$

—— Conformal invariance

$$g'_{\mu\nu} = \Lambda g_{\mu\nu}$$

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu \equiv f(x) g_{\mu\nu}$$

cartesian metric

$$g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(+,+,\cdots,+)$$

Contract μ, ν.

$$2\partial_\mu \varepsilon^{\mu} = f(x) \cdot (d)$$

—— Further simplification ($g_{\mu\nu} = \mathrm{diag}(+,+,\cdots)$)

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x) g_{\mu\nu}$$

$$\partial_\rho \partial_\mu \varepsilon_\nu + \partial_\nu \partial_\rho \varepsilon_\mu = \partial_\rho f(x) g_{\mu\nu} \quad -(1)$$

$\mu \leftrightarrow \rho$
$$\partial_\rho \partial_\mu \varepsilon_\nu + \partial_\nu \partial_\mu \varepsilon_\rho = \partial_\mu f(x) g_{\rho\nu} \quad -(2)$$

$\nu \leftrightarrow \rho$
$$\partial_\mu \partial_\nu \varepsilon_\rho + \partial_\nu \partial_\rho \varepsilon_\mu = \partial_\nu f(x) g_{\mu\rho} \quad -(3)$$

linear combination equation $(3)+(2)-(1)$

$$2\partial_\mu \partial_\nu \varepsilon_\rho = \partial_\mu f(x) g_{\rho\nu} + \partial_\nu f(x) g_{\mu\rho} - \partial_\rho f(x) g_{\mu\nu} \quad -(0)$$

Contract μ, ν

$$2\partial^2 \varepsilon_\rho = 2\partial_\rho f(x) - d\,\partial_\rho f(x)$$

$$2\partial^2 \varepsilon_\mu = (2-d)\partial_\mu f$$

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x) g_{\mu\nu}$$

$$\Downarrow \partial^2$$

$$\partial^2 \partial_\mu \varepsilon_\nu + \partial^2 \partial_\nu \varepsilon_\mu = \partial^2 f(x) g_{\mu\nu} \quad -(1)$$

Apply ∂_ν to above equation

$$2\partial_\nu \partial^2 \varepsilon_\mu = (2-d)\partial_\nu \partial_\mu f \quad -(2)$$

(1) − (2)

$$\partial_\mu \partial^2 \varepsilon_\nu - \partial_\nu \partial^2 \varepsilon_\mu = \partial^2 f(x) g_{\mu\nu} - (2-d)\partial_\nu \partial_\mu f \Rightarrow$$

left: Anti-symmetric, Right symmetric
LHS = RHS = 0

$g_{\mu\nu}\partial^2 f = (2-d)\partial_\mu\partial_\nu f.$ — (3)

Contract with μ, ν

$$0 = d\,\partial^2 f - (2-d)\partial^2 f$$

$$(d-1)\partial^2 f = 0 \qquad -(4)$$

° Solution of coordinate transformation.

—— Constraint on f. (for $d \geq 3$)

(3): $\partial^2 f = 0$

(4): $g_{\mu\nu}\partial^2 f = (2-d)\partial_\mu\partial_\nu f$

$$\partial_\mu \partial_\nu f = 0$$

$$f = A + B_\mu x^\mu$$

—— Solution of coordinate transformation

From above equation (0).

$$2\partial_\mu\partial_\nu \varepsilon_\rho = \partial_\mu f(x) g_{\rho\nu} + \partial_\nu f(x) g_{\mu\rho} - \partial_\rho f(x) g_{\mu\nu}$$

$$2\partial_\mu\partial_\nu \varepsilon_\rho = B_\mu g_{\rho\nu} + B_\nu g_{\mu\rho} - B_\rho g_{\mu\nu} = \text{const}$$

$$\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + C_{\mu\nu\rho} x^\nu x^\rho \qquad C_{\mu\nu\rho} = C_{\mu\rho\nu}$$

—— Constraint on coordinate transformation

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x) g_{\mu\nu} \qquad (1)$$

$$2\partial_\mu \varepsilon^\mu = f(x)\cdot(d) \qquad (2)$$

$$2\partial_\mu\partial_\nu \varepsilon_\rho = \partial_\mu f(x) g_{\rho\nu} + \partial_\nu f(x) g_{\mu\rho} - \partial_\rho f(x) g_{\mu\nu} \qquad (3)$$

1° No constraint on 0th order a_μ (Transformation)

2° From first 2 equation, constraint on $b_{\mu\nu}$.

$$b_{\nu\mu} + b_{\mu\nu} = f(x) g_{\mu\nu} = \frac{2}{d} b^\alpha{}_\alpha\, g_{\mu\nu}$$

$b_{\mu\nu}$ is pure anti-symmetric part and trace

$$b_{\mu\nu} = \alpha\, g_{\mu\nu} + m_{\mu\nu} \qquad m_{\mu\nu} = -m_{\nu\mu}$$

m represents rigid rotation, α reps infinitesimal scale trans.

3° from (1), (3) equation,

$$4C_{\rho\mu\nu} = \partial_\mu(\partial_\rho \varepsilon_\nu + \partial_\nu \varepsilon_\rho) + \partial_\nu(\partial_\mu \varepsilon_\rho + \partial_\rho \varepsilon_\mu) - \partial_\rho(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)$$

$$2C_{\rho\mu\nu} = 2C_{\rho\mu\nu}$$

是恒等式.

4° from (2) (3).

$$4C_{\rho\mu\nu} = g_{\rho\nu} b_\mu + g_{\mu\rho} b_\nu - g_{\mu\nu} b_\rho$$

$$b_\rho \equiv \frac{2}{d}\partial_\rho \partial_\mu \varepsilon^\mu$$

$$= \frac{4}{d} C^\lambda{}_{\lambda\rho}$$

corresponding infinitesimal transformation.

$$x'^{\mu} = x^{\mu} + 2(x \cdot b)x^{\mu} - b^{\mu}x^2$$

called special conformal transformation (SCT)

—— Finite transformation

transformation $x'^{\mu} = x^{\mu} + a^{\mu}$

dialation $x'^{\mu} = \alpha x^{\mu}$

rotation $x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$

SCT $x'^{\mu} = \dfrac{x^{\mu} - b^{\mu}x^2}{1 - 2b\cdot x + b^2x^2}$

—— relation between SCT and translation

$$x'^2 = (x - bx^2)(x - bx^2)\frac{1}{(1-2b\cdot x + b^2x^2)^2}$$

$$= (x^2 - 2(b\cdot x)\cdot x^2 + b^2x^2\cdot x^2)\frac{1}{(1-2b\cdot x + b^2x^2)^2}$$

$$= x^2(1 - 2b\cdot x + b^2x^2)^{-1}$$

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu} - b^{\mu}x^2}{x^2} = \frac{x^{\mu}}{x^2} - b^{\mu}$$

which looks like translation.

—— transformation operator with respect to functions

$$\phi'(x') = \phi(x)$$

$$\phi'(T(x)) = \phi(x)$$

$$\phi'(x) = \phi(T^{-1}(x))$$

1° translation.

$$T^{-1}(x) \approx x^{\mu} - a^{\mu}$$

$$\phi'(x) = \phi(x - a)$$

$$= \phi(x) - a^{\mu}\partial_{\mu}\phi$$

$$= (1 - i a^{\mu}(-i)\partial_{\mu})\phi$$

$$= (1 - i a^{\mu}P_{\mu})\phi \qquad P_{\mu} \equiv -i\partial_{\mu}$$

2° dialation.

$$T^{-1}(x) \doteq x^{\mu} - \alpha x^{\mu}$$

$$\phi'(x) = \phi(x - \alpha x)$$

$$\simeq (1 - i\alpha\, x^{\mu}(-i)\partial_{\mu})\phi(x)$$

$$D \equiv -i x^{\mu}\partial_{\mu}$$

3° rotation

$$x'_{\mu} = x_{\mu} + m_{\mu\nu}x^{\nu}$$

$$T^{-1}(x) = x^\mu - m^{\mu\nu} x_\nu$$

$$\begin{aligned}
\phi'(x) &= \phi(T^{-1}(x)) \\
&= \phi(x^\mu - m^{\mu\nu} x_\nu) \\
&= \phi(x) - m^{\mu\nu} x_\nu \partial_\mu \phi \quad (m \text{ is anti-symmetric}) \\
&= \phi(x) - \tfrac{1}{2} m^{\mu\nu}(x_\nu \partial_\mu - x_\mu \partial_\nu)\phi \\
&= (1 + \tfrac{1}{2} m^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu))\phi \\
&= (1 - i \tfrac{m^{\mu\nu}}{2} i (x_\mu \partial_\nu - x_\nu \partial_\mu))\phi
\end{aligned}$$

$$L_{\mu\nu} \equiv i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

4° SCT

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$$

$$\approx x^\mu - b^\mu x^2 + 2 b\cdot x\, x^\mu$$

$$T^{-1}(x) = x^\mu + b^\mu x^2 - 2 b\cdot x\, x^\mu$$

$$\begin{aligned}
\phi'(x) &= \phi(x^\mu + b^\mu x^2 - 2b\cdot x\, x^\mu) \\
&= (1 + (x^2 b^\mu - 2 b\cdot x\, x^\mu)\partial_\mu)\phi \\
&= (1 - i b^\mu\, i(x^2 \partial_\mu - 2 x_\mu x^\nu \partial_\nu))\phi
\end{aligned}$$

$$K_\mu = -i(2 x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$$

- anharmonic ratio

SCT effects on distance of two points.

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$$

$$|x'_i - x'_j| = \left| \frac{x_i - b x_i^2}{1 - 2b\cdot x_i + b^2 x_i^2} - \frac{x_j - b x_j^2}{1 - 2b\cdot x_j + b^2 x_j^2} \right|$$

$$= \left| \frac{(x_i - b x_i^2)(1 - 2b\cdot x_j + b^2 x_j^2) - (x_j - b x_j^2)(1 - 2b\cdot x_i + b^2 x_i^2)}{(1 - 2b\cdot x_i + b^2 x_i^2)(1 - 2b\cdot x_j + b^2 x_j^2)} \right|$$

$$= \left| \frac{\begin{aligned} x_i &- 2b\cdot x_j\, x_i + b^2 x_j^2 x_i - b x_i^2 + 2b\cdot x_j\, x_i^2 b - b^2 x_j^2 x_i^2 b \\ - x_j &+ 2b\cdot x_i\, x_j - b^2 x_i^2 x_j + b x_j^2 - 2b\cdot x_i\, x_j^2 b + b^2 x_j^2 x_i^2 b \end{aligned}}{(\cdots)} \right|$$

$$= \sqrt{\frac{\qquad\qquad}{(\cdots)^2}}$$

$$= \sqrt{\left(\frac{x_i - b x_i^2}{1 - 2b\cdot x_i + b^2 x_i^2} - \frac{x_j - b x_j^2}{1 - 2b\cdot x_j + b^2 x_j^2} \right)^2}$$

$$= \sqrt{\frac{(x_i - b x_i^2)^2}{(1 - 2b\cdot x_i + b^2 x_i^2)^2} + \frac{(x_j - b x_j^2)^2}{(1 - 2b\cdot x_j + b^2 x_j^2)^2} - \frac{2(x_i - b x_i^2)(x_j - b x_j^2)}{(1 - 2b\cdot x_i + b^2 x_i^2)(1 - 2b\cdot x_j + b^2 x_j^2)}}$$

$$= \sqrt{\frac{x_i^2 + b^2 x_i^4 - 2b\cdot x_i\, x_i^2}{(1-2b\cdot x_i + b^2 x_i^2)^2} + \frac{\cdots}{(1-2b\cdot x_j + b^2 x_j^2)^2} - \frac{2(x_i - b x_i^2)(x_j - b x_j^2)}{(1-2b\cdot x_i + b^2 x_i^2)(1-2b x_j + b^2 x_j^2)}}$$

$$= \sqrt{\frac{x_i^2(1-2b\cdot x_i + b^2 x_i^2)}{(1-2b\cdot x_i + b^2 x_i^2)^2} + \frac{\cdots}{(1-2b\cdot x_j + b^2 x_j^2)^2} - \frac{2(x_i - b x_i^2)(x_j - b x_j^2)}{(1-2b\cdot x_i + b^2 x_i^2)(1-2b x_j + b^2 x_j^2)}}$$

$$= \sqrt{\frac{x_i^2}{(1-2b x_i + b^2 x_i^2)} + \frac{x_j^2}{(1-2b x_j + b^2 x_j^2)} - \frac{2(x_i - b x_i^2)(x_j - b x_j^2)}{(1-2b\cdot x_i + b^2 x_i^2)(1-2b\cdot x_j + b^2 x_j^2)}}$$

$$= \frac{\sqrt{x_i^2(1-2b\cdot x_j + b^2 x_j^2) + x_j^2(1-2b\cdot x_i + b^2 x_i^2) - 2x_i\cdot x_j + 2x_i\cdot b\, x_j^2 + 2x_j\cdot b\, x_i^2 - b^2 x_i^2 x_j^2}}{\sqrt{(1-2b\cdot x_i + b^2 x_i^2)(1-2b x_j + b^2 x_j^2)}}$$

$$= \frac{\sqrt{x_i^2(1-2b\cdot x_j + b^2 x_j^2) + x_j^2(1-2b\cdot x_i + b^2 x_i^2) - 2x_i\cdot x_j + 2x_i\cdot b\, x_j^2 + 2x_j\cdot b\, x_i^2 - 2b^2 x_i^2 x_j^2}}{\sqrt{(1-2b\cdot x_i + b^2 x_i^2)(1-2b x_j + b^2 x_j^2)}}$$

$$= \frac{\sqrt{x_i^2 + x_j^2 - 2x_i\cdot x_j}}{\sqrt{(1-2b\cdot x_i + b^2 x_i^2)(1-2b x_j + b^2 x_j^2)}} = \frac{|x_i - x_j|}{(1-2b\cdot x_i + b^2 x_i^2)^{\frac{1}{2}}(1-2b\cdot x_j + b^2 x_j^2)^{1/2}}$$

—— Simplest conformal invariance

$$\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \qquad \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|}$$

- Representation of conformal group

—— commutation relations.

generators

translations $P_\mu = -i\partial_\mu$

dialation $D = -ix^\mu \partial_\mu$

rotation $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

SCT $K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

$[D, P_\mu] = iP_\mu$ $\qquad [D, L_{\mu\nu}] = 0$

$[D, K_\mu] = -iK_\mu$

$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu})$

$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu)$

$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu)$

$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho})$

denote representation of this algebra as $\hat{D}$, $\hat{P}_\mu$, $\hat{K}_\mu$, $\hat{L}_{\mu\nu}$

Exists a subalgebra of conformal algebra $\hat{D}$, $\hat{K}_\mu$, $\hat{L}_{\mu\nu}$

Commutation relation of subalgebra / representation of Subalgebra

$$[\hat{D}, \hat{L}_{\mu\nu}] = 0$$

$$[\hat{D}, \hat{K}_\mu] = -iK_\mu$$

$$[\hat{K}_\rho, \hat{L}_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu)$$

$$[\hat{L}_{\mu\nu}, \hat{L}_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho})$$

transformation of field by representation of generator

$$\Phi'(x) = (1 - i\omega_a \hat{T}_a)\Phi(x)$$

Suppose

$$\Phi'(x') = (1 - \tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu})\Phi(x)$$

$$\hat{L}_{\mu\nu}\Phi = (S_{\mu\nu} + L_{\mu\nu})\Phi$$

Noticed For field at 0 point,

$$\hat{L}_{\mu\nu}\Phi_0(0) = S_{\mu\nu}\Phi_0(0)$$

Suppose

$$\hat{L}_{\mu\nu}\Phi_0(0) = S_{\mu\nu}\Phi_0(0)$$

$$\hat{K}_\mu \Phi_0(0) = k_\mu \Phi_0(0)$$

$$D\Phi_0(0) = \tilde{\Delta}\Phi_0(0)$$

Commutation of generator requires.

$$[\tilde{\Delta}, S_{\mu\nu}] = 0$$

$$[\tilde{\Delta}, k_\mu] = -ik_\mu$$

$$[k_\rho, S_{\mu\nu}] = i(\eta_{\rho\mu}k_\nu - \eta_{\rho\nu}k_\mu)$$

$$[S_{\mu\nu}, S_{\rho\sigma}] = i(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho})$$

Generator act on fields at nonzero point

$$\hat{L}_{\mu\nu}\Phi_0(x) = e^{i\hat{P}_\mu x^\mu}\hat{L}_{\mu\nu}\Phi_0(0)$$

$$= \{e^{i\hat{P}_\mu x^\mu}\hat{L}_{\mu\nu}e^{-i\hat{P}_\mu x^\mu}\}e^{i\hat{P}_\mu x^\mu}\Phi_0(0)$$

Define field Φ' as

$$\Phi'_0(x') \equiv e^{i\hat{P}_\mu x^\mu}\Phi_0(x')$$

Action of generator $\hat{L}_{\mu\nu}$ on field Φ can be represent as

$$\hat{L}_{\mu\nu}\Phi_0(x) = \{e^{i\hat{P}\cdot x}\hat{L}_{\mu\nu}e^{-i\hat{P}\cdot x}\}\Phi'_0(0)$$

Baker - Hausdroff equation

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \cdots$$

$$e^{i\hat{P}\cdot x}\hat{L}_{\mu\nu}e^{-i\hat{P}\cdot x} \approx \hat{L}_{\mu\nu} + ix^\rho[\hat{P}_\rho, L_{\mu\nu}]\cdots$$

Noticed

$$[\hat{P}_\rho, \hat{L}_{\mu\nu}] = i(\eta_{\rho\mu}\hat{P}_\nu - \eta_{\rho\nu}\hat{P}_\mu)$$

$$e^{i\hat{P}\cdot x}\hat{L}_{\mu\nu}e^{-i\hat{P}\cdot x} \simeq \hat{L}_{\mu\nu} + ix^\rho i(\eta_{\rho\mu}\hat{P}_\nu - \eta_{\rho\nu}\hat{P}_\mu)$$
$$= \hat{L}_{\mu\nu} - x_\mu\hat{P}_\nu + x_\nu\hat{P}_\mu$$

$$\hat{L}_{\mu\nu}\Phi_0(x) = \{e^{i\hat{P}\cdot x}\hat{L}_{\mu\nu}e^{-i\hat{P}\cdot x}\}\Phi'_0(0)$$
$$= \{\hat{L}_{\mu\nu} - x_\mu\hat{P}_\nu + x_\nu\hat{P}_\mu\}\Phi'_0(0)$$
$$= (S_{\mu\nu} - x_\mu\hat{P}_\nu + x_\nu\hat{P}_\mu)\Phi'_0(0)$$
$$= (S_{\mu\nu} - x_\mu\hat{P}_\nu + x_\nu\hat{P}_\mu)e^{i\hat{P}_\mu x^\mu}\Phi_0(0)$$

$$= e^{i\hat{P}_\mu x^\mu}(S_{\mu\nu} - x_\mu\hat{P}_\nu + x_\nu\hat{P}_\mu)\Phi_0(0)$$

$$= (S_{\mu\nu} - x_\mu\hat{P}_\nu + x_\nu\hat{P}_\mu)\Phi_0(x)$$
$$\hat{L}_{\mu\nu} = S_{\mu\nu} - x_\mu\hat{P}_\nu + x_\nu\hat{P}_\mu$$

Similarily, Dialation operator

$$\hat{D} = \tilde{\Delta} + ix^\rho[\hat{P}_\rho, \hat{D}]$$
$$= \tilde{\Delta} + ix^\rho(-i\hat{P}_\rho)$$
$$= \tilde{\Delta} + x^\rho\hat{P}_\rho$$

SCT operator.

$$\hat{K}_\mu\Phi_0(x) = e^{ix\cdot\hat{P}}\hat{K}_\mu\Phi_0(0)$$
$$= e^{ix\cdot\hat{P}}\hat{K}_\mu e^{-ix\cdot\hat{P}}e^{ix\cdot\hat{P}}\Phi_0(0)$$

Baker Housdroff formula.

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2!}[[B,A],A] + \cdots$$

As for $e^{ix\cdot\hat{P}}\hat{K}_\mu e^{-ix\cdot\hat{P}}$

$$[\hat{K}_\mu, \hat{P}_\nu] = 2i(\eta_{\mu\nu}\hat{D} - \hat{L}_{\mu\nu})$$
$$[B, A] = -ix^\nu[\hat{K}_\mu, \hat{P}_\nu] = -ix^\nu 2i(\eta_{\mu\nu}\hat{D} - \hat{L}_{\mu\nu})$$
$$= 2x_\mu\hat{D} - 2x^\nu\hat{L}_{\mu\nu}$$

$$\frac{1}{2!}[[B,A],A] = \frac{1}{2!}[2x_\mu\hat{D} - 2x^\nu\hat{L}_{\mu\nu}, -ix^\rho\hat{P}_\rho]$$
$$= -ix^\rho x_\mu[\hat{D}, \hat{P}_\rho] + ix^\rho x^\nu[\hat{L}_{\mu\nu}, \hat{P}_\rho]$$

$$[\hat{P}_\rho, \hat{L}_{\mu\nu}] = i(\eta_{\rho\mu}\hat{P}_\nu - \eta_{\rho\nu}\hat{P}_\mu)$$
$$[\hat{P}_\rho, \hat{D}] = -i\hat{P}_\rho$$

$$\frac{1}{2!}[[B,A],A] = -ix^\rho x_\mu i\hat{P}_\rho + ix^\rho x^\nu(-i)(\eta_{\rho\mu}\hat{P}_\nu - \eta_{\rho\nu}\hat{P}_\mu)$$
$$= x^\rho x_\mu\hat{P}_\rho + x^\rho x^\nu(\eta_{\rho\mu}\hat{P}_\nu - \eta_{\rho\nu}\hat{P}_\mu)$$

$$e^{ix\cdot\hat{P}}\hat{K}_\mu e^{-ix\cdot\hat{P}} = \hat{K}_\mu + 2x_\mu\hat{D} - 2x^\nu\hat{L}_{\mu\nu} + x_\mu x^\rho\hat{P}_\rho + x_\mu x^\nu\hat{P}_\nu - x^2\hat{P}_\mu$$

$$\hat{K}_\mu\Phi_0(x) = (\hat{K}_\mu + 2x_\mu\hat{D} - 2x^\nu\hat{L}_{\mu\nu} + x_\mu x^\rho\hat{P}_\rho + x_\mu x^\nu\hat{P}_\nu - x^2\hat{P}_\mu)e^{ix\cdot\hat{P}}\Phi_0(0)$$

$$= e^{ix\cdot\hat{P}} (k_\mu + 2x_\mu \tilde{\Delta} - 2x^\nu S_{\mu\nu} + 2x_\mu x^\rho \hat{P}_\rho - x^2 \hat{P}_\mu) \Phi(0)$$
$$= (k_\mu + 2x_\mu \tilde{\Delta} - 2x^\nu S_{\mu\nu} + 2x_\mu x^\rho \hat{P}_\rho - x^2 \hat{P}_\mu) \Phi(x)$$

In all

$$\hat{K}_\mu = k_\mu + 2x_\mu \tilde{\Delta} - 2x^\nu S_{\mu\nu} + 2x_\mu x^\rho \hat{P}_\rho - x^2 \hat{P}_\mu$$
$$\hat{L}_{\mu\nu} = S_{\mu\nu} - x_\mu \hat{P}_\nu + x_\nu \hat{P}_\mu$$
$$\hat{D} = \tilde{\Delta} + x^\rho \hat{P}_\rho$$

$k=0$, $\tilde{\Delta} = \text{const}$

若 $S_{\mu\nu}$ 是 Lorentz 群不可约表示, 则 $[S_{\mu\nu}, \tilde{\Delta}] = 0 \Rightarrow \tilde{\Delta} = \text{const}$

由于 $[k, \tilde{\Delta}] = ik \Rightarrow k_\mu = 0$

0 Quasi-primary field. 准基本场.

conformal transformation for field

$$\exp(-i\alpha\hat{D} - ia^\mu \hat{P}_\mu - i\tfrac{1}{2}m^{\mu\nu}L_{\mu\nu} - ib_\mu \hat{K}^\mu)$$

$$= \exp(-i\alpha\tilde{\Delta} - i\tfrac{1}{2}m^{\mu\nu}S_{\mu\nu} - 2ib^\mu x_\mu \tilde{\Delta} + 2ib^\mu x^\nu S_{\mu\nu})$$

$\times$ terms with derivatives.

$$\Phi'(x') = \exp(-i\alpha\tilde{\Delta} - i\tfrac{1}{2}m^{\mu\nu}S_{\mu\nu} - 2ib^\mu x_\mu \tilde{\Delta} + 2ib^\mu x^\nu S_{\mu\nu}) \Phi(x)$$

for spinless field

$$\Phi'(x') = \exp(-i\alpha\tilde{\Delta} - 2ib^\mu x_\mu \tilde{\Delta}) \Phi(x)$$

denote $\tilde{\Delta} \equiv -i\Delta$

$$\Phi'(x') = \exp(-\alpha\Delta - 2b\cdot x\,\Delta) \Phi(x)$$

coordinate transformation

$$x'^\mu = x^\mu + \frac{a^\mu}{N} + \frac{\alpha}{N}x^\mu + \frac{m^{\mu\nu}}{N}x_\nu + (-b^\mu x^2 + 2b\cdot x\, x^\mu)\frac{1}{N}$$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \frac{\alpha}{N}\delta^\mu_\nu + \frac{m^\mu{}_\nu}{N} + (-b^\mu 2x_\nu + 2b_\nu x^\mu + 2b\cdot x\,\delta^\mu_\nu)\frac{1}{N}$$

$$\left|\frac{\partial x'}{\partial x}\right| = \det(\mathbb{I} + A) = 1 + \operatorname{tr}(A)$$
$$= 1 + \frac{\alpha}{N}d + (-2b\cdot x + 2b\cdot x + 2b\cdot x\cdot d)\frac{1}{N}$$

$$= 1 + \frac{\alpha}{N}d + 2b\cdot x\frac{1}{N}d$$

反复做 N 次变换 $\det(A^N) = \det^N(A)$

$$\left|\frac{\partial x'}{\partial x}\right| = \exp(\alpha d + 2b\cdot x\, d)$$

field transformation can be written as

$$\Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \Phi(x)$$

Fields transform like this are called quasi primary

Conserved current of conformal transformation

- translation $\Rightarrow$ Energy-momentum tensor

translation of coordinate and field.

$$x'^{\mu} = x^{\mu} + a^{\mu} = x^{\mu} + W_a \frac{\delta x^{\mu}}{\delta W_a} \qquad \phi' - \phi = (1 - i a^{\mu} P_{\mu}) \phi$$

$$\phi'(x') = \phi(x) = \phi(x) + W_a \frac{\delta F}{\delta W_a}(x) \qquad \Delta S = \int d^d x \; a_\nu \partial_\mu T^{\mu\nu}$$

conserved current

$$j^{\mu} = \left(W_a \frac{\delta x^{\nu}}{\delta W_a}\right)\left(\delta^{\mu}_{\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \partial_\nu \phi(x)\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} W_a \frac{\delta F}{\delta W_a}$$

$$= a^{\nu}\left(\delta^{\mu}_{\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \partial_\nu \phi(x)\right)$$

Define canonical energy-momentum tensor as

$$T_C^{\mu\nu} \equiv -\eta^{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \partial^{\nu} \phi$$

Satisfies conservation relation

$$\partial_\mu T_C^{\mu\nu} = 0$$

Belinfante tensor

Define modified Belinfante energy momentum tensor as

$$T_B^{\mu\nu} \equiv T_C^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} \qquad B^{\rho\mu\nu} = -B^{\mu\rho\nu}$$

Still satisfies conservation relation

$$\partial_\mu T_B^{\mu\nu} = \partial_\mu \partial_\rho B^{\rho\mu\nu} = 0$$

- rigid rotation $\rightarrow$ conserved current $j^{\mu\nu\rho}$

rigid rotation transformation of coordinate

$$x'^{\mu} = x^{\mu} + m^{\mu\nu} x_\nu = x^{\mu} + W_a \frac{\delta x^{\mu}}{\delta W_a} \qquad \phi' - \phi = \left\{1 - i \frac{m^{\mu\nu}}{2}\left[i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}\right]\right\}\phi$$

$$\Phi'(x') = \left(1 - \frac{i}{2} m_{\mu\nu} S^{\mu\nu}\right)\Phi(x) = \Phi(x) + W_a \frac{\delta F}{\delta W_a}(x) \qquad \Delta S = \int d^d x \, \frac{1}{2} m_{\nu\rho} \partial_\mu \left(j^{\mu\nu\rho}\right) \qquad j^{\mu\nu\rho} = T^{\mu\nu} x^{\rho} - T^{\mu\rho} x^{\nu}$$

conserved current

$$j^{\mu} = \left(W_a \frac{\delta x^{\nu}}{\delta W_a}\right)\left(\delta^{\mu}_{\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \partial_\nu \phi(x)\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} W_a \frac{\delta F}{\delta W_a}$$

$$= m^{\nu\rho} x_\rho \left(\delta^{\mu}_{\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \partial_\nu \phi(x)\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\left(-\frac{i}{2}\right) m_{\nu\rho} S^{\nu\rho} \Phi(x)$$

$$= m_{\nu\rho} x^{\rho}\left(\eta^{\mu\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \partial_\nu \phi(x)\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\left(-\frac{i}{2}\right) m_{\nu\rho} S^{\nu\rho} \Phi(x)$$

$$= -m_{\nu\rho} x^{\rho} T_C^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\left(-\frac{i}{2}\right) m_{\nu\rho} S^{\nu\rho} \Phi(x)$$

$$= -\frac{1}{2} m_{\nu\rho}\left(T_C^{\mu\nu} x^{\rho} - T_C^{\mu\rho} x^{\nu} + i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\nu\rho} \Phi(x)\right)$$

Define associated conserved current as

$$j^{\mu\nu\rho} = T_C^{\mu\nu} x^{\rho} - T_C^{\mu\rho} x^{\nu} + i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\nu\rho} \Phi(x)$$

Define belinfante form letting

$$j^{\mu\nu\rho} = T_B^{\mu\nu} x^\rho - T_B^{\mu\rho} x^\nu$$

$$B^{\mu\rho\nu} \equiv \frac{1}{2} i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\nu\rho} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} S^{\mu\nu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\mu\rho} \Phi \right\} \Leftarrow \text{Antisymmetric index } \mu \leftrightarrow \rho$$

$$T_B^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

$$+ \frac{1}{2} i \left\{ \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} S^{\nu\mu} \Phi \right) + \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\rho\nu} \Phi \right) + \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\rho\mu} \Phi \right) \right\}$$

Direct calculation

$$T_B^{\mu\nu} x^\rho - T_B^{\mu\rho} x^\nu = T_c^{\mu\nu} x^\rho - T_c^{\mu\rho} x^\nu$$

$$+ \frac{1}{2} i \left\{ \partial_\gamma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\gamma \phi)} S^{\nu\mu} \Phi \right) + \partial_\gamma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\gamma\nu} \Phi \right) + \partial_\gamma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\gamma\mu} \Phi \right) \right\} x^\rho$$

$$- \frac{1}{2} i \left\{ \partial_\gamma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\gamma \phi)} S^{\rho\mu} \Phi \right) + \partial_\gamma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\gamma\rho} \Phi \right) + \partial_\gamma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\gamma\mu} \Phi \right) \right\} x^\nu$$

Ignore divergence term

$$= T_c^{\mu\nu} x^\rho - T_c^{\mu\rho} x^\nu$$

$$- \frac{1}{2} i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\gamma \phi)} S^{\nu\mu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\gamma\nu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\gamma\mu} \Phi \right\} \partial_\gamma (x^\rho)$$

$$+ \frac{1}{2} i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\gamma \phi)} S^{\rho\mu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\gamma\rho} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\gamma\mu} \Phi \right\} \partial_\gamma (x^\nu)$$

$$= T_c^{\mu\nu} x^\rho - T_c^{\mu\rho} x^\nu$$

$$- \frac{1}{2} i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} S^{\nu\mu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\rho\nu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\rho\mu} \Phi \right\}$$

$$+ \frac{1}{2} i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} S^{\rho\mu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\nu\rho} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\nu\mu} \Phi \right\}$$

$$= T_c^{\mu\nu} x^\rho - T_c^{\mu\rho} x^\nu$$

$$- \frac{1}{2} i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} S^{\nu\mu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\rho\nu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\rho\mu} \Phi \right\}$$

$$+ \frac{1}{2} i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} S^{\rho\mu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\nu\rho} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\nu\mu} \Phi \right\}$$

$$= T_c^{\mu\nu} x^\rho - T_c^{\mu\rho} x^\nu + i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} S^{\nu\rho} \Phi$$

In all, conserved current of rotation can be represented as

$$j^{\mu\nu\rho} = T_B^{\mu\nu} x^\rho - T_B^{\mu\rho} x^\nu$$

—— Belinfante energy momentum tensor is symmetric

▽ 必要性 of Belinfante EM tensor to be symmetric

$$j^{\mu\nu\rho} = T_B^{\mu\nu} x^\rho - T_B^{\mu\rho} x^\nu$$

$$\partial_\mu T_B^{\mu\nu} = 0 \qquad \partial_\mu j^{\mu\nu\rho} = 0$$

$$T_B^{\rho\nu} - T_B^{\nu\rho} = 0$$

▽ 验证 T_B is symmetric

$$j^{\mu\nu\rho} = T_c^{\mu\nu} x^\rho - T_c^{\mu\rho} x^\nu + i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} S^{\nu\rho} \Phi(x)$$

$$\partial_\mu j^{\mu\nu\rho} = 0 \qquad \partial_\mu T_c^{\mu\nu} = 0$$

$$T_c^{\rho\nu} - T_c^{\nu\rho} + i\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} S^{\nu\rho} \Phi \right) = 0$$

$$T_c^{\rho\nu} - T_c^{\nu\rho} = -i\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} S^{\nu\rho} \Phi \right)$$

$$B^{\mu\rho\nu} \equiv \frac{1}{2} i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} S^{\nu\rho} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\mu\nu} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\mu\rho} \Phi \right\}$$

$$T_B^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}$$

$$\begin{aligned}
T_B^{\mu\nu} - T_B^{\nu\mu} &= T_c^{\mu\nu} - T_c^{\nu\mu} + \partial_\rho B^{\rho\mu\nu} - \partial_\rho B^{\rho\nu\mu} \\
&= -i\, \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\nu\mu} \Phi \right) \\
&\quad + \frac{1}{2} i \left\{ \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\nu\mu} \Phi \right) + \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} S^{\rho\nu} \Phi \right) + \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\rho\mu} \Phi \right) \right\} \\
&\quad - \frac{1}{2} i \left\{ \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\mu\nu} \Phi \right) + \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\rho\mu} \Phi \right) + \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} S^{\rho\nu} \Phi \right) \right\} \\
&= -i\, \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\nu\mu} \Phi \right) + i\, \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\nu\mu} \Phi \right) \\
&= 0
\end{aligned}$$

∘ conserved current of dialation.

dialation transformation

$$x'^{\mu} = (1+\alpha)x^{\mu}$$

$$\Phi'(x') = \exp\left(-i\alpha\tilde{\Delta} - i\frac{1}{2}m^{\mu\nu}S_{\mu\nu} - 2i\beta^{\mu}x_{\mu}\tilde{\Delta} + 2ib^{\mu}x^{\nu}S_{\mu\nu}\right)\Phi(x)$$

$$\phi'(x') = (1 - i\alpha(-i\Delta))\phi(x) \qquad \phi' - \phi = [1 - i\alpha(\tilde{\Delta} - ix^{\rho}\partial_{\rho})]\phi$$

$$= (1 - \alpha\Delta)\phi(x) \qquad \Delta S = \int d^dx\, \alpha\, \partial_{\mu}(j^{\mu}_D) \quad j^{\mu}_D = T^{\mu}{}_{\nu}x^{\nu}$$

conserved current of dialation

$$j^{\mu} = \left(w_a\frac{\delta x^{\nu}}{\delta w_a}\right)\left(\delta^{\mu}_{\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi(x))}\partial_{\nu}\phi(x)\right) + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi(x))}w_a\frac{\delta F}{\delta w_a}$$

$$= \alpha x^{\nu}\left(\delta^{\mu}_{\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi(x))}\partial_{\nu}\phi(x)\right) + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi(x))}(-\alpha\Delta)\phi(x)$$

$$= \alpha\cdot\left\{\mathcal{L}x^{\mu} - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi(x))}x^{\nu}\partial_{\nu}\phi - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\Delta\phi\right\}$$

$$j^{\mu}_D = \left(-\mathcal{L}\delta^{\mu}_{\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi(x))}\partial_{\nu}\phi\right)x^{\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\Delta\phi$$

$$= T_c{}^{\mu}{}_{\nu}x^{\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\Phi)}\Delta\Phi$$

define viral of the field Φ

$$V^{\mu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial^{\rho}\Phi)}(\eta^{\mu\rho}\Delta + iS^{\mu\rho})\Phi$$

assume viral is the divergence of another tensor

$$V^{\mu} = \partial_{\alpha}\sigma^{\alpha\mu}$$

$$\sigma_+^{\mu\nu} \equiv \frac{1}{2}(\sigma^{\mu\nu} + \sigma^{\nu\mu})$$

$$X^{\lambda\rho\mu\nu} \equiv \frac{2}{d-2}\left\{\eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\lambda\nu}\sigma_+^{\mu\rho} + \eta^{\mu\nu}\sigma_+^{\lambda\rho} - \frac{1}{d-1}(\eta^{\lambda\rho}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\rho\nu})\sigma_{+\alpha}^{\alpha}\right\}$$

Modified energy momentum Belinfante tensor

$$T^{\mu\nu} = T_c^{\mu\nu} + \partial_{\rho}B^{\rho\mu\nu} + \frac{1}{2}\partial_{\lambda}\partial_{\rho}X^{\lambda\rho\mu\nu}$$

$$X^{\lambda\mu\rho\nu} = \frac{2}{d-2}\left\{\eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\nu}\sigma_+^{\rho\mu} + \eta^{\rho\nu}\sigma_+^{\lambda\mu} - \frac{1}{d-1}(\eta^{\lambda\mu}\eta^{\rho\nu} - \eta^{\lambda\rho}\eta^{\mu\nu})\sigma_{+\alpha}^{\alpha}\right\}$$

$\partial_{\lambda}\partial_{\mu}(\eta^{\lambda\rho}\eta^{\mu\nu}\sigma_{+\alpha}^{\alpha}) = \partial^{\rho}\partial^{\nu}\sigma_{+\alpha}^{\alpha}$ Symmetric $\nu \leftrightarrow \rho \Rightarrow T^{\mu\nu} = T^{\nu\mu}$!

$$\sigma_+^{\mu\nu} = \sigma_+^{\nu\mu}$$

$$X^{\lambda\mu\rho\nu} = -X^{\lambda\rho\mu\nu} \quad (\text{Antisymmetric } \rho \leftrightarrow \mu)$$

Modified E-M tensor still conserved

$$\partial_\mu T^{\mu\nu} = \frac{1}{2}\partial_\lambda \partial_\rho \partial_\mu X^{\lambda\rho\mu\nu}$$

$$= -\frac{1}{2}\partial_\lambda \partial_\rho \partial_\mu X^{\lambda\mu\rho\nu}$$

$$= 0$$

Viral 的 divergent 写为

$$\partial_\mu V^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\rho\Phi)}(\eta^{\mu\rho}\Delta + i S^{\mu\rho})\Phi\right)$$

$$= \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\Phi)}\Phi\Delta\right) + i\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\rho\Phi)}S^{\mu\rho}\Phi\right)$$

$$= \partial_\mu\partial_\alpha \sigma^{\alpha\mu}$$

$$= \partial_\mu\partial_\alpha \frac{1}{2}(\sigma^{\alpha\mu} + \sigma^{\mu\alpha})$$

$$= \partial_\mu\partial_\alpha \sigma_+^{\alpha\mu}$$

Noticed trace of modified E-M tensor

$$T^\mu{}_\mu = T_c{}^\mu{}_\mu + \partial_\rho B^{\rho\mu}{}_\mu + \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu}{}_\mu$$

$$= T_c{}^\mu{}_\mu + \frac{1}{2}i\left\{\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}S^\mu{}_\mu\Phi\right) + \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}S^{\rho\mu}\Phi\right) + \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi)}S^{\rho\mu}\Phi\right)\right\}$$

$$+ \frac{1}{2}\partial_\lambda\partial_\rho\frac{2}{d-2}\left\{\eta^{\lambda\rho}\sigma_+{}^\mu{}_\mu - \eta^\lambda{}_\mu\sigma_+^{\rho\mu} - \eta^\lambda{}_\mu\sigma_+^{\mu\rho} + \eta^\mu{}_\mu\sigma_+^{\lambda\rho} - \frac{1}{d-1}(\eta^{\lambda\rho}\eta^\mu{}_\mu - \eta^\lambda{}_\mu\eta^{\rho\mu})\sigma_+{}^\alpha{}_\alpha\right\}$$

$$= T_c{}^\mu{}_\mu + i\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}S^{\rho\mu}\Phi\right)$$

$$+ \frac{1}{2}\frac{2}{d-2}\left\{\partial^2(\sigma_+{}^\mu{}_\mu) - \partial_\mu\partial_\rho\sigma_+^{\rho\mu} - \partial_\mu\partial_\rho\sigma_+^{\mu\rho} + d\,\partial_\lambda\partial_\rho\sigma_+^{\lambda\rho} - \frac{1}{d-1}(d\,\partial^2 - \partial^2)\sigma_+{}^\alpha{}_\alpha\right\}$$

$$= T_c{}^\mu{}_\mu + i\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}S^{\rho\mu}\Phi\right)$$

$$+ \frac{1}{2}\frac{2}{d-2}\left\{(d-2)\partial_\mu\partial_\rho\sigma_+^{\rho\mu}\right\}$$

$$= T_c{}^\mu{}_\mu + i\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}S^{\rho\mu}\phi\right) + \partial_\mu\partial_\rho(\sigma_+^{\rho\mu})$$

$$= T_c{}^\mu{}_\mu + i\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}S^{\rho\mu}\phi\right) + \partial_\mu V^\mu$$

$$= T_c{}^\mu{}_\mu + i\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}S^{\rho\mu}\phi\right) + \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\Phi)}\Phi\Delta\right) + i\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\rho\Phi)}S^{\mu\rho}\Phi\right)$$

$$= T_c{}^\mu{}_\mu + \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\Phi)}\Phi\Delta\right)$$

Noticed.

$$\partial_\mu j^\mu_D = \partial_\mu\left(T_c{}^\mu{}_\nu x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\Delta\Phi\right)$$

$$= T_c{}^\mu{}_\mu + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\Delta\Phi\right)$$

$$= T^\mu{}_\mu = 0$$

Means, modified EM tensor trace $= 0$.

—— Traceless relation means $j^\mu_D = T^\mu{}_\nu x^\nu$

—— Use modified E-M tensor represents rotation conserved current knowing that

$$j^{\mu\nu\rho} = T_B^{\mu\nu}x^\rho - T_B^{\mu\rho}x^\nu$$

noticed

$$T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu = T_B^{\mu\nu}x^\rho - T_B^{\mu\rho}x^\nu + \frac12 \partial_\lambda\partial_\gamma (X^{\lambda\gamma\mu\nu})x^\rho - \frac12\partial_\lambda\partial_\gamma(X^{\lambda\gamma\mu\rho})x^\nu$$

$$= T_B^{\mu\nu}x^\rho - T_B^{\mu\rho}x^\nu + \frac12\partial_\lambda[\partial_\gamma(X^{\lambda\gamma\mu\nu})x^\rho] - \frac12 \partial_\gamma(X^{\rho\gamma\mu\nu}) - \frac12\partial_\lambda[\partial_\gamma(X^{\lambda\gamma\mu\rho})x^\nu] + \frac12\partial_\gamma(X^{\nu\gamma\mu\rho})$$

$$= T_B^{\mu\nu}x^\rho - T_B^{\mu\rho}x^\nu + \frac12\partial_\lambda\partial_\gamma(X^{\lambda\gamma\mu\nu}x^\rho) - \frac12\partial_\gamma(X^{\rho\gamma\mu\nu}) - \frac12\partial_\lambda(X^{\lambda\rho\mu\nu}) - \frac12\partial_\lambda\partial_\gamma(X^{\lambda\gamma\mu\rho}x^\nu) + \frac12\partial_\gamma(X^{\nu\gamma\mu\rho}) + \frac12\partial_\lambda(X^{\lambda\nu\mu\rho})$$

Noticed

$$X^{\lambda\rho\mu\nu} = \frac{2}{d-2}\left\{\eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\lambda\nu}\sigma_+^{\mu\rho} + \eta^{\mu\nu}\sigma_+^{\lambda\rho} - \frac{1}{d-1}(\eta^{\lambda\rho}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\rho\nu})\sigma_+{}^\alpha{}_\alpha\right\}$$

$$\partial_\lambda(X^{\lambda\rho\mu\nu}) = \frac{2}{d-2}\left\{\partial^\rho\sigma_+^{\mu\nu}\ \boxed{-\partial^\mu\sigma_+^{\rho\nu}} - \partial^\nu\sigma_+^{\mu\rho} + \eta^{\mu\nu}\partial_\lambda\sigma_+^{\lambda\rho} - \frac{1}{d-1}(\eta^{\mu\nu}\partial^\rho - \eta^{\rho\nu}\partial^\mu)\sigma_+{}^\alpha{}_\alpha\right\}$$

$$X^{\rho\gamma\mu\nu} = \frac{2}{d-2}\left\{\eta^{\rho\gamma}\sigma_+^{\mu\nu} - \eta^{\rho\mu}\sigma_+^{\gamma\nu} - \eta^{\rho\nu}\sigma_+^{\mu\gamma} + \eta^{\mu\nu}\sigma_+^{\rho\gamma} - \frac{1}{d-1}(\eta^{\rho\gamma}\eta^{\mu\nu} - \eta^{\rho\mu}\eta^{\gamma\nu})\sigma_+{}^\alpha{}_\alpha\right\}$$

$$\partial_\gamma(X^{\rho\gamma\mu\nu}) = \frac{2}{d-2}\left\{\partial^\rho\sigma_+^{\mu\nu} - \eta^{\rho\mu}\partial_\gamma(\sigma_+^{\gamma\nu}) - \eta^{\rho\nu}\partial_\gamma(\sigma_+^{\mu\gamma}) + \eta^{\mu\nu}\partial_\gamma(\sigma_+^{\rho\gamma}) - \frac{1}{d-1}(\eta^{\mu\nu}\partial^\rho - \eta^{\rho\mu}\partial^\nu)\sigma_+{}^\alpha{}_\alpha\right\}$$

$$X^{\lambda\nu\mu\rho} = \frac{2}{d-2}\left\{\eta^{\lambda\nu}\sigma_+^{\mu\rho} - \eta^{\lambda\mu}\sigma_+^{\nu\rho} - \eta^{\lambda\rho}\sigma_+^{\mu\nu} + \eta^{\mu\rho}\sigma_+^{\lambda\nu} - \frac{1}{d-1}(\eta^{\lambda\nu}\eta^{\mu\rho} - \eta^{\lambda\mu}\eta^{\nu\rho})\sigma_+{}^\alpha{}_\alpha\right\}$$

$$\partial_\lambda(X^{\lambda\nu\mu\rho}) = \frac{2}{d-2}\left\{\partial^\nu\sigma_+^{\mu\rho}\ \boxed{-\partial^\mu\sigma_+^{\nu\rho}} - \partial^\rho\sigma_+^{\mu\nu} + \eta^{\mu\rho}\partial_\lambda\sigma_+^{\lambda\nu} - \frac{1}{d-1}(\eta^{\mu\rho}\partial^\nu - \eta^{\nu\rho}\partial^\mu)\sigma_+{}^\alpha{}_\alpha\right\}$$

$$X^{\nu\gamma\mu\rho} = \frac{2}{d-2}\left\{\eta^{\nu\gamma}\sigma_+^{\mu\rho} - \eta^{\nu\mu}\sigma_+^{\gamma\rho} - \eta^{\nu\rho}\sigma_+^{\mu\gamma} + \eta^{\mu\rho}\sigma_+^{\nu\gamma} - \frac{1}{d-1}(\eta^{\nu\gamma}\eta^{\mu\rho} - \eta^{\nu\mu}\eta^{\gamma\rho})\sigma_+{}^\alpha{}_\alpha\right\}$$

$$\partial_\gamma(X^{\nu\gamma\mu\rho}) = \frac{2}{d-2}\left\{\partial^\gamma\sigma_+^{\mu\rho} - \eta^{\nu\mu}\partial_\gamma(\sigma_+^{\gamma\rho}) - \eta^{\nu\rho}\partial_\gamma(\sigma_+^{\mu\gamma}) + \eta^{\mu\rho}\partial_\gamma(\sigma_+^{\nu\gamma}) - \frac{1}{d-1}(\eta^{\mu\rho}\partial^\nu - \eta^{\nu\mu}\partial^\rho)\sigma_+{}^\alpha{}_\alpha\right\}$$

为全导数项，对能动量张量无影响.

In all

$$j^{\mu\nu\rho} = T_B^{\mu\nu} x^\rho - T_B^{\mu\rho} x^\nu$$

$$\Rightarrow T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu$$

Correlator with conformal symmetry

- conformal invariance in quantum field theory

Two point correlation function

Quasi-primary field conformal transformation

$$\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x) = F(\phi(x))$$

two-point correlation function

$$\langle \phi(x_1') \phi(x_2') \rangle = \frac{1}{Z}\int D\phi\ \phi(x_1') \phi(x_2') e^{-S[\phi]}$$

$$= \frac{1}{Z}\int D\phi'\ \phi'(x_1') \phi'(x_2') e^{-S[\phi']}$$

$$= \left|\frac{\partial x_1'}{\partial x_1}\right|^{-\Delta_1/d} \left|\frac{\partial x_2'}{\partial x_2}\right|^{-\Delta_2/d} \frac{1}{Z}\int D\phi\ \phi_1(x_1) \phi_2(x_2) e^{-S[\phi]}$$

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\Delta_1/d} \left|\frac{\partial x_2'}{\partial x_2}\right|^{\Delta_2/d} \langle \phi_1(x_1') \phi_2(x_2') \rangle$$

- Choose conformal transformation be dialation.

$$x' = \lambda x$$

$$\langle \phi(x_1) \phi(x_2) \rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1) \phi_2(\lambda x_2) \rangle$$

rotation invariance

$$\langle \phi(x_1) \phi(x_2) \rangle = f(|x_1 - x_2|)$$

conformal invariance of correlation function implies

$$f(|x_1 - x_2|) = \lambda^{\Delta_1+\Delta_2} f(\lambda|x_1 - x_2|)$$

Set $\lambda = \dfrac{1}{|x_1 - x_2|}$

$$f(|x_1 - x_2|) = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

▽ Choose conformal transformation to be SCT.

$$|x_i' - x_j'| = \frac{|x_i - x_j|}{(1-2b\cdot x_i + b^2 x_i^2)^{\frac{1}{2}} (1-2b\cdot x_j + b^2 x_j^2)^{1/2}} \equiv \frac{|x_i - x_j|}{\gamma_1^{\frac{1}{2}} \gamma_2^{\frac{1}{2}}}$$

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1-2b\cdot x + b^2 x^2}$$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \frac{\delta^\mu_\nu - b^\mu 2x_\nu}{1-2b\cdot x + b^2 x^2} - \frac{x^\mu - b^\mu x^2}{(1-2b\cdot x + b^2 x^2)^2}(-2b_\nu + b^2\cdot 2x_\nu)$$

$$= \frac{(\delta^\mu_\nu - b^\mu 2x_\nu)(1-2b\cdot x + b^2 x^2) - (x^\mu - b^\mu x^2)(-2b_\nu + 2b^2 x_\nu)}{(1-2b\cdot x + b^2 x^2)^2}$$

$$= \frac{\delta^\mu_\nu (1-2b\cdot x + b^2 x^2) - 2b^\mu x_\nu + 4b\cdot x\, b^\mu x_\nu - 2b^\mu x_\nu b^2 x^2 + 2x^\mu b_\nu - 2b^2 x^\mu x_\nu - 2b^\mu x^2 b_\nu + 2b^2 x^2 b^\mu x_\nu}{(1-2b\cdot x + b^2 x^2)^2}$$

书上说 $\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1-2b\cdot x + b^2 x^2)^d} \equiv \frac{1}{\gamma(x)^d}$

Transformation of coorlation function

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\Delta_1/d} \left|\frac{\partial x_2'}{\partial x_2}\right|^{\Delta_2/d} \langle \phi(x_1')\phi_2(x_2')\rangle$$

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{1}{\gamma_1^{\Delta_1}} \frac{1}{\gamma_2^{\Delta_2}} \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} (\gamma_1^{\frac{1}{2}} \gamma_2^{\frac{1}{2}})^{\Delta_1+\Delta_2}$$

$$\Downarrow$$

$$\langle \phi(x_1)\phi(x_2)\rangle = \begin{cases} \dfrac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} & \Delta_1 = \Delta_2 \\ 0 & \Delta_1 \neq \Delta_2 \end{cases}$$

Ward identity for conformal symmetry

- Basic information

Ward identity: $\partial_\mu \langle j_a^\mu(x)\phi(x_1)\cdots\phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle\phi(x_1)\cdots G_a\phi(x_i)\cdots\phi(x_n)\rangle$

Action invariant: $\Delta S = \int d^dx\, w_a(x)\,\partial_\mu j_a^\mu(x) = -\int d^dx\,\partial_\mu(w_a j^\mu(x))$ （注意正负号问题）

Field invariant: $\Phi'(x) = (1 - i w_a G_a)\Phi(x)$

1° translation $\Rightarrow$ EM tensor

$$x'^\mu = x^\mu + a^\mu = x^\mu + w_a\frac{\delta x^\mu}{\delta w_a} \qquad \phi'-\phi = \{-i a_\mu(-i\partial^\mu)\}\phi$$

$$\phi'(x') = \phi(x) = \phi(x) + w_a\frac{\delta F}{\delta w_a}(x) \qquad \Delta S = \int d^dx\; a_\nu\,\partial_\mu T^{\mu\nu}$$

Ward identity

$$\partial_\mu\langle T^{\mu\nu}(x)\phi(x_1)\cdots\phi(x_n)\rangle = -\sum_{i=1}\delta(x-x_i)\langle\phi(x_1)\cdots\partial^\nu\phi(x_i)\cdots\phi(x_n)\rangle \quad - \boxed{1}$$

2° rigid rotation

$$x'^\mu = x^\mu + m^{\mu\nu}x_\nu = x^\mu + w_a\frac{\delta x^\mu}{\delta w_a} \qquad \phi'-\phi = \{-i\tfrac{m^{\mu\nu}}{2}[i(x_\mu\partial_\nu - x_\nu\partial_\mu)+S_{\mu\nu}]\}\phi$$

$$\Phi'(x') = (1 - \tfrac{i}{2}m_{\mu\nu}S^{\mu\nu})\Phi(x) = \Phi(x) + w_a\frac{\delta F}{\delta w_a}(x) \qquad \Delta S = \int d^dx\,\tfrac{1}{2}m_{\nu\rho}\,\partial_\mu\big(j^{\mu\nu\rho}\big) \qquad j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$$

Ward identity

$$\partial_\mu\langle(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)\phi(x_1)\cdots\phi(x_n)\rangle = \sum_{i=1}^{n}\delta(x-x_i)\langle\phi(x_1)\cdots((x^\nu\partial^\rho - x^\rho\partial^\nu) - iS_i^{\nu\rho})\phi(x_i)\cdots\phi(x_n)\rangle \quad - \boxed{2}$$

combine (1), (2)

$$\langle(T^{\rho\nu} - T^{\nu\rho})\phi(x_1)\cdots\phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\,S_i^{\nu\rho}(X)$$

3° dialation

$$x'^\mu = (1+\alpha)x^\mu \qquad \phi'-\phi = [-i\alpha(\tilde{\Delta} - i x^\rho\partial_\rho)]\phi$$

$$\phi'(x') = (1 - i\alpha(-i\Delta))\phi(x) \qquad \Delta S = \int d^dx\,\alpha\,\partial_\mu(j_D^\mu) \qquad j_D^\mu = T^\mu{}_\nu x^\nu$$

$$= (1-\alpha\Delta)\phi(x)$$

Ward identity.

$$\partial_\mu\langle T^\mu{}_\nu x^\nu\phi(x_1)\cdots\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle\phi(x_1)\cdots(\tilde{\Delta} - i x^\rho\partial_\rho)\phi(x_i)\cdots\phi(x_n)\rangle$$

$$\partial_\mu\langle T^\mu{}_\nu x^\nu\phi(x_1)\cdots\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\langle\phi(x_1)\cdots(\Delta + x^\rho\partial_\rho)\phi(x_i)\cdots\phi(x_n)\rangle \quad - \boxed{3}$$

combine (1) (3)

$$\langle T^\mu{}_\mu\,\phi(x_1)\cdots\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\,\Delta_i\,\langle\cdots\rangle$$

Conformal invariance in two dimension.

- Coordinate transformation leads to metric transformation

—— from $g_{\alpha\beta}(z)$ to $g_{\mu\nu}(w)$

$$g_{\alpha\beta}(z)\, dz^\alpha dz^\beta = g_{\mu\nu}(w)\, dw^\mu dw^\nu$$

$$g_{\mu\nu}(w) = g_{\alpha\beta}(z) \frac{\partial z^\alpha}{\partial w^\mu} \frac{\partial z^\beta}{\partial w^\nu}$$

$$g_{\mu\nu}(w) = J^T \cdot g \cdot J \qquad J^\beta{}_\nu = \frac{\partial z^\beta}{\partial w^\nu} \qquad (J^T)_\mu{}^\alpha = \frac{\partial z^\alpha}{\partial w^\mu}$$

$$(J^{-1})^\nu{}_\beta = \frac{\partial w^\nu}{\partial z^\beta}$$
$$(J^{T-1})_\alpha{}^\mu = \frac{\partial w^\mu}{\partial z^\alpha}$$

$$(g^{-1}(w))^{\mu\nu} = (J^T g J)^{-1}$$
$$= J^{-1} g^{-1} (J^{T-1})$$
$$= \frac{\partial w^\mu}{\partial z^\alpha} g^{\alpha\beta} \frac{\partial w^\nu}{\partial z^\beta}$$

$$g^{\mu\nu}(w) = \frac{\partial w^\mu}{\partial z^\alpha} \frac{\partial w^\nu}{\partial z^\beta} g^{\alpha\beta}$$

—— Check $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$

$$g^{\mu\gamma} g_{\gamma\nu} = \frac{\partial w^\mu}{\partial z^a} \frac{\partial w^\gamma}{\partial z^b} g^{ab} \frac{\partial z^c}{\partial w^\gamma} \frac{\partial z^d}{\partial w^\nu} g_{cd}$$

$$= \frac{\partial w^\mu}{\partial z^a} \delta^c_b\, g^{ab} g_{cd} \frac{\partial z^d}{\partial w^\nu}$$

$$= \frac{\partial w^\mu}{\partial z^a} \delta^a_d \frac{\partial z^d}{\partial w^\nu}$$

$$= \delta^\mu_\nu$$

- Conformal symmetry requirement on coordinate transformation

$$g_{\mu\nu}(w) \propto g_{\alpha\beta}(z)$$

—— 1° $\quad g^{00}(w)/g^{11}(w) = g^{00}(z)/g^{11}(z)$

$$\Downarrow$$

$$\frac{\frac{\partial w^0}{\partial z^0}\frac{\partial w^0}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^0}{\partial z^1}}{\frac{\partial w^1}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^1}{\partial z^1}\frac{\partial w^1}{\partial z^1}} = 1$$

$$\Downarrow$$

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^0}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^0}{\partial z^1} = \frac{\partial w^1}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^1}{\partial z^1}\frac{\partial w^1}{\partial z^1} \qquad \text{—— (1)}$$

—— 2° $\quad g^{01}(w) = g^{10}(w) = 0$

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0 \qquad \text{—— (2)}$$

- Define complex coordinates z and $\bar{z}$

$$z = z^0 + i z^1 \qquad \bar z = z^0 - i z^1$$

$$z^0 = \frac{1}{2}(z + \bar z) \qquad z^1 = i\frac{1}{2}(\bar z - z)$$

$$\partial_z = \frac{\partial z^0}{\partial z}\frac{\partial}{\partial z^0} + \frac{\partial z^1}{\partial z}\frac{\partial}{\partial z^1}$$

$$= \frac{1}{2}\partial_0 - \frac{i}{2}\partial_1$$

$$\partial_{\bar z} = \frac{\partial z^0}{\partial \bar z}\partial_0 + \frac{\partial z^1}{\partial \bar z}\partial_1$$

$$= \frac{1}{2}\partial_0 + \frac{i}{2}\partial_1$$

$$\partial_0 = \partial_z + \partial_{\bar z}$$

$$\partial_1 = i(\partial_z - \partial_{\bar z})$$

---

metric of complex coordinate

$$g_{\mu\nu}\, dz'^{\mu} dz'^{\nu} = g_{\alpha\beta}\, dz^\alpha dz^\beta \qquad z = (z^0, z^1)$$

$$g_{\mu\nu} = g_{\alpha\beta}\frac{\partial z^\alpha}{\partial z'^\mu}\frac{\partial z^\beta}{\partial z'^\nu} \qquad z' = (z, \bar z)$$

$$\Lambda_{\alpha\mu} = \frac{\partial z^\alpha}{\partial z'^\mu} = \begin{pmatrix} \frac12 & \frac12 \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac12 & \frac12 \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac12 & \frac12 \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix} = \begin{pmatrix} \frac12 & -\frac{i}{2} \\ \frac12 & \frac{i}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac12 & \frac12 \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac12 & -\frac{i}{2} \\ \frac12 & \frac{i}{2} \end{pmatrix}\begin{pmatrix} \frac12 & \frac12 \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac12 \\ \frac12 & 0 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Anti symmetric tensor of complex coordinate

$$\epsilon_{\mu\nu}\, dz'^{\mu} dz'^{\mu} = \epsilon_{\alpha\beta}\, dz^\alpha dz^\beta$$

$$\epsilon_{\mu\nu} = \epsilon_{\alpha\beta}\frac{\partial z^\alpha}{\partial z'^\mu}\frac{\partial z}{\partial z'^\nu}$$

$$= \begin{pmatrix} \frac12 & -\frac{i}{2} \\ \frac12 & \frac{i}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac12 & \frac12 \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{i}{2} & \frac12 \\ -\frac{i}{2} & \frac12 \end{pmatrix}\begin{pmatrix} \frac12 & \frac12 \\ -\frac{i}{2} & \frac{i}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}$$

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Lower case coordinates

$$z_0 = g_{0\alpha} z^\alpha = z^0 \qquad z_1 = g_{1\alpha} z^\alpha = z^1$$

$$z'_0 = g_{0\mu} z^\mu = \frac12 \bar z \qquad z'_1 = g_{1\mu} z^\mu = \frac12 z$$

$$= \frac12 (z_0 - i z_1) \qquad = \frac12 (z_0 + i z_1)$$

$$z_0 = z_0' + z_1' \qquad z_1 = i(z_0' - z_1')$$

$$\frac{\partial z_\beta}{\partial z'_\nu} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

first index

uppercase antisymmetric tensor

$$\varepsilon^{\mu\nu} = \varepsilon^{\alpha\beta} \frac{\partial z_\alpha}{\partial z'_\mu} \frac{\partial z_\beta}{\partial z'_\nu}$$

$$= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$= \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}$$

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coordinate transformation relation & Cauchy Riemann relation

coordinate transformation constraint for conformal symmetry.

$$\left(\frac{\partial W^0}{\partial z^0}\right)^2 + \left(\frac{\partial W^0}{\partial z^1}\right)^2 = \left(\frac{\partial W^1}{\partial z^0}\right)^2 + \left(\frac{\partial W^1}{\partial z^1}\right)^2$$

$$\frac{\partial W^0}{\partial z^0}\frac{\partial W^1}{\partial z^0} + \frac{\partial W^0}{\partial z^1}\frac{\partial W^1}{\partial z^1} = 0$$

Two possible solution for these coordinate transformation

$$\frac{\partial W^1}{\partial z^0} = \frac{\partial W^0}{\partial z^1} \qquad \frac{\partial W^0}{\partial z^0} = -\frac{\partial W^1}{\partial z^1} \qquad (1)$$

$$\frac{\partial W^1}{\partial z^0} = -\frac{\partial W^0}{\partial z^1} \qquad \frac{\partial W^0}{\partial z^0} = \frac{\partial W^1}{\partial z^1} \qquad (2)$$

Noticed

$$\partial_z = \frac{1}{2}\partial_0 - \frac{i}{2}\partial_1$$

$$\partial_{\bar z} = \frac{1}{2}\partial_0 + \frac{i}{2}\partial_1$$

situation (2) means

$$\frac{\partial (W^0 + iW^1)}{\partial \bar z} = \left(\frac{1}{2}\partial_0 + \frac{i}{2}\partial_1\right)(W^0 + iW^1)$$

$$= \frac{1}{2}\partial_0 W^0 - \frac{1}{2}\partial_1 W^1 + \frac{i}{2}\partial_1 W^0 + \frac{i}{2}\partial_0 W^1$$

$$= 0$$

$\partial_{\bar z} W(z, \bar z) = 0$ cauchy-riemann equation Holomorphic

equation (1) means

$$\frac{\partial (W^0 + iW^1)}{\partial z} = \frac{1}{2}(\partial_0 - i\partial_1)(W^0 + iW^1)$$

$$= \frac{1}{2}\partial_0 W^0 + \frac{1}{2}\partial_1 W^1 + \frac{i}{2}\partial_0 W^1 - \frac{i}{2}\partial_1 W^0$$

$$= 0$$

Anti-Holomorphic

$$\partial_z W(z, \bar z) = 0$$

- special conformal group

  all Global conformal transformation form conformal group

$$f(z) = \frac{az+b}{cz+d} \quad \text{with} \quad ab-bc=1$$

(可逆全纯函数) ⇒ $\partial_{\bar z} W(z,\bar z)=0$, $W(z)$ 可逆!

分子一次 ⇒ 无割线. 若为 $z^2 = y$ $\quad y^{\frac{1}{2}} = z \Rightarrow (y_0 e^{i2\pi})^{\frac{1}{2}} = y_0^{\frac{1}{2}} \cdot e^{i\pi} = -y_0^{\frac{1}{2}}$.

分母一次 ⇒ 奇点为极点.

- conformal generator

  infinitesimal coordinate transformation

$$z' = z + \varepsilon(z) \qquad \varepsilon(z) = \sum_{-\infty}^{+\infty} C_n z^{n+1}$$

spinless — Dimension Less field trans

$$\begin{aligned}
\phi'(z', \bar z') &= \phi(z, \bar z) \\
&= \phi(z', \bar z') - \varepsilon(z')\partial\phi(z, \bar z') - \bar\varepsilon(\bar z')\bar\partial\phi(z', \bar z') \\
\delta\phi &= -\varepsilon(z)\partial\phi(z,\bar z) - \bar\varepsilon(\bar z)\bar\partial\phi(z,\bar z) \\
&= \sum_n \{ c_n l_n + \bar c_n \bar l_n \} \phi(z, \bar z) \\
l_n &= -z^{n+1}\partial \\
\bar l_n &= -\bar z^{n+1}\bar\partial
\end{aligned}$$

commutation relation

$$[l_n, l_m] = (n-m) l_{n+m}$$
$$[\bar l_n, \bar l_m] = (n-m)\bar l_{n+m}$$
$$[l_n, \bar l_m] = 0$$

- Quasi-primary field. 准基本场.

  conformal transformation for field

$$\exp(-i\alpha\hat D - i a^\mu \hat P_\mu - i\tfrac{1}{2} m^{\mu\nu} L_{\mu\nu} - i b_\mu \hat K^\mu)$$

$$= \exp(-i\alpha\hat\Delta - i\tfrac{1}{2} m^{\mu\nu} S_{\mu\nu} - 2i b^\mu x_\mu \hat\Delta + 2i b^\mu x^\nu S_{\mu\nu})$$

$\times$ terms with derivatives.

$$\Phi'(x') = \exp(-i\alpha\hat\Delta - i\tfrac{1}{2} m^{\mu\nu} S_{\mu\nu} - 2i b^\mu x_\mu \hat\Delta + 2i b^\mu x^\nu S_{\mu\nu}) \Phi(x)$$

For field with spin $S_{\mu\nu} = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}$ $\quad m^{\mu\nu} = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}$ called field with spin s.

$$\Phi'(x') = \exp(-i\alpha\hat\Delta - 2i b^\mu x_\mu \hat\Delta - i m S)\Phi(x)$$

denote $\hat{\Delta} \equiv -i\Delta$

$$\Phi'(x') = \exp(-\alpha\Delta - 2b\cdot x\,\Delta - imS)\,\Phi(x)$$

coordinate transformation

$$x'^{\mu} = x^{\mu} + a^{\mu} + \alpha x^{\mu} + m^{\mu\nu}x_{\nu} + (-b^{\mu}x^2 + 2b\cdot x\, x^{\mu})\frac{1}{N}$$

in two dimensions $m^{\mu\nu} = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}$

$$x'^0 = x^0 + a^0 + \alpha x^0 + m x^1 + (-b^0 x^2 + 2b\cdot x\, x^0)$$

$$x'^1 = x^1 + a^1 + \alpha x^1 - m x^0 + (-b^1 x^2 + 2b\cdot x\, x^1)$$

Jacobi matrix for infinitesimal transformation

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \begin{pmatrix} 1+\alpha - 2b^0x^0 + 2b\cdot x + 2b^0x^0 & m - 2b^0x^1 + 2b^1x^0 \\ -m - 2b^1x^0 + 2b^0x^1 & 1+\alpha - 2b^1x^1 + 2b\cdot x + 2b^1x^1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\alpha+2b\cdot x & m - 2b^0x^1 + 2b^1x^0 \\ -m - 2b^1x^0 + 2b^0x^1 & 1+\alpha+2b\cdot x \end{pmatrix}$$

$$\left|\frac{\partial x'}{\partial x}\right| = (1+\alpha+2b\cdot x)^2 - (m - 2b^0x^1 + 2b^1x^0)(-m + 2b^0x^1 - 2b^1x^0)$$

$$= (1+\alpha+2b\cdot x)^2 + (m - 2b^0x^1 + 2b^1x^0)^2$$

$$= 1 + 2\alpha + 4b\cdot x + 2m(2b^1x^0 - 2b^0x^1)$$

$$= 1 + 2\alpha + 4b\cdot x + 4m(b^1x^0 - b^0x^1)$$

也不知书上怎么算的 ⇒ 思路： $w = w(z,\bar z)$　$\bar w = \bar w(z,\bar z)$，求 $\frac{d(x'^0 + i x'^1)}{d(x^0 + i x^1)}$ ⇒ $\frac{dw}{dz}$.

$h \equiv \frac{1}{2}(\Delta + S)$　$\bar h = \frac{1}{2}(\Delta - S)$

$$\phi'(w,\bar w) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar w}{d\bar z}\right)^{-\bar h}\phi(z,\bar z)$$

variation of quasi-primary field

for

$$w = z + \epsilon(z) \qquad \bar w = \bar z + \bar\epsilon(\bar z)$$

$$\phi'(z+\epsilon, \bar z+\bar\epsilon) = (1+\partial_z\epsilon)^{-h}(1+\partial_{\bar z}\bar\epsilon(\bar z))^{-\bar h}\phi(z,\bar z)$$

$$\phi'(z,\bar z) = (1 - h\partial_z\epsilon)(1 - \bar h\partial_{\bar z}\bar\epsilon(\bar z))\,\phi(z-\epsilon, \bar z-\bar\epsilon)$$

$$= \phi(z,\bar z) - \epsilon\partial_z\phi(z,\bar z) - \bar\epsilon\partial_{\bar z}\phi(z,\bar z) - h\phi\partial_z\epsilon - \bar h\phi\partial_{\bar z}\epsilon$$

Ward identities in two dimensions

- Basic tools — delta function in two dimension

consider integration $x = (z^0, z^1)$ $z = z^0 + i z^1$

$$\frac{1}{\pi}\int_M d^2x \, f(z)\, \partial_{\bar z} \frac{1}{z}$$

$$= \frac{1}{\pi}\int_M d^2x \, \partial_{\bar z}\left(\frac{f(z)}{z}\right)$$

Gauss integration law in two dimensions

$$\int_M d^2x \, \partial_\mu F^\mu = \int_{\partial M} \{ dz\, \varepsilon_{\bar z z} F^{\bar z} + d\bar z\, \varepsilon_{z \bar z} F^{z} \}$$ ← Gauss equation in $(z, \bar z)$ coor system.

$$= \frac{1}{2} i \int_{\partial M} \{ - dz\, F^{\bar z} + d\bar z \, F^{z} \}$$

$$F^{\bar z} = \frac{f(z)}{z} \qquad F^{z} = 0$$

$$\frac{1}{\pi}\int_M d^2x \, f(z)\, \partial_{\bar z} \frac{1}{z}$$

$$= \frac{1}{\pi}\int_M d^2x \, \partial_{\bar z}\left(\frac{f(z)}{z}\right)$$

$$= \frac{1}{\pi}\frac{1}{2} i \int_{\partial M} (-dz) \frac{f(z)}{z}$$

$$= \frac{1}{2\pi i}\int_{\partial M} dz\, \frac{f(z)}{z}$$

$$= f(0)$$

$$= \int_M d^2x \, f(z)\, \delta(x)$$

$$\boxed{\begin{aligned} \delta(x) &= \frac{1}{\pi}\partial_{\bar z}\frac{1}{z} \\ &= \frac{1}{\pi}\partial_{z}\frac{1}{\bar z} \end{aligned}}$$

- Ward identity in two dimension

Original ward id
$$\frac{\partial}{\partial x^\mu}\langle T^\mu{}_\nu(x) X\rangle = -\sum_{i=1}^{n} \delta(x - x_i)\frac{\partial}{\partial x_i^\nu}\langle X\rangle$$

$$\langle (T^{\rho\nu} - T^{\nu\rho})\,\phi(x_1)\cdots\phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x - x_i)\, S_i^{\nu\rho}\langle X\rangle$$

$$\langle T^\mu{}_\mu\, \phi(x_1)\cdots\rangle = -\sum_{i=1}^{n}\delta(x - x_i)\,\Delta_i\langle\cdots\rangle$$

Two dimensional angular momentum $S_i^{\mu\nu} = -s\,\varepsilon^{\mu\nu} = s\begin{pmatrix} 0 & ? \\ -? & 0\end{pmatrix}$

$$\langle (T^{\rho\nu} - T^{\nu\rho})\,\phi(x_1)\cdots\phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x - x_i)\,\varepsilon^{\nu\rho}\langle X\rangle \qquad \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0\end{pmatrix}$$

$$\boxed{\begin{aligned} &\varepsilon_{\mu\nu}\langle T^{\mu\nu}(x) X\rangle = -i\sum_{i=1}^{n} s_i\,\delta(x - x_i)\langle X\rangle \\ &\frac{\partial}{\partial x^\mu}\langle T^\mu{}_\nu(x) X\rangle = -\sum_{i=1}^{n}\delta(x - x_i)\frac{\partial}{\partial x_i^\nu}\langle X\rangle \\ &\langle T^\mu{}_\mu\,\phi(x_1)\cdots\rangle = -\sum_{i=1}^{n}\delta(x - x_i)\,\Delta_i\langle\cdots\rangle \end{aligned}}$$

- insert δ function representation

$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0\end{pmatrix}$ $\quad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0\end{pmatrix}$ $\quad \varepsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0\end{pmatrix}$

$$\partial_z\langle T^z{}_z X\rangle + \partial_{\bar z}\langle T^{\bar z}{}_z X\rangle = -\sum_{i=1}^{n}\delta(x - x_i)\,\partial_{z_i}\langle X\rangle$$

$$2\partial_z\langle T_{\bar z z} X\rangle + 2\partial_{\bar z}\langle T_{z z} X\rangle = -\sum_{i=1}^{n}\frac{1}{\pi}\partial_{\bar z}\frac{1}{z - z_i}\,\partial_{z_i}\langle X\rangle \qquad —(1)$$

$$\partial_z\langle T^z{}_{\bar z} X\rangle + \partial_{\bar z}\langle T^{\bar z}{}_{\bar z} X\rangle = -\sum_{i=1}^{n}\frac{1}{\pi}\partial_z\frac{1}{\bar z - \bar z_i}\,\partial_{\bar z_i}\langle X\rangle$$

$$2\partial_z\langle T_{\bar z\bar z} X\rangle + 2\partial_{\bar z}\langle T_{z\bar z} X\rangle = -\sum_{i=1}^{n}\frac{1}{\pi}\partial_z\frac{1}{\bar z - \bar z_i}\,\partial_{\bar z_i}\langle X\rangle \qquad —(2)$$

$$\langle T^{z}{}_{z} X\rangle + \langle T^{\bar z}{}_{\bar z} X\rangle = -\sum_i \delta(x-x_i)\Delta_i \langle X\rangle$$

$$2\langle T_{\bar z z} X\rangle + 2\langle T_{z\bar z} X\rangle = -\sum_i \delta(x-x_i)\Delta_i \langle X\rangle \qquad \text{— (3)}$$

$$\frac{i}{2}\langle T^{z\bar z} X\rangle - \frac{i}{2}\langle T^{\bar z z} X\rangle = -i\sum_{i=1}^{n} s_i \delta(x-x_i)\langle X\rangle$$

$$2\langle T_{\bar z z} X\rangle - 2\langle T_{z\bar z} X\rangle = -\sum_{i=1}^{n} s_i \delta(x-x_i)\langle X\rangle \qquad \text{— (4)}$$

Add (3) (4)

$$4\langle T_{\bar z z} X\rangle = -\sum_i (s_i+\Delta_i)\frac{1}{\pi}\partial_{\bar z}\frac{1}{z-z_i}\langle X\rangle$$

Subtract (3) (4)

...

$$2\pi\langle T_{\bar z z} X\rangle = -\sum_i \partial_{\bar z}\frac{1}{z-z_i} h_i \langle X\rangle \qquad h_i = \frac{1}{2}(s_i+\Delta_i)$$

$$2\pi\langle T_{z\bar z} X\rangle = -\sum_{i=1}^{n} \partial_z \frac{1}{\bar z-\bar z_i}\bar h_i \langle X\rangle \qquad \bar h_i = \frac{1}{2}(s_i-\Delta_i) \qquad \text{— (5)}$$

insert (5) to (1) (2).

$$\partial_{\bar z}\left\{\langle T X\rangle - \sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right]\right\} = 0$$

$$\partial_z\left\{\langle \bar T X\rangle - \sum_{i=1}^{n}\left[\frac{1}{\bar z-\bar w_i}\partial_{\bar w_i}\langle X\rangle + \frac{\bar h_i}{(\bar z-\bar w_i)^2}\langle X\rangle\right]\right\} = 0$$

$$T \equiv -2\pi T_{zz}$$

$$\bar T \equiv -2\pi T_{\bar z\bar z}$$

Expression in the upper braces are holomorphic or antiholomorphic

$$\langle TX\rangle = \sum_{i=1}^{n}\left\{\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right\} + \underbrace{\text{reg}}$$ stands for holomorphic function of z regular at $z=w_i$

- Conformal ward identity

$$\partial_\mu(\varepsilon_\nu T^{\mu\nu}) = \varepsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu)T^{\mu\nu} + \frac{1}{2}(\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu)T^{\mu\nu}$$

$$\overset{\downarrow}{=} \varepsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\rho\varepsilon^\rho)\eta_{\mu\nu}T^{\mu\nu} + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta\ \varepsilon_{\mu\nu}T^{\mu\nu}$$

conformal transformation 条件:

$$\left.\begin{array}{r} 2\partial_\mu\varepsilon^\mu = f(x)\cdot(d) \\ \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu = f(x)\, g_{\mu\nu}\end{array}\right\} \Rightarrow \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu = (\partial_\mu\varepsilon^\mu)\eta_{\mu\nu}$$

$$\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu = k(x)\cdot\varepsilon_{\mu\nu} \Rightarrow 2\varepsilon^{\mu\nu}\partial_\mu\varepsilon_\nu = k(x)\cdot 2 \Rightarrow k(x) = \varepsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta \Rightarrow \partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu = \varepsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta\,\varepsilon_{\mu\nu}$$

Ward identity for translation, rotation, dialation derived before

$$\varepsilon_{\mu\nu} \langle T^{\mu\nu}(x) X \rangle = -i \sum_{i=1}^{n} s_i \delta(x - x_i) \langle X \rangle$$

$$\frac{\partial}{\partial x^\mu} \langle T^\mu{}_\nu(x) X \rangle = - \sum_{i=1}^{n} \delta(x - x_i) \frac{\partial}{\partial x_i^\nu} \langle X \rangle$$

$$\langle T^\mu{}_\mu \phi(x_1) \cdots \rangle = - \sum_{i=1}^{n} \delta(x - x_i) \Delta_i \langle \cdots \rangle$$

Consider ward identity

$$\partial_\mu \langle \varepsilon_\nu T^{\mu\nu} X \rangle = \varepsilon_\nu \partial_\mu \langle T^{\mu\nu} X \rangle + \frac{1}{2}(\partial_\rho \varepsilon^\rho) \eta_{\mu\nu} \langle T^{\mu\nu} X \rangle + \frac{1}{2} \varepsilon^{\alpha\beta} \partial_\alpha \varepsilon_\beta \quad \varepsilon_{\mu\nu} \langle T^{\mu\nu} X \rangle$$

$$= \varepsilon^\nu (-1) \sum_{i=1}^{n} \delta(x - x_i) \frac{\partial}{\partial x_i^\nu} \langle X \rangle + \frac{1}{2}(\partial_\rho \varepsilon^\rho)(-1) \sum_{i=1}^{n} \delta(x - x_i) \Delta_i \langle X \rangle + \frac{1}{2} \varepsilon^{\alpha\beta} \partial_\alpha \varepsilon_\beta (-i) \sum_{i=1}^{n} s_i \delta(x - x_i) \langle X \rangle$$

for coordinate transformation

$$x'^\mu = x^\mu + a^\mu + m^\mu{}_\nu x^\nu + \alpha x^\mu$$

$$= x^\mu + a^\mu + \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix} x^\nu + \alpha x^\mu$$

$$= x^\mu + a^\mu + \begin{pmatrix} 0 & m x^1 \\ -m x^0 & 0 \end{pmatrix} + \alpha x^\mu$$

dialation :

$$\alpha = \frac{1}{2} \partial_\rho \varepsilon^\rho$$

rotation

$$m = - \frac{1}{2} \varepsilon^{\alpha\beta} \partial_\alpha \varepsilon_\beta = - \varepsilon^{01} \partial_0 \varepsilon_1$$

$$\partial_\mu \langle \varepsilon_\nu T^{\mu\nu} X \rangle = (-1) \varepsilon^\nu \sum_{i=1}^{n} \delta(x - x_i) \frac{\partial}{\partial x_i} \langle X \rangle - \alpha \sum_{i=1}^{n} \delta(x - x_i) \Delta_i \langle X \rangle + i m \sum_{i=1}^{n} s_i \delta(x - x_i) \langle X \rangle$$

for quasi-primary field (no SCT. $b = 0$) $S_{\mu\nu} = -s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $m_{\mu\nu} = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}$

$$\Phi'(x') = \exp\Big(-i\alpha \underset{(-i\Delta)}{\tilde{\Delta}} - i \frac{1}{2} m^{\mu\nu} S_{\mu\nu} - 2i b^\mu x_\mu \tilde{\Delta} + 2i b^\mu x^\nu S_{\mu\nu} \Big) \Phi(x)$$

$$= \exp(-\alpha \Delta + i m s) \phi$$

$$\Phi'(x) = - \varepsilon^\nu \partial_\nu \Phi - \alpha \Delta \Phi + i m s \Phi$$

Denote as.

$$\delta_\varepsilon \langle X \rangle = \int_M d^2x \Big\{ (-1) \varepsilon^\nu \sum_{i=1}^{n} \delta(x - x_i) \frac{\partial}{\partial x_i} \langle X \rangle - \alpha \sum_{i=1}^{n} \delta(x - x_i) \Delta_i \langle X \rangle + i m \sum_{i=1}^{n} s_i \delta(x - x_i) \langle X \rangle \Big\}$$

$$= \int_M d^2x \, \partial_\mu \langle T^{\mu\nu} \varepsilon_\nu X \rangle$$

Gauss integration law in two dimensions

$$\int_M d^2x \, \partial_\mu F^\mu = \int_{\partial M} \{ dz \, \varepsilon_{\bar{z}z} F^{\bar{z}} + d\bar{z} \, \varepsilon_{z\bar{z}} F^z \}$$

$$= \frac{1}{2} i \int_{\partial M} \{ -dz F^{\bar{z}} + d\bar{z} F^z \}$$

$$\delta_{\varepsilon, \bar{\varepsilon}} \langle X \rangle = \frac{1}{2} i \int_{\partial M} \{ -d\bar{z} \langle T^{\bar{z}\bar{z}} \varepsilon_{\bar{z}} X \rangle + d\bar{z} \langle T^{zz} \varepsilon_z X \rangle \} \qquad (\varepsilon = \varepsilon^z, \bar{\varepsilon} = \varepsilon^{\bar{z}})$$

These expression contains no $\langle T^{\bar{z}z}X\rangle$ or $\langle T^{z\bar{z}}X\rangle$

1° $T^{\bar{z}z} = T^{z\bar{z}}$, symmetric of EM tensor

2° (4.68) ward identity for dilation.

$$\langle T^{\mu}{}_{\mu} X\rangle = -\sum_{i=1}^{n} \delta(x-x_i)\Delta_i \langle\cdots\rangle$$

3° metric in two dimension

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$\langle T^{\mu}{}_{\mu} X\rangle = \langle T^{z}{}_{z} X\rangle + \langle T^{\bar{z}}{}_{\bar{z}} X\rangle = \frac{1}{2}\langle T^{z\bar{z}}X\rangle + \frac{1}{2}\langle T^{\bar{z}z}X\rangle$$

$\Rightarrow$ $\langle T^{z\bar{z}}X\rangle$, $\langle T^{\bar{z}z}X\rangle \neq 0$ at $x = x_i$ $\Rightarrow$ $\int_{\partial M}$ 不经过 x_i.

denote

$$T = -2\pi T_{zz} \qquad \bar{T} = -2\pi T_{\bar{z}\bar{z}} \qquad T^{\bar{z}\bar{z}} = 4T_{zz} \qquad T^{zz} = 4T_{\bar{z}\bar{z}}$$

Conformal ward identity

$$\delta_{\varepsilon\bar{\varepsilon}}\langle X\rangle = \frac{1}{2} i \int_{\partial M} \{ -d\bar{z} \langle T^{\bar{z}\bar{z}} \varepsilon_{\bar{z}} X\rangle + dz \langle T^{zz}\varepsilon_{z} X\rangle \}$$

$$= 2i \int_{\partial M} \{ -dz \langle T_{zz} \tfrac{1}{2}\varepsilon^{z} X\rangle + d\bar{z}\langle T_{\bar{z}\bar{z}} \tfrac{1}{2}\varepsilon^{\bar{z}} X\rangle \}$$

$$= i\int_{\partial M} \{ -dz \langle T_{zz}\varepsilon X\rangle + d\bar{z}\langle T_{\bar{z}\bar{z}}\varepsilon^{\bar{z}} X\rangle \}$$

$$= -\frac{1}{2\pi i}\oint_C dz\, \varepsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\, \bar{\varepsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle$$

° variation of primary field under infinitesimal transformation

suasi-primary 准基本场

$$\phi'(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z})$$

$$w = z+\varepsilon \qquad \bar{w} = \bar{z}+\bar{\varepsilon}$$

$$\phi'(w,\bar{w}) = (1 - h\frac{dw}{dz})(1-\bar{h}\frac{d\bar{w}}{d\bar{z}})\phi(w-\varepsilon, \bar{w}-\bar{\varepsilon})$$

$$\phi'(z,\bar{z}) = \phi(z,\bar{z}) - (h\,\partial_z w + \bar{h}\,\partial_{\bar{z}}\bar{w})\phi - (\varepsilon\,\partial_z\phi + \bar{\varepsilon}\,\partial_{\bar{z}}\phi)$$

$$\delta_{\varepsilon,\bar{\varepsilon}}\phi \equiv \phi'(z,\bar{z}) - \phi(z,\bar{z})$$

$$= -(h\phi\,\partial_z\varepsilon + \varepsilon\partial_z\phi) - (\bar{h}\phi\,\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}}\phi)$$

by integral conformal ward id

$$\delta_\varepsilon\langle X\rangle = -\sum_i (\varepsilon(w_i)\partial_{w_i} + \partial\varepsilon(w_i) h_i)\langle X\rangle = 0 \qquad -(1)$$

Global conformal

$$f(z) = \frac{(1+\alpha)z+\beta}{\gamma z + (1-\alpha)} \approx z + \beta + 2\alpha z - \gamma z^2 \Rightarrow \varepsilon = \beta + 2\alpha z - \gamma z^2 \qquad -(2)$$

from (1) (2).

β) $\sum_i \partial_{w_i} \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle = 0$

α: $\sum_i (w_i \partial_{w_i} + h_i) \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle = 0$

γ) $\sum_i (\gamma w_i^2 \partial_{w_i} + 2 \gamma w_i h_i) \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle = 0$

$\sum_i (w_i^2 \partial_{w_i} + 2 w_i h_i) \langle \phi_1(w_1) \cdots \phi_n(w_n) \rangle = 0$

Free fields and operator product expansion

Free Boson

- OPE of holomorphic / antiholomorphic

$$S = \frac{1}{2} g \int d^2x \; \partial_\mu \varphi \partial^\mu \varphi \qquad \mathcal{L} = \frac{1}{2} g \, \partial_\mu \varphi \partial^\mu \varphi$$

$$\langle \varphi(x) \varphi(y) \rangle = -\frac{1}{4\pi g} \ln (x-y)^2 + \text{const}$$

complex coordinate

$$\langle \varphi(z,\bar{z}) \varphi(w,\bar{w}) \rangle = -\frac{1}{4\pi g} \ln\left((x_1-y_1)^2 + (x_2-y_2)^2\right) + \text{const}$$

$$= -\frac{1}{4\pi g} \ln\left((z+\bar{z}-w-\bar{w})^2 - (z-\bar{z}-w+\bar{w})^2 \right) + \text{const}$$

$$= -\frac{1}{4\pi g} \ln\left([(z-w)+(\bar{z}-\bar{w})]^2 - [(z-w)-(\bar{z}-\bar{w})]^2 \right) + \text{const}$$

$$= -\frac{1}{4\pi g} \ln\left(4(z-w)(\bar{z}-\bar{w}) \right) + \text{const}$$

$$= -\frac{1}{4\pi g} \left[\ln(z-w) + \ln(\bar{z}-\bar{w}) \right]$$

—— holomorphic correlator antiholomorphic correlator

holo $\langle \partial_z \varphi(z,\bar{z}) \partial_w \varphi(w,\bar{w}) \rangle = -\frac{1}{4\pi g} \frac{1}{(z-w)^2}$ $\Rightarrow$ OPE of this field with itself

Anti holo $\langle \partial_{\bar{z}} \varphi(z,\bar{z}) \partial_{\bar{w}} \varphi(w,\bar{w}) \rangle = -\frac{1}{4\pi g} \frac{1}{(\bar{z}-\bar{w})^2}$

$$\partial\varphi(z) \partial\varphi(w) \sim -\frac{1}{4\pi g} \frac{1}{(z-w)^2}$$

- Energy momentum tensor

$$T_c^{\mu\nu} \equiv -\eta^{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi(x))} \partial^\nu \varphi$$

$$= -\eta^{\mu\nu}\left(\frac{1}{2} g \, \partial_\mu \varphi \partial^\mu \varphi \right) + g \, \partial^\mu \varphi \partial^\nu \varphi$$

$$= g\left(\partial^\mu \varphi \partial^\nu \varphi - \frac{1}{2} \partial_\alpha \varphi \partial^\alpha \varphi \, \eta^{\mu\nu} \right)$$

renormalized energy momentum tensor defined as

$$T \equiv -2\pi T_{zz}$$

$$= -2\pi \left\{ g \left(\partial_z \varphi \partial_z \varphi - \frac{1}{2} \partial_\alpha \varphi \partial^\alpha \varphi \, \eta_{zz} \right) \right\}$$

$$= -2\pi g \, \partial_z \varphi \partial_z \varphi$$

After normal ordering

$$T(z) = -2\pi g :\partial\varphi \partial\varphi:$$

—— OPE of $T(z)$ with $\partial\varphi$

$$\langle T(z) \partial\varphi(w) \rangle = -2\pi g \langle 0| T(:\partial\varphi \partial\varphi: \partial\varphi(w)) |0\rangle$$

$$= -4\pi g \langle 0| : \partial\varphi(z) \underbrace{\partial\varphi(z) \partial\varphi(w)} : |0\rangle + \underbrace{\langle 0| : \partial\varphi \partial\varphi \partial\varphi(w) : |0\rangle}_{=0}$$

$$\sim +4\pi g \, \partial\varphi(z) \frac{1}{4\pi g} \frac{1}{(z-w)^2} = \frac{\partial\varphi(z)}{(z-w)^2}$$

expand around w

$$T(z) \partial\varphi(w) = \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2 \varphi}{(z-w)}$$

compare

$$\langle T(z) X \rangle = \sum_{i=1}^{n} \left\{ \frac{1}{z-w_i} \partial_{w_i} \langle X \rangle + \frac{h_i}{(z-w_i)^2} \langle X \rangle \right\} + \text{reg}$$

$\partial\varphi$ is primary field with conformal dimension $h=1$

—— OPE of energy momentum tensor with itself (Use wick's Theorem) $T(:ABC::DE\cdots:) = :ABC\cdots DE:$ + contract between normal order term.

$$\langle T(z)T(w)\rangle = \langle +4\pi^2 g^2 :\partial\varphi(z)\partial\varphi(z)::\partial\varphi(w)\partial\varphi(w):\rangle$$
$$= \underbrace{4\pi^2 g^2\langle 0|:\partial\varphi(z)\partial\varphi(z)\partial\varphi(w)\partial\varphi(w):|0\rangle}_{} = 0$$
$$+8\pi^2 g^2 \langle\partial\varphi(z)\partial\varphi(w)\rangle\langle\partial\varphi(z)\partial\varphi(w)\rangle$$
$$+16\pi^2 g^2 \langle\partial\varphi(z)\partial\varphi(w)\rangle\langle 0|:\partial\varphi(z)\partial\varphi(w):|0\rangle$$

Q why orange term vanished while $T(w)$ not vanished? Since they are all normal ordering!

$$\sim 8\pi^2 g^2 \frac{1}{16\pi^2 g^2}\frac{1}{(z-w)^4} - 16\pi^2 g^2 \frac{1}{4\pi g}\frac{1}{(z-w)^2}\underbrace{\langle 0|:\partial\varphi(w)\partial\varphi(w):|0\rangle}_{T(w)}$$
$$-16\pi^2 g^2\frac{1}{4\pi g}\frac{1}{(z-w)}\underbrace{\langle 0|:\partial^2\varphi(w)\partial\varphi(w):|0\rangle}_{\frac{1}{2}\partial T(w)}$$
$$\sim \frac{1}{2}\frac{1}{(z-w)^4} + 2\frac{1}{(z-w)^2}\langle 0|T(w)|0\rangle$$
$$-4\pi g\frac{1}{(z-w)}\frac{1}{2}\partial\{\langle 0|:\partial\varphi(w)\partial\varphi(w):|0\rangle\}$$

$$T(z)T(w) \sim \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{1}{(z-w)}\partial T(w)$$

EM tensor is not a primary field.

Free Fermion

- Action

$$S = \frac{1}{2}g\int d^2x\, \Psi^\dagger\gamma^0\gamma^\mu\partial_\mu\Psi \qquad \Psi = \begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}$$
$$\gamma^0 = \begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \gamma^1 = i\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$
$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \qquad \eta^{\mu\nu} = \mathrm{diag}(1,1)$$
$$\partial_z = \frac{1}{2}\partial_0 - \frac{i}{2}\partial_1 \qquad \partial_{\bar z} = \frac{1}{2}\partial_0 + \frac{i}{2}\partial_1$$

$$S = \frac{1}{2}g\int d^2x\,(\bar\psi,\psi)\begin{pmatrix}0&1\\1&0\end{pmatrix}\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}\partial_0 + i\begin{pmatrix}0&-1\\1&0\end{pmatrix}\partial_1\right\}\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}$$
$$= \frac{1}{2}g\int d^2x\,(\bar\psi,\psi)\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\partial_0 + i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\partial_1\right\}\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}$$
$$= \frac{1}{2}g\int d^2x\,(\bar\psi,\psi)\left\{\begin{pmatrix}\partial_0\psi\\ \partial_0\bar\psi\end{pmatrix} + i\begin{pmatrix}\partial_1\psi\\ -\partial_1\bar\psi\end{pmatrix}\right\}$$
$$= \frac{1}{2}g\int d^2x\,(\bar\psi,\psi)\begin{pmatrix}\partial_0\psi + i\partial_1\psi\\ \partial_0\bar\psi - i\partial_1\bar\psi\end{pmatrix}$$
$$= \frac{1}{2}g\int d^2x\,(\bar\psi,\psi)\begin{pmatrix}2\partial_{\bar z}\psi\\ 2\partial_z\bar\psi\end{pmatrix}$$
$$= g\int d^2x\,(\bar\psi\partial_{\bar z}\psi + \psi\partial_z\bar\psi)$$

这个好和书上不同.

Classical equation of motion

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = \frac{\partial\mathcal{L}}{\partial\varphi} \Rightarrow \boxed{\begin{aligned}\partial_z\bar\psi &= \partial_{\bar z}\bar\psi\\ \partial_{\bar z}\psi &= \partial_z\psi\end{aligned}}$$

若按书上所说

$$S = g\int d^2x\,(\bar\psi\partial_z\bar\psi + \psi\partial_{\bar z}\psi)$$

EOM

$$\partial_z\bar\psi = \partial_z\bar\psi \overset{\text{set}}{=} 0 \qquad \partial_{\bar z}\psi = 0$$
$$\partial_{\bar z}\psi = \partial_{\bar z}\psi = 0$$

—— Use coordinate (x^0, x^1) obtain equation of motion

$$S = \frac{1}{2} g \int d^2x \; (\bar{\psi}, \psi) \begin{pmatrix} \partial_0 \psi + i \partial_1 \psi \\ \partial_0 \bar{\psi} - i \partial_1 \bar{\psi} \end{pmatrix}$$

$$= \frac{1}{2} g \int d^2x \left(\bar{\psi} \partial_0 \psi + i \bar{\psi} \partial_1 \psi + \psi \partial_0 \bar{\psi} - i \psi \partial_1 \bar{\psi} \right)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = \partial_0 \bar{\psi} - i \partial_1 \bar{\psi} = \partial_0 (\bar{\psi}) + \partial_1 (i \bar{\psi})$$

$$\partial_1 \bar{\psi} = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) = \partial_0 \psi + i \partial_1 \psi = \partial_0 \psi - i \partial_1 \psi$$

$$\partial_1 \psi = 0 \qquad (2)$$

结果很怪!

- Generating function and correlation function

Generating function

$$Z[J, J^\dagger] = \int D\psi \, D\psi^\dagger \; \exp(-S +$$

$$S = \frac{1}{2} g \int d^2x \; \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi$$

$$= \frac{1}{2} g \int d^2x \, d^2y \; \Psi^\dagger \; \gamma^0 \gamma^\mu \partial_\mu \Psi$$

$$= \frac{1}{2} g \int d^2x \, d^2y \; \Psi^\dagger(x) \left\{ \delta(x-y) \gamma^0 \gamma^\mu \partial_\mu \right\} \Psi(y)$$

$$= \frac{1}{2} g \int d^2x \, d^2y \; \Psi^\dagger(x) \, A(x,y) \, \Psi(y)$$

Green function 方法

$$g \gamma^0 \gamma^\mu \partial_\mu K(\vec{r}) = \delta(\vec{r})$$

$$g (\gamma^0 \gamma^\mu)_{ik} \frac{\partial}{\partial x^\mu} K_{kj}(x) = \delta(x) \delta_{ij}$$

$$2g \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \begin{pmatrix} \langle \psi(z,\bar{z}) \psi(w,\bar{w}) \rangle & \langle \psi(z,\bar{z}) \bar{\psi}(w,\bar{w}) \rangle \\ \langle \bar{\psi}(z,\bar{z}) \psi(w,\bar{w}) \rangle & \langle \bar{\psi}(z,\bar{z}) \bar{\psi}(w,\bar{w}) \rangle \end{pmatrix} = \begin{pmatrix} \overbrace{\frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z-w}}^{\delta(x)} & \\ & \underbrace{\frac{1}{\pi} \partial_z \frac{1}{\bar{z}-\bar{w}}}_{\delta(x)} \end{pmatrix}$$

solution

$$\langle \psi(z,\bar{z}) \psi(w,\bar{w}) \rangle = \frac{1}{2\pi g} \frac{1}{z-w}$$

$$\langle \bar{\psi}(z,\bar{z}) \bar{\psi}(w,\bar{w}) \rangle = \frac{1}{2\pi g} \frac{1}{\bar{z}-\bar{w}}$$

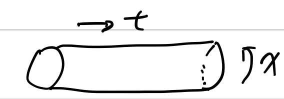
$$\langle \psi(z,\bar{z}) \bar{\psi}(w,\bar{w}) \rangle = 0$$

Operator formalism

The operator formalism of conformal field theory

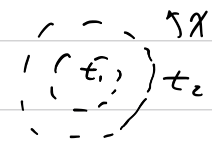
- infinite spacetime cylinder

$$\xi = \tau - i x \quad \Rightarrow \quad x \in [0, L), \ t \in (-\infty, +\infty)$$



$$z = e^{2\pi\xi/L} = e^{2\pi\frac{1}{L}(\tau - i x)}$$

$$\Rightarrow$$



- Hermitian product

Noticed $\tau = it$ $\Rightarrow$ Hermitian conjugate 保持t不变：$\tau \to -\tau \Rightarrow$ Hermitian conjugate： $z \to \frac{1}{z^*} = e^{-\frac{2\pi}{L}\tau} e^{-\frac{2\pi}{L} i x}$

assume field ϕ be quasi-primary

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z})$$

on the real surface, $z^* = \bar{z}$, justifies definition of hermitian conjugate.

$$w = \frac{1}{\bar{z}} \quad \bar{w} = \frac{1}{z}$$

$$\frac{dw}{dz} = -\frac{1}{\bar{z}^2}\frac{d\bar{z}}{dz} \qquad \frac{d\bar{w}}{d\bar{z}} = -\frac{1}{z^2}\frac{dz}{d\bar{z}}$$

$$\phi'(w, \bar{w}) = \left(\frac{d\bar{w}}{dz}\right)^{-h} \left(\frac{dw}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z})$$ 有点怪，这个式子可得结果，但为何？

$$\phi'\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) = \left(\frac{1}{z}\right)^{-2h} \left(\frac{1}{\bar{z}}\right)^{-2\bar{h}} \phi(z, \bar{z})$$

$$\phi^\dagger(z, \bar{z}) = (\bar{z})^{-2h} (z)^{-2\bar{h}} \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right)$$ 直接将此式当作定义

- inner product of $|in\rangle, |out\rangle$ state

$$|\phi_{in}\rangle \equiv \lim_{z, \bar{z} \to 0} \phi(z, \bar{z})|0\rangle$$

$$|\phi_{out}\rangle \equiv (|\phi_{in}\rangle)^{DC.}$$

$$\langle \phi_{out} | \phi_{in} \rangle = \lim_{w, \bar{w}, z, \bar{z} \to 0} \langle 0 | \phi^\dagger(w, \bar{w}) \phi(z, \bar{z}) | 0 \rangle$$

$$= \lim_{w, \bar{w}, z, \bar{z} \to 0} (\bar{w})^{-2h} (w)^{-2\bar{h}} \langle 0 | \phi\left(\frac{1}{\bar{w}}, \frac{1}{w}\right) \phi(z, \bar{z}) | 0 \rangle$$

$$= \lim_{w, \bar{w} \to 0} (\bar{w})^{-2h} (w)^{-2\bar{h}} \langle 0 | \phi\left(\frac{1}{\bar{w}}, \frac{1}{w}\right) \phi(0, 0) | 0 \rangle$$

$$= \lim_{\xi, \bar{\xi} \to +\infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \langle 0 | \phi(\bar{\xi}, \xi) \phi(0, 0) | 0 \rangle$$

$\Updownarrow$ Time ordered $\Leftrightarrow$ Radial ordered

correlation function restricted by conformal symmetry (5.25)

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \quad \text{if } \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{cases} \quad \text{otherwise} = 0$$

$$\langle \phi_{out} | \phi_{in} \rangle = C_{12} = \text{const}.$$

自洽！

- Mode expansion

quasi-primary conformal field of dimension $h, \bar{h}$。 mode expand，也将 mode expand 当作假设.

$$\phi(z, \bar{z}) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n}$$

$$\phi_{m,n} = \frac{1}{2\pi i} \oint dz\, z^{m+h-1} \frac{1}{2\pi i} \oint d\bar{z}\, \bar{z}^{n+\bar{h}-1} \phi(z, \bar{z})$$

——— hermitian conjugate

$$\phi^\dagger(z,\bar z) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \bar z^{-m-h} z^{-n-\bar h} \phi^\dagger_{m,n}$$

on the other hand.

$$(\bar z)^{-2h} (z)^{-2\bar h} \phi\left(\frac{1}{\bar z}, \frac{1}{z}\right)$$

$$= (\bar z)^{-2h} (z)^{-2\bar h} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \bar z^{+m+h} z^{+n+\bar h} \phi_{m,n}$$

$$= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \bar z^{m-h} z^{n-\bar h} \phi_{m,n}$$

$$= \sum_{m,n\in\mathbb{Z}} \bar z^{-m-h} z^{-n-\bar h} \phi_{-m,-n}$$

$$\phi^\dagger_{m,n} = \phi_{-m,-n}.$$

To make sure in / out state well defined

$$|\phi_{in}\rangle = \lim_{z,\bar z\to 0} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h} \bar z^{-n-\bar h} \phi_{m,n} |0\rangle$$

$$\Downarrow \qquad (-m-h<0 \text{ or } -n-\bar h<0)$$

$$\phi_{m,n}|0\rangle = 0 \qquad (m>-h \text{ or } n>-\bar h)$$

Drop antiholomorphic dependence, lighten expression

$$\phi(z) = \sum_{m\in\mathbb{Z}} z^{-m-h} \phi_m$$

$$\phi_m = \frac{1}{2\pi i} \oint dz\, z^{m+h-1} \phi(z)$$

- Radial ordering

$$T(\Phi_1(z)\Phi_2(w)) = R(\Phi_1(z)\Phi_2(w)) = \begin{cases} \Phi_1(z)\Phi_2(w) & \text{for } |z|>|w| \\ \Phi_2(w)\Phi_1(z) & |z|<|w| \end{cases}$$

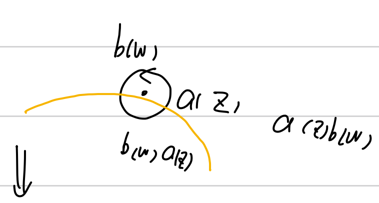
- commutation relation relate to circle integral

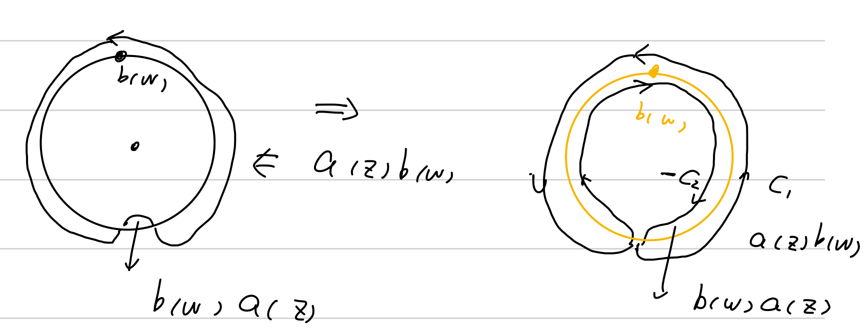
$$R\left\{ \oint_w dz\, a(z) b(w) \right\}$$

$$= \oint_{C_1} dz\, a(z)b(w) - \oint_{C_2} dz\, b(w) a(z)$$

$$= [A, b(w)]$$

$$A \equiv \oint dz\, a(z)$$





- commutator of operator (省略径向 ordering)

$$[A,B] = R\left\{\oint_0 dw \oint_w a(z)b(w)\right\} \qquad A = \oint a(z)dz \qquad B = \oint b(z)dz.$$

Q：暗含：先积对A的积分，区间：C_1 or C_2 取决于 $[A,B]$ 中的哪一项
再积对B的积分.

Virasoro algebra

- Conformal ward identity

$$\delta_{\varepsilon,\bar\varepsilon}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dz\, \varepsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar z\, \bar\varepsilon(\bar z)\langle \bar T(\bar z)X\rangle$$

$$Q_\varepsilon = \frac{1}{2\pi i}\oint dz\, \varepsilon(z)T(z)$$

$$\delta_\varepsilon\langle \Phi(w)\rangle = -\frac{1}{2\pi i}\oint_w dz\, \varepsilon(z)\langle T(z)\Phi(w)\rangle$$

$$= -[Q_\varepsilon, \Phi(w)]$$

Q_ε is the generator of conformal transformation

- Virasoro algebra

Basic definition

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad \xrightarrow{} \text{mode operator.} \qquad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z)$$

$$\bar T(\bar z) = \sum_{n\in\mathbb{Z}} \bar z^{-n-2}\bar L_n \qquad \bar L_n = \frac{1}{2\pi i}\oint d\bar z\, \bar z^{n+1}\bar T(\bar z)$$

Generator

$$\varepsilon(z) \equiv \sum_{n\in\mathbb{Z}} z^{n+1}\varepsilon_n$$

$$Q_\varepsilon = \frac{1}{2\pi i}\oint dz\, \varepsilon(z)T(z)$$

$$= \frac{1}{2\pi i}\oint dz \sum_{n\in\mathbb{Z}} z^{n+1}\varepsilon_n \sum_{m\in\mathbb{Z}} z^{-m-2}L_m$$

$$= \frac{1}{2\pi i}\sum_{n,m\in\mathbb{Z}}\oint dz\, z^{n-m-1}\varepsilon_n L_m$$

$$= \sum_n \varepsilon_n L_n$$

commutation relation

$$[L_n, L_m] = \left[\frac{1}{2\pi i}\oint dz\, z^{n+1}T(z),\ \frac{1}{2\pi i}\oint dw\, w^{m+1}T(w)\right]$$

$$= \frac{1}{(2\pi i)^2}\oint_0 dw\, w^{m+1}\oint_w dz\, z^{n+1} R\{T(z)T(w)\}$$

$$T\{T(z)T(w)\} \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

$$= \frac{1}{(2\pi i)^2}\oint_0 dw\, w^{m+1}\oint_w dz\, z^{n+1}\left\{\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\right\}$$

$$\frac{1}{2\pi i}\oint_w dz\, z^{n+1}\frac{c/2}{(z-w)^4} = \frac{1}{2\pi i}\oint_w d\alpha\, (w+\alpha)^{n+1}\frac{c/2}{\alpha^4}$$

$$= \frac{1}{2\pi i}\oint_w d\alpha\, \frac{1}{3!}(n+1)n(n-1)w^{n-2}\cdot\frac{c}{2}\frac{1}{\alpha} = \frac{c}{12}(n+1)(n)(n-1)w^{n-2}$$

...

...

$$= \frac{1}{2\pi i}\oint_0 dw\, w^{m+1}\left\{\frac{1}{12}c(n+1)n(n-1)w^{n-2} + 2(n+1)w^n T(w) + w^{n+1}\partial T(w)\right\}$$

$$\downarrow \text{integral by part}$$

$$= \frac{1}{12}cn(n^2-1)\delta_{n+m,0} + 2(n+1)L_{m+n} - \frac{1}{2\pi i}\oint_0 dw\,(n+m+2)w^{n+m+1}T(w)$$

$$= \frac{1}{12}cn(n^2-1)\delta_{n+m,0} + 2(n+1)L_{n+m} - (n+m+2)L_{n+m}$$

$$= \frac{1}{12}cn(n^2-1)\delta_{n+m,0} + (n-m)L_{m+n}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$$

$$[L_n, \bar{L}_m] = 0$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$$