Bethe Ansatz 以及量子计算

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摘要. 没有什么想写的，就是个学习笔记...

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CHAPTER 1

模型

在这一章中，我们将会推导基本算符之间满足的性质。首先，我们定义了可积系统中的基本算符 R，K^+，K^-，V。并且声明了它们之间满足的基本性质：规则性 (Regularity)，Yang-Baxter 方程，反射方程，对偶反射方程。之后，根据这些基本性质，能够得到一些推论。利用这些基本性质和推论，可以进行聚合操作以及计算转移算符的乘积表达式。

1.1. 基本性质推导

1.1.1. 定理. 定义 R 算符，作用在 5×5 维空间 $V_1 \times V_2$ 中:

$$R_{12}(u), \tag{1.1}$$

定义 K^+ 与 K^- 算符, 作用在空间 V_1 中:

$$K_1^+(u) \quad K_1^-(u) \tag{1.2}$$

下面讲述 R 算符和 K 算符满足一些性质。

Proposition 1 (性质 1:Regularity).

$$R_{12}(0) \sim f^{\frac{1}{2}}(0)\mathcal{P}_{12} \tag{1.3}$$

性质中的 $\mathcal{P}_{12}$ 是交换算符 (并且 1,2 空间的维度相同)

$$\mathcal{P}_{12}\binom{i1,i2}{j1,j2} = \delta_{j2}^{i1}\delta_{j1}^{i2} \tag{1.4}$$

交换算符可以交换算符作用的空间:

$$A_{21}(u) = \mathcal{P}_{12}A_{12}(u)\mathcal{P}_{12} \tag{1.5}$$

并且满足:

$$\mathcal{P}_{12}\mathcal{P}_{12} = \mathbb{I}_{1,2} \tag{1.6}$$

这里的 f(u) 是下面幺正性中定义的。

□

Proposition 2 (性质 2: Unitary).

$$R_{12}(u)R_{21}(-u) \sim f(u)\mathbb{I}. \tag{1.7}$$

□

Proposition 3 (性质: PT Symmetry). *这里的 P 是 permutation (交换算符) 的意思。T 是转置 transpose matrix 的意思。R 算符具有 PT symmetry:*

$$R_{21}(u) = \mathcal{P}_{12}R_{12}(u)\mathcal{P}_{12} = R_{12}^{t_1 t_2}(u). \tag{1.8}$$

Proposition 4 (性质 3: Crossing Symmetry). *可以定义算符 V such that,*

$$R_{12}(u) = V_1^{-1} R_{21}^{t_1}(-u-k) V_1 = V_1^{-1} R_{12}^{t_2}(-u-k) V_1 \tag{1.9}$$

$$= (V_2^{t_2})^{-1} R_{21}^{t_2}(-u-k) V_2^{t_2} = (V_2^{t_2})^{-1} R_{12}^{t_1}(-u-k) V_2^{t_2}. \tag{1.10}$$

$$M \equiv V^t V^{-1} \quad M^{-1} \equiv V(V^t)^{-1} \quad M^t \equiv (V^t)^{-1} V. \tag{1.11}$$

Notice, 黄色的部分是在没有 PT 对称性时用的。V 算符始终满足 $V_1 = V_1^{-1}$ (可以通过作用两次这个式子得到) 进一步可以得到 $M = M^t$。有 PT 对称性的时候 $V = V^t$.(这个证明在 ipad 上面)。在之后的所有性质推导过程中，都只使用没有 PT 对称性的后两个性质 (推导有些地方还没有改过来，考完期末需要改过来)

□

Proposition 5 (性质 4: Yang-Baxter 方程).

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v) \tag{1.12}$$

Proposition 6 (性质 5: Reflection Function).

$$R_{12}(u-v) K_1^-(u) R_{21}(u+v) K_2^-(v) = K_2^-(v) R_{12}(u+v) K_1^-(u) R_{21}(u-v) \tag{1.13}$$

Proposition 7 (性质 6: Dual Reflection Function).

$$\begin{aligned} &R_{21}(u-v) K_2^+(v) M_2^{-1} R_{12}(-u-v-2k) M_2 K_1^+(u) \\ &= K_1^+(u) M_2 R_{21}(-u-v-2k) M_2^{-1} K_2^+(v) R_{12}(u-v) \end{aligned} \tag{1.14}$$

Proposition 8 (性质 7: 混合关系 1).

$$(V_0^{-1})^{t_0} \left\{ K_0^+(-u-k) \right\}^{t_0} V_0 = \frac{1}{g(u)} tr_1 \left\{ R_{01}(0) R_{01}(2u) K_1^+(u) \right\} \tag{1.15}$$

Proposition 9 (性质 8: 混合关系 2).

$$tr_0 \left\{ R_{01}(0) R_{01}(2u) V_0 \left\{ K_0^-(-u-k) \right\}^{t_0} V_0^{t_0} \right\} = g(u) K_1^-(u) \tag{1.16}$$

1.1.2. 推论.

Corollary 1 (在一些退化点，可以导出投影算符).

$$R_{12}(\delta) = P_{12}^{(d)} \times S_{12} \tag{1.17}$$

于是，在退化点，R 算符满足

$$P_{12}^{(d)} R_{12}(\delta) = R_{12}(\delta) \tag{1.18}$$

□

下面是证明

证明. 这一点是考虑到了 R 矩阵的幺正性2:

$$R_{12}(u) R_{21}(-u) \sim f(u) \mathbb{I} \tag{1.19}$$

当 $f(\delta) = 0$ 的时候，

$$Img[R_{21}(-\delta)] \subset ker[R_{12}(\delta)] \tag{1.20}$$

认为:

$$ker[R_{12}(\delta)] \neq \emptyset \tag{1.21}$$

于是:

$$R_{12}(\delta) = P_{12}^{(d)} \times S \tag{1.22}$$

□

寻找投影算符 在退化点构建的投影算符满足:

$$P_{12}^{(d)} R_{12}(\delta) = R_{12}(\delta) \tag{1.23}$$

可以通过寻找 $R_{12}(\delta)$ 中线性无关的列向量 v_i 来寻找投影矩阵, 并且将投影矩阵定义为

$$P_{12}^{d} = \begin{pmatrix} v_1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} v_1^t \\ \vdots \\ 0 \end{pmatrix}. \tag{1.24}$$

Corollary 2 (VVP 关系). *由 Crossing Symmetry 4:*

$$\begin{cases} R_{12}(u) & = V_1^{-1} R_{12}^{t_2}(-u-k) V_1, \\ & = (V_2^{t_2})^{-1} R_{21}^{t_2}(-u-k) V_2^{t_2}. \end{cases} \tag{1.25}$$

得到

$$V_1^{-1} R_{12}(u) V_1 = V_2 R_{21}(u) V_2^{-1}. \tag{1.26}$$

由于在退化点, R operator 满足 $P_{12}R_{12} = R_{12}$, 于是

$$R_{12}(\delta) V_1 V_2 = V_1 V_2 R_{21}(\delta), \tag{1.27}$$

$$P_{12} V_1 V_2 R_{21}(\delta) = V_1 V_2 R_{21}(\delta), \tag{1.28}$$

$$P_{12} V_1 V_2 P_{21} = V_1 V_2 P_{21}. \tag{1.29}$$

Corollary 3 (RT 交换关系 1).

$$R_{10}(0) T_0(u) = T_1(u) R_{01}(0) \tag{1.30}$$

下面是证明

证明. 由于 R 算符的 Regularity 1:

$$R_{10}(0) = f(0)^{1/2} \mathcal{P}_{10} = R_{01}(0) \quad \mathcal{P}_{10}\mathcal{P}_{10} = \mathbb{I} \tag{1.31}$$

并且:

$$\mathcal{P}_{10} T_0(u) \mathcal{P}_{10} = T_1(u) \tag{1.32}$$

于是:

$$R_{10}(0) T_0(u) = T_1(u) R_{01}(0) \tag{1.33}$$

□

Corollary 4 (RT 交换关系 2).

$$\hat{T}_0(u)R_{01}(0) = R_{10}(0)\hat{T}_1(u) \tag{1.34}$$

证明方式和上面一样

Corollary 5 (RTT 关系). *定义 T 算符（monodromy 算符）:*

$$T_0(u) \equiv R_{01}(u-\theta_1)R_{02}(u-\theta_2)R_{03}(u-\theta_3)\cdots R_{0N}(u-\theta_N), \tag{1.35}$$

满足 RTT 关系

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v) \tag{1.36}$$

首先，可以证明，如果

$$\begin{aligned} R_{12}(u-v)X_1(u)X_2(v) = X_2(v)X_1(u)R_{12}(u-v) \\ R_{12}(u-v)Y_1(u)Y_2(v) = Y_2(v)Y_1(u)R_{12}(u-v), \end{aligned} \tag{1.37}$$

其中, X 作用在空间 $V \times V_x$，Y 作用在空间 $V \times V_y$，XY 作用在空间 $V \times V_x \times V_y$ 则:

$$R_{12}(u-v)(X(u)Y(u))_1(u)(X(u)Y(u))_2(v) = (X(u)Y(u))_2(v)(X(u)Y(u))_1(u)R_{12}(u-v) \tag{1.38}$$

证明.

$$\begin{aligned} R_{12}(u-v)\left(X_1(u)Y_1(u)\right)\left(X_2(v)Y_2(v)\right) &= R_{12}(u-v)X_1(u)X_2(v)Y_1(u)Y_2(v) \\ &= X_2(v)X_1(u)R_{12}(u-v)Y_1(u)Y_2(v) \\ &= X_2(v)X_1(u)Y_2(v)Y_1(u)R_{12}(u-v) \\ &= \left(X(v)Y(v)\right)_2\left(X(u)Y(u)\right)_1 R_{12}(u-v) \end{aligned} \tag{1.39}$$

□

由 Yang-Baxter 方程 (5):

$$R_{00'}(u-v)R_{0i}(u-\theta_i)R_{0'i}(v-\theta_i) = R_{0'i}(v-\theta_i)R_{0i}(u-\theta_i)R_{00'}(u-v) \tag{1.40}$$

相当于 $X^{(i)}(u) = R_{0i}(u-\theta_i)$ 作用于空间 $V \times V_i$，那么

$$\begin{aligned} T_0(u) &= R_{01}(u-\theta_1)R_{02}(u-\theta_2)R_{03}(u-\theta_3)\cdots R_{0N}(u-\theta_N) \\ &= X^{(1)}(u)X^{(2)}(u)\cdots X^{(N)}(u) \end{aligned} \tag{1.41}$$

满足 RTT 关系

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v) \tag{1.42}$$

Corollary 6 (RTT 关系 2). *定义 T 算符（Monodromy 算符）:*

$$\hat{T}_0(u) = R_{N0}(u+\theta_N)\cdots R_{20}(u+\theta_2)R_{10}(u+\theta_1), \tag{1.43}$$

满足 RTT 关系 2

$$R_{21}(u-v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{12}(u-v) \tag{1.44}$$

下面是证明:

证明. 由于 Yang-Baxter 方程5:

$$R_{12}(u-(u-v))R_{13}(u)R_{23}(u-v)=R_{23}(u-v)R_{13}(u)R_{12}(u-(u-v)) \tag{1.45}$$

方程左右换边，

$$R_{23}(u-v)R_{13}(u)R_{12}(u-(u-v))=R_{12}(u-(u-v))R_{13}(u)R_{23}(u-v) \tag{1.46}$$

指标替换:$(1,2,3)\to(3,1,2)$

$$R_{12}(u-v)R_{32}(u)R_{31}(u-(u-v))=R_{31}(u-(u-v))R_{32}(u)R_{12}(u-v) \tag{1.47}$$

再次指标替换:

$$R_{0'0}(u-v)R_{i0}(u+\theta_i)R_{i0'}(v+\theta_i)=R_{i0'}(v+\theta_i)R_{i0}(u+\theta_i)R_{0'0}(u-v) \tag{1.48}$$

如果有作用在 $V_x\times V$ 空间中的 X 算符与作用在 $V_y\times V$ 空间中的 Y 算符。他们满足:

$$\begin{aligned} R_{0'0}(u-v)X_0(u)X_{0'}(v)&=X_{0'}(v)X_0(u)R_{0'0}(u-v)\\ R_{0'0}(u-v)Y_0(u)Y_{0'}(v)&=Y_{0'}(v)Y_0(u)R_{0'0}(u-v)\end{aligned} \tag{1.49}$$

可以推导出 XY 也满足这个条件:

$$R_{0'0}(u-v)(XY)_0(u)(XY)_{0'}(v)=(XY)_{0'}(v)(XY)_0(u)R_{0'0}(u-v) \tag{1.50}$$

相当于 $X^{(i)}(u)=R_{i0}(u+\theta_i)$ 作用于空间 $V_i\times V$，那么

$$\begin{aligned} \hat{T}_0(u)&=R_{N0}(u+\theta_N)\cdots R_{20}(u+\theta_2)R_{10}(u+\theta_1)\\ &=X^{(N)}(u)X^{(N-1)}(u)\cdots X^{(1)}(u)\end{aligned} \tag{1.51}$$

满足 RTT 关系 2

$$R_{21}(u-v)\hat{T}_1(u)\hat{T}_2(v)=\hat{T}_2(v)\hat{T}_1(u)R_{12}(u-v) \tag{1.52}$$

□

Corollary 7 (RTT 关系 3). *定义:*

$$T_0(u)=R_{01}(u-\theta_1)R_{02}(u-\theta_2)R_{03}(u-\theta_3)\cdots R_{0N}(u-\theta_N), \tag{1.53}$$

$$\hat{T}_0(u)=R_{N0}(u+\theta_N)\cdots R_{20}(u+\theta_2)R_{10}(u+\theta_1), \tag{1.54}$$

那么:

$$\hat{T}_a(u)R_{ba}(2u+k)T_b(u+k)=T_b(u+k)R_{ba}(2u+k)\hat{T}_a(u) \tag{1.55}$$

更普遍一点

$$\hat{T}_a(v)R_{ba}(u+v)T_b(u)=T_b(u)R_{ba}(u+v)\hat{T}_a(v) \tag{1.56}$$

下面是证明:

证明. 采用归纳法证明，当 N=1 时。考虑 R 算符满足的 Yang-Baxter 方程5。

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{1.57}$$

改变指标与变量:

$$R_{1a}(v+\theta_1)R_{ba}(u+v)R_{b1}(u-\theta_1) = R_{b1}(u-\theta_1)R_{ba}(u+v)R_{1a}(v+\theta_1) \tag{1.58}$$

假设 $N-1$ 时候成立, 也就是:

$$\begin{aligned} &[R_{N-1,a}(v+\theta_{N-1})\cdots R_{1a}(v+\theta_1)]\,R_{ba}(u+v)\,[R_{b1}(u-\theta_1)\cdots R_{b,N-1}(u-\theta_{N-1})] \\ =&\,[R_{b,1}(u-\theta_1)\cdots R_{b,N-1}(u-\theta_{N-1})]\,R_{ba}(u+v)\,[R_{N-1,a}(v+\theta_{N-1})\cdots R_{1a}(v+\theta_1)] \end{aligned} \tag{1.59}$$

于是:

$$\begin{aligned} &R_{N,a}(v+\theta_N)\,[R_{N-1,a}(v+\theta_{N-1})\cdots]\,R_{ba}(u+v)\,[\cdots R_{b,N-1}(u-\theta_{N-1})]\,R_{b,N}(u-\theta_N) \\ &=[R_{N-1,a}(v+\theta_{N-1})\cdots]\,R_{N,a}(v+\theta_N)R_{ba}(u+v)R_{b,N}(u-\theta_N)\,[\cdots R_{b,N-1}(u-\theta_{N-1})] \\ &=[R_{N-1,a}(v+\theta_{N-1})\cdots]\,R_{b,N}(u-\theta_N)R_{ba}(u+v)R_{N,a}(v+\theta_N)\,[\cdots R_{b,N-1}(u-\theta_{N-1})] \\ &=[R_{b,1}(u-\theta_1)\cdots R_{b,N}(u-\theta_N)]\,R_{ba}(u+v)\,[R_{N,a}(v+\theta_N)R_{N-1,a}(v+\theta_{N-1})\cdots] \end{aligned} \tag{1.60}$$

也就是在 N 的时候，也成立。 □

Corollary 8 (单值算符的 Cross Symmetry 1).

$$T_0^{t_0}(-u-k) = V_0^{t_0}\hat{T}_0(u)(V_0^{-1})^{t_0} \tag{1.61}$$

下面是证明:

证明. 定义单值算符 T 为 (5, 6):

$$\begin{cases} T_0(u) = R_{01}(u-\theta_1)R_{02}(u-\theta_2)R_{03}(u-\theta_3)\cdots R_{0N}(u-\theta_N) \\ \hat{T}_0(u) = R_{N0}(u+\theta_N)\cdots R_{20}(u+\theta_2)R_{10}(u+\theta_1) \end{cases} \tag{1.62}$$

于是:

$$T_0^{t_0}(-u-k) = \{R_{01}(-u-k-\theta_1)R_{02}(-u-k-\theta_2)R_{03}(-u-k-\theta_3)\cdots R_{0N}(-u-k-\theta_N)\}^{t_0} \tag{1.63}$$

由于交叉幺正性 (4):

$$\Big\{ R_{12}(u) = V_1^{-1}R_{21}^{t_1}(-u-k)V_1 = (V_2^{t_2})^{-1}R_{21}^{t_2}(-u-k)V_2^{t_2} \tag{1.64}$$

于是:

$$\begin{aligned} \text{上式} &= \left\{V_0^{-1}R_{10}^{t_0}(u+\theta_1)V_0\;V_0^{-1}R_{20}^{t_0}(u+\theta_2)V_0\cdots V_0^{-1}R_{N0}^{t_0}(u+\theta_N)V_0\right\}^{t_0} \\ &= V_0^{t_0}\left\{R_{N0}(u+\theta_N)\cdots R_{10}(u+\theta_1)\right\}(V_0^{-1})^{t_0} \\ &= V_0^{t_0}\hat{T}_0(u)(V_0^{-1})^{t_0} \end{aligned} \tag{1.65}$$

□

Corollary 9 (单值算符的 Cross Symmetry 2).

$$\hat{T}_0^{t_0}(-u-k) = V_0T_0(u)V_0^{-1} \tag{1.66}$$

证明. 定义单值算符 T 为 (5, 6):

$$
\begin{cases}
T_0(u) = R_{01}(u-\theta_1)R_{02}(u-\theta_2)R_{03}(u-\theta_3)\cdots R_{0N}(u-\theta_N) \\
\hat{T}_0(u) = R_{N0}(u+\theta_N)\cdots R_{20}(u+\theta_2)R_{10}(u+\theta_1)
\end{cases}
\tag{1.67}
$$

于是:

$$
\hat{T}_0^{t_0}(-u-k) = \{R_{N0}(-u-k+\theta_N)\cdots R_{20}(-u-k+\theta_2)R_{10}(-u-k+\theta_1)\}^{t_0}
\tag{1.68}
$$

由于交叉幺正性 (4):

$$
\begin{cases}
R_{12}(u) = V_1^{-1}R_{21}^{t_1}(-u-k)V_1 = (V_2^{t_2})^{-1}R_{21}^{t_2}(-u-k)V_2^{t_2}
\end{cases}
\tag{1.69}
$$

于是:

$$
\begin{aligned}
\text{上式} &= \left\{(V_0^{t_0})^{-1}R_{0N}^{t_0}(u-\theta_N)V_0^{t_0}\cdots(V_0^{t_0})^{-1}R_{02}^{t_0}(u-\theta_2)V_0^{t_0}\ (V_0^{t_0})^{-1}R_{01}^{t_0}(u-\theta_1)V_0^{t_0}\right\}^{t_0} \\
&= V_0\left\{R_{01}(u-\theta_1)\cdots R_{0N}(u-\theta_N)\right\}V_0^{-1} \\
&= V_0T_0(u)V_0^{-1}
\end{aligned}
\tag{1.70}
$$

□

Corollary 10 (M_1M_2 和 R_{12} 的对易关系).

$$
M_2^{t_2}M_1^{t_1}R_{12}(u) = R_{12}(u)M_2^{t_2}M_1^{t_1}
\tag{1.71}
$$

$$
(M_2^{t_2})^{-1}R_{12}(u)M_2^{t_2} = M_1^{t_1}R_{12}(u)(M_1^{t_1})^{-1}
\tag{1.72}
$$

并且注意到 $M = M^t$, 所以这个式子就是 M 和 R 的对易关系

$$
M_1M_2R_{12}(u) = R_{12}(u)M_1M_2.
\tag{1.73}
$$

□

下面是证明:

证明. 由 Crossing-Symmetry，(4) 得到 M 算符的定义以及交叉对称性关系:

$$
\begin{cases}
R_{12}(u) = V_1^{-1}R_{12}^{t_2}(-u-k)V_1 = (V_2^{t_2})^{-1}R_{12}^{t_1}(-u-k)V_2^{t_2} \\
M = V^tV^{-1} \quad M^{-1} = V(V^t)^{-1} \quad M^t = (V^t)^{-1}V \quad (M^t)^{-1} = V^{-1}V^t \quad V = V^{-1},
\end{cases}
\tag{1.74}
$$

需要证明的式子，等价于证明:

$$
V_2^{-1}V_2^{t_2}R_{12}(u)(V_2^{t_2})^{-1}V_2 = (V_1^{t_1})^{-1}V_1R_{12}(u)V_1^{-1}V_1^{t_1}
\tag{1.75}
$$

由于 Crossing-Symmetry:

$$
\begin{cases}
R_{12}(u) = (V_2^{t_2})^{-1}R_{12}^{t_1}(-u-k)V_2^{t_2} \\
R_{12}(u) = V_1^{-1}R_{12}^{t_2}(-u-k)V_1
\end{cases}
\tag{1.76}
$$

相当于要证明:

$$
V_2^{-1}R_{12}^{t_1}(-u-k)V_2 = (V_1^{t_1})^{-1}R_{12}^{t_2}(-u-k)V_1^{t_1}
\tag{1.77}
$$

由 Crossing-Symmetry:

$$\begin{cases} R_{12}(u) = V_1^{-1} R_{12}^{t_2}(-u-k) V_1, \\ R_{12}(u) = (V_2^{t_2})^{-1} R_{12}^{t_1}(-u-k) V_2^{t_2}. \end{cases} \tag{1.78}$$

对上面的式子取 transpose，相当于要证明

$$V_2^{t_2} R_{12}^{t_2}(-u-k)(V_2^{t_2})^{-1} = V_1 R_{12}^{t_1}(-u-k) V_1^{-1}, \tag{1.79}$$

$$V_1^{-1} R_{12}^{t_2}(-u-k) V_1 = (V_2)^{t_2} R_{12}^{t_1}(-u-k)(V_2). \tag{1.80}$$

等式成立！ □

Corollary 11 (聚合 R 算符 1).

$$P_{12}^{(d)} R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)} = R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)} \tag{1.81}$$

下面是证明:

证明. 由于 Yang-Baxter 方程5得到:

$$R_{12}(\delta) R_{13}(u+\delta) R_{23}(u) = R_{23}(u) R_{13}(u+\delta) R_{12}(\delta) \tag{1.82}$$

根据投影算符的导出1，由于在退化点，R 算符可以写为:

$$R_{12}(\delta) = P_{12}^{(d)} S_{12} \tag{1.83}$$

于是:

$$P_{12}^{(d)} R_{12}(\delta) = P_{12}^{(d)} P_{12}^{(d)} S_{12} = P_{12}^{(d)} S_{12} = R_{12}(\delta) \tag{1.84}$$

那么:

$$\begin{aligned} P_{12}^{(d)} R_{12}(\delta) R_{13}(u+\delta) R_{23}(u) &= P_{12}^{(d)} R_{23}(u) R_{13}(u+\delta) R_{12}(\delta) \\ P_{12}^{(d)} R_{12}(\delta) R_{13}(u+\delta) R_{23}(u) &= R_{23}(u) R_{13}(u+\delta) R_{12}(\delta) \end{aligned} \tag{1.85}$$

相当于:

$$P_{12}^{(d)} R_{23}(u) R_{13}(u+\delta) R_{12}(\delta) = R_{23}(u) R_{13}(u+\delta) R_{12}(\delta) \tag{1.86}$$

$$P_{12}^{(d)} R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)} = R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)} \tag{1.87}$$

□

Corollary 12 (聚合后的 R 算符 1). *可以定义聚合后的 R matrix 为*

$$R_{<1,2>3} \equiv P_{12}^{(d)} R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)} \tag{1.88}$$

由前面的性质，以及原本的 R 满足 Yang-Baxter 方程，直接计算可以证明，聚合后的 R matrix 满足 Yang-Baxter 方程。

$$R_{<1,2>3}(u-v) R_{<1,2>4}(u) R_{3,4}(v) = R_{3,4}(v) R_{<1,2>4}(u) R_{<1,2>3}(u-v). \tag{1.89}$$

□

证明. 直接将聚合后的 R operator 的定义式带入

$$LHS = P_{12}^{(d)} R_{23}(u-v) R_{13}(u-v+\delta) P_{12}^{(d)}\ P_{12}^{(d)} R_{24}(u) R_{14}(u+\delta) P_{12}^{(d)}\ R_{34}(v). \tag{1.90}$$

利用聚合关系以及 Yang-Baxter 方程

$$\begin{cases} P_{12}^{(d)} R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)} = R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)}, \\ R_{13}(u-v+\delta) R_{14}(u+\delta) R_{34}(v) = R_{34}(v) R_{14}(u+\delta) R_{13}(u-v+\delta), \\ R_{23}(u-v) R_{24}(u) R_{34}(v) = R_{34}(v) R_{24}(u) R_{23}(u-v). \end{cases} \tag{1.91}$$

将左侧化简

$$\begin{aligned} LHS =& R_{23}(u-v) R_{13}(u-v+\delta) R_{24}(u) R_{14}(u+\delta) R_{34}(v) P_{12}^{(d)} && (1.92)\\ =& R_{23}(u-v) R_{24}(u) R_{13}(u-v+\delta) R_{14}(u+\delta) R_{34}(v) P_{12}^{(d)} && (1.93)\\ =& R_{23}(u-v) R_{24}(u) R_{34}(v) R_{14}(u+\delta) R_{13}(u-v+\delta) P_{12}^{(d)} && (1.94)\\ =& R_{34}(v) R_{24}(u) R_{23}(u-v) R_{14}(u+\delta) R_{13}(u-v+\delta) P_{12}^{(d)} && (1.95)\\ =& R_{34}(v) P_{12}^{(d)} R_{24}(u) R_{14}(u+\delta) P_{12}^{(d)}\ P_{12}^{(d)} R_{23}(u-v) R_{13}(u-v+\delta) P_{12}^{(d)} && (1.96)\\ =& R_{34}(v) R_{<1,2>4}(u) R_{<1,2>3}(u-v) = RHS. && (1.97) \end{aligned}$$

□

Corollary 13 (聚合 R 算符 2).

$$P_{21}^{(d)} R_{32}(u) R_{31}(u+\delta) P_{21}^{(d)} = R_{32}(u) R_{31}(u+\delta) P_{21}^{(d)} \tag{1.98}$$

下面是证明:

证明. 由 Yang-Baxter 方程5:

$$R_{12}(u) R_{13}(u+\delta) R_{23}(\delta) = R_{23}(\delta) R_{13}(u+\delta) R_{12}(u) \tag{1.99}$$

变量代换: $(1,2,3) \to (3,2,1)$

$$R_{32}(u) R_{31}(u+\delta) R_{21}(\delta) = R_{21}(\delta) R_{31}(u+\delta) R_{32}(u) \tag{1.100}$$

考虑到:

$$P_{21}^{(d)} R_{21}(\delta) = R_{21}(\delta) \tag{1.101}$$

$$P_{21}^{(d)} R_{32}(u) R_{31}(u+\delta) R_{21}(\delta) = R_{32}(u) R_{31}(u+\delta) R_{21}(\delta) \tag{1.102}$$

也就是:

$$P_{21}^{(d)} R_{32}(u) R_{31}(u+\delta) P_{21}^{(d)} = R_{32}(u) R_{31}(u+\delta) P_{21}^{(d)} \tag{1.103}$$

□

Corollary 14 (聚合后的 R 算符 2). *利用这个聚合性质，可以构造聚合后的 R 算符。构造为*

$$R_{3,<12>}(u) \equiv P_{21}^{(d)} R_{32}(u) R_{31}(u+\delta) P_{21}^{(d)}. \tag{1.104}$$

可以证明，这样构造的 R 算符也满足 Yang-Baxter 方程

$$R_{3,4}(u-v) R_{3,<12>}(u) R_{4,<12>}(v) = R_{4,<12>}(v) R_{3,<12>}(u) R_{3,4}(u-v) \tag{1.105}$$

□

证明. 将定义式带入

$$LHS = R_{34}(u-v)P_{21}^{(d)}R_{32}(u)R_{31}(u+\delta)P_{21}^{(d)}\ P_{21}^{(d)}R_{42}(v)R_{41}(v+\delta)P_{21}^{(d)} \tag{1.106}$$

□

利用 R matrix 在退化点满足的性质以及 R matrix 满足的 Yang-Baxter 方程:

$$\begin{cases} P_{21}^{(d)}R_{32}(u)R_{31}(u+\delta)P_{21}^{(d)} = R_{32}(u)R_{31}(u+\delta)P_{21}^{(d)}, \\ R_{34}(u-v)R_{32}(u)R_{42}(v) = R_{42}(v)R_{32}(u)R_{34}(u-v), \\ R_{34}(u-v)R_{31}(u+\delta)R_{41}(v+\delta) = R_{41}(v+\delta)R_{31}(u+\delta)R_{34}(u-v). \end{cases} \tag{1.107}$$

化简左式为

$$\begin{aligned} LHS =&(u-v)R_{32}(u)R_{31}(u+\delta)R_{42}(v)R_{41}(v+\delta)P_{21}^{(d)}R_{34} &(1.108)\\ =&R_{34}(u-v)R_{32}(u)R_{42}(v)R_{31}(u+\delta)R_{41}(v+\delta)P_{21}^{(d)} &(1.109)\\ =&R_{42}(v)R_{32}(u)R_{34}(u-v)R_{31}(u+\delta)R_{41}(v+\delta)P_{21}^{(d)} &(1.110)\\ =&R_{42}(v)R_{32}(u)R_{41}(v+\delta)R_{31}(u+\delta)R_{34}(u-v)P_{21}^{(d)} &(1.111)\\ =&P_{21}^{(d)}R_{42}(v)R_{41}(v+\delta)P_{21}^{(d)}\ P_{21}^{(d)}R_{32}(u)R_{31}(u+\delta)P_{21}^{(d)}R_{34}(u-v) &(1.112)\\ =&R_{4<1,2>}(v)R_{3<1,2>}(u)R_{34}(u-v) = RHS. &(1.113) \end{aligned}$$

Corollary 15 (聚合后的 V 算符). *Crossing symmetry* 本来有四个式子，但是在聚合操作之后，*PT symmetry* 无法再满足，于是 *Crossing Symmetry* 变为了两个

$$\begin{aligned} R_{12}(u) =&V_1^{-1}R_{12}^{t_2}(-u-k)V_1 &(1.114)\\ =&(V_2^{t_2})^{-1}R_{12}^{t_1}(-u-k)V_2^{t_2} &(1.115) \end{aligned}$$

推广为聚合后的 *Crossing symmetry,*

$$\begin{aligned} R_{<12>3}(u) =& V_{<12>}^{-1}R'^{t_3}_{<12>3}(-u-k)V_{<12>} &(1.116)\\ =& (V_3^{t_3})^{-1}R^{t_1t_2}_{<12>3}(-u-k)V_3^{t_3}. &(1.117) \end{aligned}$$

其中的另一种 R matrix 定义为 (这两种 R matrix 一般可以通过 Gauge Transformation 联系在一起，这个词好奇怪，为什么叫做 gauge transformation 呢？)

$$R'_{<12>3}(u) = R_{<21>3}(u-\delta), \tag{1.118}$$

证明. 1) 考虑到聚合后的 R matrix 的定义式:

$$\begin{cases} R_{<12>3} = P_{12}^{(d)}R_{23}(u)R_{13}(u+\delta)P_{12}^{(d)} \\ R'_{<12>3}(-u-k) = P_{21}^{(d)}R_{13}(-u-k-\delta)R_{23}(-u-k)P_{21}^{(d)}. \end{cases} \tag{1.119}$$

直接带入

$$\begin{aligned} RHS1 =&V_{<12>}^{-1}R'^{t_3}_{<12>3}(-u-k)V_{<12>} &(1.120)\\ =& V_{<12>}^{-1}\left\{P_{21}^{(d)}R_{13}(-u-k-\delta)R_{23}(-u-k)P_{21}^{(d)}\right\}^{t_3}V_{<12>} &(1.121)\\ =& V_{<12>}^{-1}\ P_{21}^{(d)}R_{13}^{t_3}(-u-k)R_{23}^{t_3}(-u-k-\delta)P_{21}^{(d)}\ V_{<12>} &(1.122) \end{aligned}$$

左侧

$$LHS = P_{12}^{(d)} R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)}. \tag{1.123}$$

利用 Crossing Symmetry

$$\begin{cases} R_{23}(u) = V_2^{-1} R_{23}^{t_3}(-u-k) V_2, \\ R_{13}(u+\delta) = V_1^{-1} R_{13}^{t_3}(-u-\delta-k) V_1. \end{cases} \tag{1.124}$$

化简为

$$\begin{aligned} LHS =& P_{12}^{(d)} V_2^{-1} R_{23}^{t_3}(-u-k) V_2 V_1^{-1} R_{13}^{t_3}(-u-\delta-k) V_1 P_{12}^{(d)}, & (1.125)\\ =& P_{12}^{(d)} V_2^{-1} V_1^{-1} R_{23}^{t_3}(-u-k) R_{13}^{t_3}(-u-\delta-k) V_2 V_1 P_{12}^{(d)} & (1.126) \end{aligned}$$

考虑到 VVP 关系

$$\begin{cases} P_{12} V_1 V_2 P_{21} \quad = V_1 V_2 P_{21}. \end{cases} \tag{1.127}$$

对比 RHS 以及 LHS, 可以定义

$$P_{21}^{(d)} V_{<12>} = V_2 V_1 P_{12}^{(d)} \tag{1.128}$$

$$V_{<12>} \equiv P_{21}^{(d)} V_2 V_1 P_{12}^{(d)} = V_2 V_1 P_{12}^{(d)}, \tag{1.129}$$

$$V_{<12>}^{-1} \equiv P_{21}^{(d)} V_2^{-1} V_1^{-1} P_{12}^{(d)} \tag{1.130}$$

2) 对于第二个等式，同样，先利用聚合后的 R matrix 的定义式

$$\begin{cases} R_{<12>3}(-u-k) = P_{12}^{(d)} R_{23}(-u-k) R_{13}(-u-k+\delta) P_{12}^{(d)}. \end{cases} \tag{1.131}$$

于是，右侧化简为

$$\begin{aligned} RHS2 =& (V_3^{t_3})^{-1} \left\{ P_{12}^{(d)} R_{23}(-u-k) R_{13}(-u-k+\delta) P_{12}^{(d)} \right\}^{t_1 t_2} V_3^{t_3} & (1.132)\\ =& (V_3^{t_3})^{-1} P_{12}^{(d)} R_{23}^{t_2}(-u-k) R_{13}^{t_1}(-u-k+\delta) P_{12}^{(d)} V_3^{t_3} & (1.133)\\ =& P_{12}^{(d)} (V_3^{t_3})^{-1} R_{23}^{t_2}(-u-k) R_{13}^{t_1}(-u-k+\delta) V_3^{t_3} P_{12}^{(d)}. & (1.134) \end{aligned}$$

利用 R matrix 的交叉对称性

$$\begin{cases} R_{23}(u) = (V_3^{t_3})^{-1} R_{23}^{t_2}(-u-k) V_3^{t_3}, \\ R_{13}(u+\delta) = (V_3^{t_3})^{-1} R_{13}^{t_1}(-u-\delta-k) V_3^{t_3}. \end{cases} \tag{1.135}$$

左侧化简为

$$\begin{aligned} LHS =& P_{12}^{(d)} R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)}, & (1.136)\\ =& P_{12}^{(d)} (V_3^{t_3})^{-1} R_{23}^{t_2}(-u-k) V_3^{t_3} \, (V_3^{t_3})^{-1} R_{13}^{t_1}(-u-\delta-k) V_3^{t_3} P_{12}^{(d)}, & (1.137)\\ =& P_{12}^{(d)} (V_3^{t_3})^{-1} R_{23}^{t_2}(-u-k) R_{13}^{t_1}(-u-\delta-k) V_3^{t_3} P_{12}^{(d)}, & (1.138) \end{aligned}$$

直接对比得到

$$LHS = RHS2. \tag{1.139}$$

□

Corollary 16 (聚合 K^- 算符).

$$P_{12}^{(d)}K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)P_{21}^{(d)} = K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)P_{21}^{(d)} \tag{1.140}$$

下面是证明:

证明. 考虑反射方程6

$$R_{12}(\delta)K_1^-(u+\delta)R_{21}(2u+\delta)K_2^-(u) = K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)R_{21}(\delta) \tag{1.141}$$

于是:

$$P_{12}^{(d)}K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v) = K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)R_{21}(\delta) \tag{1.142}$$

也就是:

$$P_{12}^{(d)}K_2^-(v)R_{12}(u+v)K_1^-(u)P_{21}^{(d)} = K_2^-(v)R_{12}(u+v)K_1^-(u)P_{21}^{(d)} \tag{1.143}$$

□

Corollary 17 (聚合后的 K^- 算符). *通过 K^- 算符的聚合关系，可以构建聚合后的 K^- 算符为*

$$K^-_{<12>}(u) \equiv P_{12}^{(d)}K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)P_{21}^{(d)} \tag{1.144}$$

并且，可以证明，聚合后的 K^- 算符满足 Reflecting Equation.

$$R_{<12>3}(u-v)K^-_{<12>}(u)R_{3<12>}(u+v)K_3^-(v) = K_3^-(v)R_{<12>3}(u+v)K^-_{<12>}(u)R_{3<12>}(u-v). \tag{1.145}$$

□

证明. 将聚合后的 K^- 算符以及 R 算符的性质带入

$$\begin{cases} R_{<12>3}(u-v) = P_{12}^{(d)}R_{23}(u-v)R_{13}(u-v+\delta)P_{12}^{(d)} \\ R_{3,<12>}(u+v) = P_{21}^{(d)}R_{32}(u+v)R_{31}(u+v+\delta)P_{21}^{(d)} \end{cases} \tag{1.146}$$

左式写为

$$\begin{aligned} LHS =& P_{12}^{(d)}R_{23}(u-v)R_{13}(u-v+\delta)P_{12}^{(d)}\ P_{12}^{(d)}K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)P_{21}^{(d)} && (1.147)\\ & P_{21}^{(d)}R_{32}(u+v)R_{31}(u+v+\delta)P_{21}^{(d)}\ K_3^-(v) && (1.148)\end{aligned}$$

利用 R 算符以及 K 算符的聚合性质，反射方程，Yang-Baxter 方程

(1.149)

$$\begin{cases} P_{12}^{(d)}K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)P_{21}^{(d)} = K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)P_{21}^{(d)} \\ P_{21}^{(d)}R_{32}(u)R_{31}(u+\delta)P_{21}^{(d)} = R_{32}(u)R_{31}(u+\delta)P_{21}^{(d)} \\ P_{12}^{(d)}K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)P_{21}^{(d)} = K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)P_{21}^{(d)} \\ R_{13}(u-v+\delta)R_{12}(2u+\delta)R_{32}(u+v) = R_{32}(u+v)R_{12}(2u+\delta)R_{13}(u-v+\delta) \\ R_{13}(u-v+\delta)K_1^-(u+\delta)R_{31}(u+v+\delta)K_3^-(v) = K_3^-(v)R_{13}(u+v+\delta)K_1^-(u+\delta)R_{31}(u+\delta-v) \end{cases}$$

来化简左式

$$LHS = R_{23}(u-v)R_{13}(u-v+\delta)K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta) \tag{1.150}$$

$$R_{32}(u+v)R_{31}(u+v+\delta)K_3^-(v)\ P_{21}^{(d)} \tag{1.151}$$
$$=R_{23}(u-v)K_2^-(u)R_{13}(u-v+\delta)R_{12}(2u+\delta)R_{32}(u+v) \tag{1.152}$$
$$K_1^-(u+\delta)R_{31}(u+v+\delta)K_3^-(v)\ P_{21}^{(d)} \tag{1.153}$$
$$=R_{23}(u-v)K_2^-(u)R_{32}(u+v)R_{12}(2u+\delta)R_{13}(u-v+\delta) \tag{1.154}$$
$$K_1^-(u+\delta)R_{31}(u+v+\delta)K_3^-(v)\ P_{21}^{(d)} \tag{1.155}$$
$$=R_{23}(u-v)K_2^-(u)R_{32}(u+v)R_{12}(2u+\delta) \tag{1.156}$$
$$K_3^-(v)R_{13}(u+v+\delta)K_1^-(u+\delta)R_{31}(u+\delta-v)\ P_{21}^{(d)} \tag{1.157}$$

利用反射方程以及 Yang-Baxter 方程做进一步化简

$$\begin{cases} R_{23}(u-v)K_2^-(u)R_{32}(u+v)K_3^-(v) = K_3^-(v)R_{23}(u+v)K_2^-(u)R_{32}(u-v) \\ R_{32}(u-v)R_{12}(2u+\delta)R_{13}(u+v+\delta) = R_{13}(u+v+\delta)R_{12}(2u+\delta)R_{32}(u-v) \end{cases} \tag{1.158}$$

得到

$$LHS = K_3^-(v)R_{23}(u+v)R_{13}(u+v+\delta)K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta) \tag{1.159}$$
$$R_{32}(u-v)R_{31}(u+\delta-v)P_{21}^{(d)} \tag{1.160}$$
$$=K_3^-(v)R_{<1,2>3}(u+v)K_{<12>}^-(u)R_{3<12>}(u-v) = RHS. \tag{1.161}$$

□

Corollary 18 (聚合 K^+ 算符).

$$\begin{aligned} &P_{21}^{(d)}K_1^+(u+\delta)M_2R_{21}(-2u-2k-\delta)M_2^{-1}K_2^+(u)P_{12}^{(d)} \\ &\quad = K_1^+(u+\delta)M_2R_{21}(-2u-2k-\delta)M_2^{-1}K_2^+(u)P_{12}^{(d)} \end{aligned} \tag{1.162}$$

下面是证明:

证明. 根据对偶反射方程7

$$\begin{aligned} &R_{21}(u-v)K_2^+(v)M_2^{-1}R_{12}(-u-v-2k)M_2K_1^+(u) \\ &\quad = K_1^+(u)M_2R_{21}(-u-v-2k)M_2^{-1}K_2^+(v)R_{12}(u-v) \end{aligned} \tag{1.163}$$

$$\begin{aligned} &K_1^+(u)M_2R_{21}(-u-v-2k)M_2^{-1}K_2^+(v)R_{12}(u-v) \\ &\quad = P_{21}^{(d)}K_1^+(u)M_2R_{21}(-u-v-2k)M_2^{-1}K_2^+(v)R_{12}(u-v) \end{aligned} \tag{1.164}$$

$$\begin{aligned} &P_{21}^{(d)}K_1^+(u+\delta)M_2R_{21}(-2u-2k-\delta)M_2^{-1}K_2^+(u)P_{12}^{(d)} \\ &\quad = K_1^+(u+\delta)M_2R_{21}(-2u-2k-\delta)M_2^{-1}K_2^+(u)P_{12}^{(d)} \end{aligned} \tag{1.165}$$

□

Corollary 19 (聚合后的 K^+ 算符). *可以定义聚合后的高维空间的 K^+ 算符为*

$$K_{<12>}^+(u) \equiv P_{21}^{(d)}K_1^+(u+\delta)M_2R_{21}(-2u-2k-\delta)M_2^{-1}K_2^+(u)P_{12}^{(d)}. \tag{1.166}$$

这样定义的 K^+ 算符满足对偶反射方程 (注意是带有 prime 的 R 算符, 它的定义在聚合后的对偶反射方程部分)

$$R_{3<12>}(u-v)K_3^+(v)M_3^{-1}R'_{<12>3}(-u-v-2k)M_3K_{<12>}^+(u) \tag{1.167}$$

$$= K^{+}_{<12>}(u) M_3 R'_{3<12>}(-u-v-2k) M_3^{-1} K_3^{+}(v) R_{<12>3}(u-v). \tag{1.168}$$

在证明的时候需要反复用到性质 $M_1 M_2 R_{12} = R_{12} M_1 M_2$.

□

Corollary 20 (聚合 T 算符 1). *定义 T 算符:*

$$T_0(u) = R_{01}(u-\theta_1) R_{02}(u-\theta_2) R_{03}(u-\theta_3) \cdots R_{0N}(u-\theta_N), \tag{1.169}$$

满足:

$$P^{(d)}_{12} T_2(u) T_1(u+\delta) P^{(d)}_{12} = T_2(u) T_1(u+\delta) P^{(d)}_{12} \tag{1.170}$$

□

下面是证明:

证明. 由 RTT 关系 15

$$R_{12}(u-v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u-v) \tag{1.171}$$

变化变量后得到:

$$R_{12}(\delta) T_1(u+\delta) T_2(u) = T_2(u) T_1(u+\delta) R_{12}(\delta) \tag{1.172}$$

由投影算符的性质1:

$$R_{12}(\delta) = P_{12} \times S_{12} \quad P^{(d)}_{12} R_{12}(\delta) = R_{12}(\delta) \tag{1.173}$$

于是:

$$P^{(d)}_{12} T_2(u) T_1(u+\delta) R_{12}(\delta) = T_2(u) T_1(u+\delta) R_{12}(\delta) \tag{1.174}$$

于是:

$$P^{(d)}_{12} T_2(u) T_1(u+\delta) P^{(d)}_{12} = T_2(u) T_1(u+\delta) P^{(d)}_{12} \tag{1.175}$$

□

Corollary 21 (聚合 T 算符 2). *定义 T 算符:*

$$\hat{T}_0(u) = R_{N0}(u+\theta_N) \cdots R_{20}(u+\theta_2) R_{10}(u+\theta_1), \tag{1.176}$$

满足:

$$P^{(d)}_{21} \hat{T}_2(u) \hat{T}_1(u+\delta) P^{(d)}_{21} = \hat{T}_2(u) \hat{T}_1(u+\delta) P^{(d)}_{21} \tag{1.177}$$

下面是证明:

证明. 由 RTT 关系 2，6:

$$R_{21}(\delta) \hat{T}_1(u+\delta) \hat{T}_2(u) = \hat{T}_2(u) \hat{T}_1(u+\delta) R_{12}(\delta) \tag{1.178}$$

由投影算符的性质1:

$$P^{(d)}_{21} R_{21}(\delta) = R_{21}(\delta) \quad R_{21}(\delta) = P^{(d)}_{21} \times S_{21} \tag{1.179}$$

于是:

$$P_{21}\hat{T}_2(u)\hat{T}_1(u+\delta)R_{12}(\delta)=\hat{T}_2(u)\hat{T}_1(u+\delta)R_{12}(\delta) \tag{1.180}$$

$$P_{12}^{(d)}P_{21}\hat{T}_2(u)\hat{T}_1(u+\delta)P_{12}^{(d)}=\hat{T}_2(u)\hat{T}_1(u+\delta)P_{12}^{(d)} \tag{1.181}$$

□

Corollary 22 (Crossing-Unitary). *–Old Version,* 新版在 *ipad* 上, 这里证明用的 *Crossing Symmetry* 和之前定义的没有 *PT* 对称性的 *Crossing Symmetry* 不一样。总之，没有 *PT* 对称性时候也是可以证明这个性质的。

$$R_{12}^{t_1}(u)M_1R_{21}^{t_1}(-u-2k)M_1^{-1}=R_{12}^{t_2}(u)M_2^{-1}R_{21}^{t_2}(-u-2k)M_2=f(u+k) \tag{1.182}$$

下面是证明，首先证明第一项和第三项相等

证明. 由 R 算符的 Crossing-Symmetry 的性质4中的第一个等式:

$$\begin{cases}R_{12}(u)=V_1^{-1}R_{21}^{t_1}(-u-k)V_1=(V_2^{t_2})^{-1}R_{21}^{t_2}(-u-k)V_2^{t_2}\\ M=V^tV^{-1}\quad M^{-1}=V(V^t)^{-1}\end{cases} \tag{1.183}$$

得到

$$\begin{cases}R_{12}(u)=V_1^{-1}R_{21}^{t_1}(-u-k)V_1\\ R_{21}^{t_1}(-u-k)=V_1R_{12}(u)V_1^{-1}\end{cases} \tag{1.184}$$

上面的第一式取转置，第二式平移自变量:

$$\begin{cases}R_{12}^{t_1}(u)=V_1^{t_1}R_{21}(-u-k)(V_1^{t_1})^{-1}\\ R_{21}^{t_1}(-u-2k)=V_1R_{12}(u+k)V_1^{-1}\end{cases} \tag{1.185}$$

于是:

$$\begin{aligned}&R_{12}^{t_1}(u)M_1R_{21}^{t_1}(-u-2k)M_1^{-1}\\ &\quad=V_1^{t_1}R_{21}(-u-k)(V_1^{t_1})^{-1}\ M_1\ V_1R_{12}(u+k)V_1^{-1}M_1^{-1}\\ &\quad=V_1^{t_1}R_{21}(-u-k)(V_1^{t_1})^{-1}\ V_1^{t_1}V_1^{-1}\ V_1R_{12}(u+k)V_1^{-1}V_1(V_1^{t_1})^{-1}\\ &\quad=V_1^{t_1}R_{21}(-u-k)R_{12}(u+k)(V_1^{t_1})^{-1}\end{aligned} \tag{1.186}$$

考虑到 R 算符的 Unitar 性质2:

$$\Big\{R_{12}(u)R_{21}(-u)\sim f(u)\mathbb{I} \tag{1.187}$$

于是

$$\text{上式}=V_1^{t_1}f(u+k)(V_1^{t_1})^{-1}=f(u+k) \tag{1.188}$$

□

下面证明，第二项和第三项相等:

证明. 由 Crossing-Symmetry 4:

$$\begin{cases} R_{12}(u) = V_1^{-1}R_{21}^{t_1}(-u-k)V_1 = (V_2^{t_2})^{-1}R_{21}^{t_2}(-u-k)V_2^{t_2} \\ M = V^tV^{-1} \quad M^{-1} = V(V^t)^{-1} \end{cases} \tag{1.189}$$

得到:

$$\begin{cases} R_{12}^{t_2}(u) = V_2R_{21}(-u-k)V_2^{-1} \\ R_{21}^{t_2}(-u-2k) = V_2^{t_2}R_{12}(u+k)(V_2^{t_2})^{-1} \end{cases} \tag{1.190}$$

于是:

$$\begin{aligned} &R_{12}^{t_2}(u)M_2^{-1}R_{21}^{t_2}(-u-2k)M_2 \\ &\quad = V_2R_{21}(-u-k)V_2^{-1}\ M_2^{-1}\ V_2^{t_2}R_{12}(u+k)(V_2^{t_2})^{-1}\ M_2 \\ &\quad = V_2R_{21}(-u-k)V_2^{-1}\ V_2(V_2^{t_2})^{-1}\ V_2^{t_2}R_{12}(u+k)(V_2^{t_2})^{-1}\ V_2^{t_2}V_2^{-1} \\ &\quad = f(u+k) \end{aligned} \tag{1.191}$$

□

Corollary 23 (单值算符生成投影算符).

$$T_a(\theta_j)T_b(\theta_j+\delta) = P_{ba}^{(d)}T_a(\theta_j)T_b(\theta_j+\delta) \tag{1.192}$$

下面是证明:

证明.

$$\begin{aligned} T_a(\theta_j)T_b(\theta_j+\delta) =& R_{a1}(\theta_j-\theta_1)\cdots R_{aj}(0)\cdots R_{aN}(\theta_j-\theta_N) \\ & R_{b1}(\theta_j+\delta-\theta_1)\cdots R_{bj}(\delta)\cdots R_{bN}(\theta_j+\delta-\theta_N) \end{aligned} \tag{1.193}$$

由 R 算符的 Unitary 性质2:

$$R_{aj}(0)R_{ja}(0) = f(0) \tag{1.194}$$

由 R 算符的 Regularity 性质1:

$$R_{aj}(0) = f(0)^{\frac{1}{2}}\mathcal{P}_{aj} \tag{1.195}$$

于是:

$$\begin{aligned} \text{上式} =& R_{a1}(\theta_j-\theta_1)\cdots R_{a,j-1}(\theta_j-\theta_{j-1})\mathcal{P}_{aj}f(0)^{\frac{1}{2}}R_{a,j+1}(\theta_j-\theta_{j+1})\cdots R_{aN}(\theta_j-\theta_N) \\ & R_{b1}(\theta_j+\delta-\theta_1)\cdots R_{bj}(\delta)\cdots R_{bN}(\theta_j+\delta-\theta_N) \\ =& R_{a1}(\theta_j-\theta_1)\cdots R_{a,j-1}(\theta_j-\theta_{j-1})\mathcal{P}_{aj}f(0)^{\frac{1}{2}}R_{a,j+1}(\theta_j-\theta_{j+1})\cdots R_{aN}(\theta_j-\theta_N) \\ & R_{b1}(\theta_j+\delta-\theta_1)\cdots P_{bj}^{(d)}S_{bj}\cdots R_{bN}(\theta_j+\delta-\theta_N) \\ =& R_{a1}(\theta_j-\theta_1)\cdots R_{a,j-1}(\theta_j-\theta_{j-1})\mathcal{P}_{aj}f(0)^{\frac{1}{2}}R_{a,j+1}(\theta_j-\theta_{j+1})\cdots R_{aN}(\theta_j-\theta_N) \\ & R_{b1}(\theta_j+\delta-\theta_1)\cdots P_{bj}^{(d)}S_{bj}R_{aj}(0)R_{ja}(0)f(0)^{-1}\cdots R_{bN}(\theta_j+\delta-\theta_N) \\ =& R_{a1}(\theta_j-\theta_1)\cdots R_{a,j-1}(\theta_j-\theta_{j-1})\mathcal{P}_{aj}f(0)^{\frac{1}{2}}R_{a,j+1}(\theta_j-\theta_{j+1})\cdots R_{aN}(\theta_j-\theta_N) \\ & R_{b1}(\theta_j+\delta-\theta_1)\cdots P_{bj}^{(d)}S_{bj}f(0)^{1/2}\mathcal{P}_{aj}R_{ja}(0)f(0)^{-1}\cdots R_{bN}(\theta_j+\delta-\theta_N) \\ =& R_{a1}(\theta_j-\theta_1)\cdots R_{a,j-1}(\theta_j-\theta_{j-1})\mathcal{P}_{aj}R_{a,j+1}(\theta_j-\theta_{j+1})\cdots R_{aN}(\theta_j-\theta_N) \\ & R_{b1}(\theta_j+\delta-\theta_1)\cdots P_{bj}^{(d)}S_{bj}\mathcal{P}_{aj}R_{ja}(0)\cdots R_{bN}(\theta_j+\delta-\theta_N) \end{aligned} \tag{1.196}$$

也就是:

$$
\begin{aligned}
\text{上式}=&R_{a1}(\theta_j-\theta_1)\cdots R_{a,j-1}(\theta_j-\theta_{j-1})R_{j,j+1}(\theta_j-\theta_{j+1})\cdots R_{jN}(\theta_j-\theta_N)\\
&R_{b1}(\theta_j+\delta-\theta_1)\cdots P^{(d)}_{ba}S_{ba}R_{ja}(0)\cdots R_{bN}(\theta_j+\delta-\theta_N)\quad \text{交换了 a,j 指标}\\
=&R_{j,j+1}(\theta_j-\theta_{j+1})\cdots R_{jN}(\theta_j-\theta_N)\\
&R_{a1}(\theta_j-\theta_1)\cdots R_{a,j-1}(\theta_j-\theta_{j-1})R_{b1}(\theta_j+\delta-\theta_1)\cdots R_{b,j-1}(\theta_j+\delta-\theta_{j-1})P^{(d)}_{ba}\\
&S_{ba}R_{ja}(0)\cdots R_{bN}(\theta_j+\delta-\theta_N)\quad \text{交换了顺序}
\end{aligned}
\tag{1.197}
$$

由于聚合 T 矩阵性质20:

$$T_0(u)=R_{01}(u-\theta_1)R_{02}(u-\theta_2)R_{03}(u-\theta_3)\cdots R_{0,j-1}(u-\theta_{j-1}),\tag{1.198}$$

满足:

$$P^{(d)}_{ba}T_a(u)T_b(u+\delta)P^{(d)}_{ba}=T_a(u)T_b(u)P^{(d)}_{ba}\tag{1.199}$$

于是:

$$
\begin{aligned}
\text{上式}=&R_{j,j+1}(\theta_j-\theta_{j+1})\cdots R_{jN}(\theta_j-\theta_N)\\
&P^{(d)}_{ba}R_{a1}(\theta_j-\theta_1)\cdots R_{a,j-1}(\theta_j-\theta_{j-1})R_{b1}(\theta_j+\delta-\theta_1)\cdots R_{b,j-1}(\theta_j+\delta-\theta_{j-1})P^{(d)}_{ba}\\
&S_{ba}R_{ja}(0)\cdots R_{bN}(\theta_j+\delta-\theta_N)\\
=&P^{(d)}_{ba}R_{j,j+1}(\theta_j-\theta_{j+1})\cdots R_{jN}(\theta_j-\theta_N)\\
&R_{a1}(\theta_j-\theta_1)\cdots R_{a,j-1}(\theta_j-\theta_{j-1})R_{b1}(\theta_j+\delta-\theta_1)\cdots R_{b,j-1}(\theta_j+\delta-\theta_{j-1})P^{(d)}_{ba}\\
&S_{ba}R_{ja}(0)\cdots R_{bN}(\theta_j+\delta-\theta_N)
\end{aligned}
\tag{1.200}
$$

也就是:

$$T_a(\theta_j)T_b(\theta_j+\delta)=P^{(d)}_{ba}T_a(\theta_j)T_b(\theta_j+\delta)\tag{1.201}$$

□

Corollary 24 (单值算符生成投影算符 2).

$$\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta)=P^{(d)}_{ab}\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta)\tag{1.202}$$

下面是证明:

证明.

$$
\begin{aligned}
\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta)=&R_{Na}(-\theta_j+\theta_N)\cdots R_{ja}(0)\cdots R_{1a}(-\theta_j+\theta_1)\\
&R_{Nb}(-\theta_j+\theta_N+\delta)\cdots R_{jb}(\delta)\cdots R_{1b}(-\theta_j+\theta_1+\delta)
\end{aligned}
\tag{1.203}
$$

由 R 算符的 Unitary 性质2:

$$R_{ja}(0)R_{aj}(0)=f(0)\tag{1.204}$$

由 R 算符的 Regularity 性质1:

$$R_{ja}(0)=f(0)^{\frac{1}{2}}\mathcal{P}_{ja}\tag{1.205}$$

于是:

$$\text{上式}=R_{Na}(-\theta_j+\theta_N)\cdots R_{j+1,a}(-\theta_j+\theta_{j+1})\mathcal{P}_{ja}f(0)^{\frac{1}{2}}R_{j-1,a}(-\theta_j+\theta_{j-1})\cdots R_{1a}(-\theta_j+\theta_1)\tag{1.206}$$

$$
\begin{align}
& R_{Nb}(-\theta_j+\theta_N+\delta)\cdots R_{jb}(\delta)\cdots R_{1b}(-\theta_j+\theta_1+\delta) \tag{1.207}\\
=& R_{Na}(-\theta_j+\theta_N)\cdots R_{j+1,a}(-\theta_j+\theta_{j+1})\mathcal{P}_{ja}f(0)^{\frac{1}{2}}R_{j-1,a}(-\theta_j+\theta_{j-1})\cdots R_{1a}(-\theta_j+\theta_1) \tag{1.208}\\
& R_{Nb}(-\theta_j+\theta_N+\delta)\cdots P^{(d)}_{jb}S_{jb}\cdots R_{1b}(-\theta_j+\theta_1+\delta) \tag{1.209}\\
=& R_{Na}(-\theta_j+\theta_N)\cdots R_{j+1,a}(-\theta_j+\theta_{j+1})\mathcal{P}_{ja}f(0)^{\frac{1}{2}}R_{j-1,a}(-\theta_j+\theta_{j-1})\cdots R_{1a}(-\theta_j+\theta_1) \tag{1.210}\\
& R_{Nb}(-\theta_j+\theta_N+\delta)\cdots P^{(d)}_{jb}S_{jb}R_{ja}(0)R_{aj}(0)f(0)^{-1}\cdots R_{1b}(-\theta_j+\theta_1+\delta) \tag{1.211}\\
=& R_{Na}(-\theta_j+\theta_N)\cdots R_{j+1,a}(-\theta_j+\theta_{j+1})\mathcal{P}_{ja}f(0)^{\frac{1}{2}}R_{j-1,a}(-\theta_j+\theta_{j-1})\cdots R_{1a}(-\theta_j+\theta_1) \tag{1.212}\\
& R_{Nb}(-\theta_j+\theta_N+\delta)\cdots P^{(d)}_{jb}S_{jb}f(0)^{1/2}\mathcal{P}_{ja}(0)R_{aj}(0)f(0)^{-1}\cdots R_{1b}(-\theta_j+\theta_1+\delta) \tag{1.213}\\
=& R_{Na}(-\theta_j+\theta_N)\cdots R_{j+1,a}(-\theta_j+\theta_{j+1})R_{j-1,j}(-\theta_j+\theta_{j-1})\cdots R_{1j}(-\theta_j+\theta_1) \tag{1.214}\\
& R_{Nb}(-\theta_j+\theta_N+\delta)\cdots P^{(d)}_{ab}S_{ab}R_{aj}(0)\cdots R_{1b}(-\theta_j+\theta_1+\delta) \tag{1.215}\\
& \tag{1.216}
\end{align}
$$

也就是:

$$
\begin{split}
\text{上式}=&R_{j-1,j}(-\theta_j+\theta_{j-1})\cdots R_{1j}(-\theta_j+\theta_1)\\
&R_{Na}(-\theta_j+\theta_N)\cdots R_{j+1,a}(-\theta_j+\theta_{j+1})R_{Nb}(-\theta_j+\theta_N+\delta)\cdots R_{j+1,b}(-\theta_j+\theta_{j+1}+\delta)P^{(d)}_{ab}\\
&S_{ab}R_{aj}(0)\cdots R_{1b}(-\theta_j+\theta_1+\delta)\quad\text{交换了顺序}
\end{split}\tag{1.217}
$$

由于聚合 T 矩阵性质20:

$$\hat{T}_0(u)=R_{N0}(u+\theta_N)\cdots R_{j+1,0}(u+\theta_{j+1}),\tag{1.218}$$

满足:

$$P^{(d)}_{21}\hat{T}_2(-\theta_j)\hat{T}_1(-\theta_j+\delta)P^{(d)}_{21}=\hat{T}_2(-\theta_j)\hat{T}_1(-\theta_j+\delta)P^{(d)}_{21}\tag{1.219}$$

于是:

$$
\begin{split}
\text{上式}=&R_{j-1,j}(-\theta_j+\theta_{j-1})\cdots R_{1j}(-\theta_j+\theta_1)P^{(d)}_{ab}\\
&R_{Na}(-\theta_j+\theta_N)\cdots R_{j+1,a}(-\theta_j+\theta_{j+1})R_{Nb}(-\theta_j+\theta_N+\delta)\cdots R_{j+1,b}(-\theta_j+\theta_{j+1}+\delta)P^{(d)}_{ab}\\
&S_{ab}R_{aj}(0)\cdots R_{1b}(-\theta_j+\theta_1+\delta)\\
=&P^{(d)}_{ab}R_{j-1,j}(-\theta_j+\theta_{j-1})\cdots R_{1j}(-\theta_j+\theta_1)\\
&R_{Na}(-\theta_j+\theta_N)\cdots R_{j+1,a}(-\theta_j+\theta_{j+1})R_{Nb}(-\theta_j+\theta_N+\delta)\cdots R_{j+1,b}(-\theta_j+\theta_{j+1}+\delta)P^{(d)}_{ab}\\
&S_{ab}R_{aj}(0)\cdots R_{1b}(-\theta_j+\theta_1+\delta)
\end{split}\tag{1.220}
$$

也就是:

$$\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta)=P^{(d)}_{ab}\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta)\tag{1.221}$$

□

COROLLARY 25 (转移算符乘积表达). 定义

$$t(u)=tr_0\left\{K^+_0(u)T_0(u)K^-_0(u)\hat{T}_0(u)\right\}\tag{1.222}$$

于是

$$t_1(u)t_2(u+\delta)\tag{1.223}$$

$$=f(2u+\delta+k)^{-1}tr_{12}\Big\{ \tag{1.224}$$

$$K_1^+(u+\delta)M_2R_{21}(-2u-\delta-2k)M_2^{-1}K_2^+(u)T_2(u)T_1(u+\delta) \tag{1.225}$$

$$K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)\hat{T}_2(u)\hat{T}_1(u+\delta)\Big\} \tag{1.226}$$

下面是证明

证明. 考虑到 Crossing-Unitary 22

$$R_{12}^{t_1}(u)M_1R_{21}^{t_1}(-u-2k)M_1^{-1}=f(u+k) \tag{1.227}$$

由转移算符的定义:

$$t_2(u)t_1(u+\delta)=tr_2\left\{K_2^+(u)T_2(u)K_2^-(u)\hat{T}_2(u)\right\} \tag{1.228}$$

$$\times\, tr_1\left\{K_1^+(u+\delta)T_1(u+\delta)K_1^-(u+\delta)\hat{T}_1(u+\delta)\right\}^{t_1} \tag{1.229}$$

$$=f(2u+\delta+k)^{-1}tr_{12}\Big\{K_2^+(u)T_2(u)K_2^-(u)\hat{T}_2(u) \tag{1.230}$$

$$\left[T_1(u+\delta)K_1^-(u+\delta)\hat{T}_1(u+\delta)\right]^{t_1} \tag{1.231}$$

$$R_{12}^{t_1}(2u+\delta)M_1R_{21}^{t_1}(-2u-\delta-2k)M_1^{-1}\ K_1^+(u+\delta)^{t_1}\Big\} \tag{1.232}$$

$$=f(2u+\delta+k)^{-1}tr_{12}\Big\{K_2^+(u)T_2(u)K_2^-(u)\hat{T}_2(u) \tag{1.233}$$

$$\left[T_1(u+\delta)K_1^-(u+\delta)\hat{T}_1(u+\delta)\right]^{t_1} \tag{1.234}$$

$$R_{12}^{t_1}(2u+\delta)\left[(M_1^{-1})^{t_1}R_{21}(-2u-\delta-2k)(M_1)^{t_1}\right]^{t_1}\ K_1^+(u+\delta)^{t_1}\Big\} \tag{1.235}$$

$$=f(2u+\delta+k)^{-1}tr_{12}\Big\{\left[K_1^+(u+\delta)(M_1^{-1})^{t_1}R_{21}(-2u-\delta-2k)\right. \tag{1.236}$$

$$\left.(M_1)^{t_1}K_2^+(u)T_2(u)K_2^-(u)\hat{T}_2(u)\ \right]^{t_1} \tag{1.237}$$

$$\left[R_{12}(2u+\delta)T_1(u+\delta)K_1^-(u+\delta)\hat{T}_1(u+\delta)\right]^{t_1}\Big\} \tag{1.238}$$

考虑到

$$\Big\{tr_{12}\{A_{12}^{t_1}B_{12}^{t_1}\}=tr_{12}\{A_{12}B_{12}\}. \tag{1.239}$$

于是:

$$\text{上式}=f(2u+\delta+k)^{-1}tr_{12}\Big\{K_1^+(u+\delta)(M_1^{-1})^{t_1}R_{21}(-2u-\delta-2k) \tag{1.240}$$

$$M_1^{t_1}K_2^+(u)T_2(u)K_2^-(u)\hat{T}_2(u) \tag{1.241}$$

$$R_{12}(2u+\delta)T_1(u+\delta)K_1^-(u+\delta)\hat{T}_1(u+\delta)\Big\} \tag{1.242}$$

考虑到 RTT 关系 3 7

$$\Big\{\hat{T}_a(u)R_{ba}(2u+\delta)T_b(u+\delta)=T_b(u+\delta)R_{ba}(2u+\delta)\hat{T}_a(u). \tag{1.243}$$

于是:

$$\text{上式}=f(2u+\delta+k)^{-1}tr_{12}\Big\{K_1^+(u+\delta)(M_1^{-1})^{t_1}R_{21}(-2u-\delta-2k) \tag{1.244}$$

$$(M_1)^{t_1}K_2^+(u)T_2(u)K_2^-(u)T_1(u+\delta) \tag{1.245}$$

$$\left.R_{12}(2u+\delta)\hat{T}_2(u)K_1^-(u+\delta)\hat{T}_1(u+\delta)\right\} \tag{1.246}$$

$$=f(2u+\delta+k)^{-1}tr_{12}\Big\{ \tag{1.247}$$

$$K_1^+(u+\delta)(M_1^{-1})^{t_1}R_{21}(-2u-\delta-2k)M_1^{t_1}K_2^+(u)T_2(u)T_1(u+\delta) \tag{1.248}$$

$$\left.K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)\hat{T}_2(u)\hat{T}_1(u+\delta)\right\} \tag{1.249}$$

考虑到 M 算符和 R 算符之间的对易关系 (10):

$$\Big\{(M_2^{t_2})^{-1}R_{12}(u)M_2^{t_2}=M_1^{t_1}R_{12}(u)(M_1^{t_1})^{-1} \tag{1.250}$$

于是:

$$\text{上式}=f(2u+\delta+k)^{-1}tr_{12}\Big\{ \tag{1.251}$$

$$K_1^+(u+\delta)M_2^{t_2}R_{21}(-2u-\delta-2k)(M_2^{t_2})^{-1}K_2^+(u)T_2(u)T_1(u+\delta) \tag{1.252}$$

$$\left.K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)\hat{T}_2(u)\hat{T}_1(u+\delta)\right\} \tag{1.253}$$

如果考虑到 M 的转置和自身相等:

$$\text{上式}=f(2u+\delta+k)^{-1}tr_{12}\Big\{ \tag{1.254}$$

$$K_1^+(u+\delta)M_2R_{21}(-2u-\delta-2k)M_2^{-1}K_2^+(u)T_2(u)T_1(u+\delta) \tag{1.255}$$

$$\left.K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)\hat{T}_2(u)\hat{T}_1(u+\delta)\right\} \tag{1.256}$$

□

Corollary 26 (转移算符 Crossing-Symmetry).

$$t(-u-k)=t(u) \tag{1.257}$$

其中，转移算符如上面的定义:

$$t(u)=tr_0\left\{K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u)\right\} \tag{1.258}$$

下面是证明:

证明.

$$\begin{aligned} t(-u-k)&=tr_0\left\{K_0^+(-u-k)T_0(-u-k)K_0^-(-u-k)\hat{T}_0(-u-k)\right\}\\ &=tr_0\left\{K_0^+(-u-k)T_0(-u-k)\right\}^{t_0}\left\{K_0^-(-u-k)\hat{T}_0(-u-k)\right\}^{t_0}\end{aligned} \tag{1.259}$$

由于单值算符的 cross symmetry (8, 9)

$$\begin{cases}T_0^{t_0}(-u-k)=V_0^{t_0}\hat{T}_0(u)(V_0^{-1})^{t_0}\\ \hat{T}_0^{t_0}(-u-k)=V_0T_0(u)V_0^{-1}\end{cases} \tag{1.260}$$

于是:

$$\begin{aligned}\text{上式}&=tr_0V_0^{t_0}\hat{T}_0(u)(V_0^{-1})^{t_0}\left\{K_0^+(-u-k)\right\}^{t_0}V_0T_0(u)V_0^{-1}\left\{K_0^-(-u-k)\right\}^{t_0}\\ &=tr_0\hat{T}_0(u)(V_0^{-1})^{t_0}\left\{K_0^+(-u-k)\right\}^{t_0}V_0T_0(u)V_0^{-1}\left\{K_0^-(-u-k)\right\}^{t_0}V_0^{t_0}\end{aligned} \tag{1.261}$$

考虑到混合关系 (8)

$$\left\{(V_0^{-1})^{t_0}\left\{K_0^+(-u-k)\right\}^{t_0}V_0 = \tfrac{1}{g(u)}tr_1\left\{R_{01}(0)R_{01}(2u)K_1^+(u)\right\}\right. \tag{1.262}$$

于是:

$$上式 = tr_0\hat{T}_0(u)tr_1\left\{R_{01}(0)R_{01}(2u)K_1^+(u)\right\}T_0(u)V_0^{-1}\left\{K_0^-(-u-k)\right\}^{t_0}V_0^{t_0}\frac{1}{g(u)} \tag{1.263}$$

考虑到 RT 置换关系 2(4):

$$\left\{\hat{T}_0(u)R_{01}(0) = R_{10}(0)\hat{T}_1(u)\right. \tag{1.264}$$

于是:

$$上式 = tr_0tr_1R_{10}(0)\hat{T}_1(u)R_{01}(2u)T_0(u)V_0^{-1}\left\{K_0^-(-u-k)\right\}^{t_0}V_0^{t_0}K_1^+(u)\frac{1}{g(u)} \tag{1.265}$$

由于 RTT 关系 3 (7):

$$\left\{\hat{T}_a(v)R_{ba}(u+v)T_b(u) = T_b(u)R_{ba}(u+v)\hat{T}_a(v)\right. \tag{1.266}$$

于是:

$$上式 = tr_0tr_1R_{10}(0)T_0(u)R_{01}(2u)\hat{T}_1(u)V_0^{-1}\left\{K_0^-(-u-k)\right\}^{t_0}V_0^{t_0}K_1^+(u)\frac{1}{g(u)} \tag{1.267}$$

由于 RT 置换关系 1 (3):

$$\left\{R_{10}(0)T_0(u) = T_1(u)R_{01}(0)\right. \tag{1.268}$$

$$上式 = tr_0tr_1T_1(u)R_{01}(0)R_{01}(2u)V_0^{-1}\left\{K_0^-(-u-k)\right\}^{t_0}V_0^{t_0}\hat{T}_1(u)K_1^+(u)\frac{1}{g(u)} \tag{1.269}$$

由混合关系 2(9):

$$\left\{tr_0R_{01}(0)R_{01}(2u)V_0\left\{K_0^-(-u-k)\right\}^{t_0}V_0^{t_0} = g(u)K_1^-(u)\right. \tag{1.270}$$

$$\begin{aligned}上式 &= tr_1T_1(u)K_1^-(u)\hat{T}_1(u)K_1^+(u)\\ &= tr_1K_1^+(u)K_1^-(u)\hat{T}_1(u)\\ &= t(u)\end{aligned} \tag{1.271}$$

COROLLARY 27 (转移算符对易). *利用 Yang-Baxter 方程，反射方程，对偶反射方程可以得到:*

$$[t(u),t(v)] = 0 \tag{1.272}$$

□

1.1.3. 小结. 这里做一个小结。可以将理论方面的公式总结为一些定理（proposition）和引理。其中定理是所定义的基本算符一定需要满足的关系，而其他的性质（引理）都可以用这些定理推导出来。**Propositions :** 基本的定理

regularity 1:

$$R_{12}(0) \sim f^{\frac{1}{2}}(0)\mathcal{P}_{12} \tag{1.273}$$

unitary 2:

$$R_{12}(u)R_{21}(-u) \sim f(u)\mathbb{I} \tag{1.274}$$

Crossing-Symmetry 4:

$$\begin{aligned} R_{12}(u) = V_1^{-1}R_{21}^{t_1}(-u-k)V_1 = (V_2^{t_2})^{-1}R_{21}^{t_2}(-u-k)V_2^{t_2} \\ M = V^tV^{-1} \quad M^{-1} = V(V^t)^{-1} \end{aligned} \tag{1.275}$$

YBE 5:

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{1.276}$$

Reflection func 6:

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v) \tag{1.277}$$

Dual Reflection func 7:

$$\begin{aligned} &R_{21}(u-v)K_2^+(v)M_2^{-1}R_{12}(-u-v-2k)M_2K_1^+(u) \\ &\quad = K_1^+(u)M_2R_{21}(-u-v-2k)M_2^{-1}K_2^+(v)R_{12}(u-v) \end{aligned} \tag{1.278}$$

$$\Downarrow$$

Corollarys: 通过定理可以导出一些引理

Projector 1: $R_{12}(\delta) = P_{12}^{(d)} \times S_{12}$
RT exchange1 3: $R_{10}(0)T_0(u) = T_1(u)R_{01}(0)$
RT exchange2 4: $\hat{T}_0(u)R_{01}(0) = R_{10}(0)\hat{T}_1(u)$
RTT 关系 5: $R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$
RTT 关系-2 (6): $R_{21}(u-v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)R_{12}(u-v)$
RTT 关系-3 (7): $\hat{T}_a(u)R_{ba}(2u+\delta)T_b(u+\delta) = T_b(u+\delta)R_{ba}(2u+\delta)\hat{T}_a(u)$
Crossing Unitary (22):

$$R_{12}^{t_1}(u)M_1R_{21}^{t_1}(-u-2k)M_1^{-1} = R_{12}^{t_2}(u)M_2^{-1}R_{21}^{t_2}(-u-2k)M_2 = f(u+k) \tag{1.279}$$

$$\Downarrow$$

聚合操作 ：

单值矩阵相乘导出投影算符-1 (23):

$$T_a(\theta_j)T_b(\theta_j+\delta)=P_{ba}^{(d)}T_a(\theta_j)T_b(\theta_j+\delta) \tag{1.280}$$

单值矩阵相乘导出投影算符-2 (24):

$$\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta)=P_{ab}^{(d)}\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta) \tag{1.281}$$

聚合 R 算符-1 (11):

$$P_{12}^{(d)}R_{23}(u)R_{13}(u+\delta)P_{12}^{(d)}=R_{23}(u)R_{13}(u+\delta)P_{12}^{(d)} \tag{1.282}$$

聚合 R 算符-2 (13):

$$P_{21}^{(d)}R_{32}(u)R_{31}(u+\delta)P_{21}^{(d)}=R_{32}(u)R_{31}(u+\delta)P_{21}^{(d)} \tag{1.283}$$

聚合 K^- 算符 (16):

$$P_{12}^{(d)}K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)P_{21}^{(d)}=K2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)P_{21}^{(d)} \tag{1.284}$$

聚合 K^+ 算符 (18):

$$\begin{aligned}&P_{21}^{(d)}K_1^+(u+\delta)M_2R_{21}(-2u-2k-\delta)M_2^{-1}K_2^+P_{12}^{(d)}\\&\quad=K_1^+(u+\delta)M_2R_{21}(-2u-2k-\delta)M_2^{-1}K_2^+(u)P_{12}^{(d)}\end{aligned} \tag{1.285}$$

$$\Downarrow$$

转移算符的乘积表达式 :

转移算符相乘 (25):

$$t_1(u)t_2(u+\delta) \tag{1.286}$$

$$=f(2u+\delta+k)^{-1}tr_{12}\Big\{ \tag{1.287}$$

$$K_1^+(u+\delta)M_2R_{21}(-2u-\delta-2k)M_2^{-1}K_2^+(u)T_2(u)T_1(u+\delta) \tag{1.288}$$

$$K_2^-(u)R_{12}(2u+\delta)K_1^-(u+\delta)\hat{T}_2(u)\hat{T}_1(u+\delta)\Big\} \tag{1.289}$$

转移算符 cross symmetry (26):

$$t(-u-k)=t(u) \tag{1.290}$$

CHAPTER 2

量子信息基础

2.1. Linear algebra basic

2.1.1. Schmidt 分解. 将一个复合系统分成 A,B 两个子空间, 则复合系统的态矢可以用这两个子空间的基矢展开。

$$|\psi\rangle = C_{mn}|A_m\rangle|B_n\rangle \tag{2.1}$$

其中 $\{|A_m\rangle\}$,$\{|B_n\rangle\}$ 是两个子空间中的任意正交基底。施密特 Schmidt 定理说，他可以写成:

$$|\psi\rangle = \Sigma_n \sqrt{\lambda_n}|a_n\rangle|b_n\rangle \tag{2.2}$$

求和指标 n 的最大值为 A,B 中较小空间的维数。$\{|a_n\rangle\}$，$\{|b_n\rangle\}$ 分别是空间的密度算符 的本征向量。对应的本征值是 $\{\lambda_n\}$。先看一下一个关于举证 SVD 定理的证明, 首先对于方阵。

Theorem 2.1 (eigendecomposition of A). *对于任意一个对称矩阵 $A \in R^{n\times n}$, $\exists$ 一个正交矩阵: $Q = [q_1, q_2 ... q_n]$ 和一个对角矩阵 $\Lambda = diag(\lambda_1, ... \lambda_n)$ s.t. $A = Q\Lambda Q^T$。*
我们指出 λ_i 是 A 的本征值, q_i 是相应的本征向量。(Q 也可以叫做本征向量矩阵 Eigenvectors matrix)(这 n 个本征向量是相互正交的, 并且有模是 1)
□

Theorem 2.2 (Singular Value Decomposition(SVD)). *$\forall$ $X \in R^{n\times d}$, $\exists$ $U \in R^{n\times n}$, $V \in R^{d\times d}$; $UU^T = I_{n\times n}$,$VV^T = I_{d\times d}$, and a nonnegatice "diagonal" matrix $\Sigma \in R^{n\times d}$ such that:*

$$X_{n\times d} = U_{n\times n}\Sigma_{n\times d}V^T_{d\times d} \tag{2.3}$$

□

SVD 分解和上面提到的对称矩阵的分解有着很大的联系。

$$\begin{aligned} XX^T &= U\Sigma V^T V\Sigma^T U^T = U(\Sigma\Sigma^T)U^T \\ X^T X &= V\Sigma^T U^T U\Sigma V^T = V(\Sigma^T\Sigma)V^T \end{aligned} \tag{2.4}$$

可以看出来 XX^T 和 X^TX 是对称矩阵。所以服从上面的定理。于是我们说

- U 是 XX^T 的本征向量矩阵。
- V 是 X^TX 的本征向量的矩阵。

我们对于给定的 X, 想要做 SVD 分解。首先我们计算 X^TX, 他是一个 $d\times d$ 维的矩阵。对他进行对角分解可以求出 $\Sigma^T\Sigma$, 也可以求出他的本征向量构成的矩阵 V。然后我们再把 SVD 方程写成 (给 SVD 方程左右同时乘以 V):

$$\begin{gathered} XV = U\Sigma \\ X[v_1 ... v_d] = [u_1 ... u_n]\Sigma_{n\times d} \end{gathered} \tag{2.5}$$

我们首先说 $\Sigma_{n\times d}$ 虽然是对角矩阵，但是我们说他的非零元素只有 r 个。$(r < min(n,d))$ 于是上面的矩阵方程写成这样:

$$X[v_1..v_r..v_d] = [u_1..u_r..u_n]\begin{bmatrix}\sigma_1 & & & \\ & . & & \\ & & . & \\ & & & \sigma_r \\ & & & \end{bmatrix}_{n\times d} \tag{2.6}$$

对于上面的式子, 我们考虑第 i 列的结果。于是:

$$Xv_i = \begin{cases}\sigma_i u_i & 1 \leq i \leq r \\ 0 & r < i \leq d\end{cases} \tag{2.7}$$

这件事情告诉我们我们的 U 矩阵一定是这样的 $u_i = \frac{1}{\sigma_i}Xv_i$ for $1 \leq i \leq r$。
当然上面只能说如果定理成立, 那么我们可以这样先找到 V, 再找到 U。
不过我们是有疑问的，比如说如果 Σ 写成了上面的那样有 r 个非零值的对角形式。$\Sigma\Sigma^T$ 和 $\Sigma^T\Sigma$ 的非零元素是完全一样的。而且 U 也要是 $\Sigma\Sigma^T$ 的本征向量, 问题才是自洽的。

SVD 定理证明. 为了简单，我们定义一个 C 矩阵 $C = X^TX \in R^{d\times d}$。于是 C 对称的，半正定的。这个为什么是半正定的呢? 在 SVD 定理中 $C = V\Lambda V^T$ 其中 $V \in R^{d\times d}$, $VV^T = V^TV = I_{d\times d}$ 对角矩阵 $\Lambda = diag(\lambda_1...\lambda_d)$ 其中，$\lambda_1 \geq ...\lambda_r > 0 = \lambda_{r+1}... = \lambda_d$。$r = rank(X) \leq d$ 这个 r 为什么等于 X 的秩呢?

于是我们简单地让 $\sigma_i = \sqrt{\lambda_i}$

$$\Sigma = \begin{bmatrix} diag(\sigma_1,...\sigma_r) & O_{r\times(d-r)} \\ O_{(n-r)\times r} & O_{(n-r)\times(d-r)}\end{bmatrix} \tag{2.8}$$

同时, 定义

$$u_i \equiv \frac{1}{\sigma_i}Xv_i \ \in R^n \quad \text{for each } 1 \leq i \leq r \tag{2.9}$$

其实直接可以看出来 u_i 确实是 X^TX 的基底。本征值是 σ_i。然后他们其实也是正交的

$$\begin{aligned} u_i^Tu_j &= (\frac{1}{\sigma_i}Xv_i)^T(\frac{1}{\sigma_j}Xv_j) = \frac{1}{\sigma_i\sigma_j}v_i^TX^TXv_j = \frac{1}{\sigma_i\sigma_j}v_i^T(\lambda_jv_j) = \frac{\sigma_j}{\sigma_i}v_i^Tv_j \\ &= \begin{cases}1 & i = j \\ 0 & i \neq j\end{cases}\end{aligned} \tag{2.10}$$

然后我们需要选取 $u_{r+1}...u_n \in R^n$ 来构建一个正交的矩阵:

$$U = [u_1...u_r, u_{r+1}...u_n] \in R^{n\times n} \tag{2.11}$$

这样, SVD 方程就得到了满足。 □

我下面说一个求 v_1 的方法我们考虑一个矩阵 $A \in R^{m\times n}$. SVD 正交化：$A = U\Sigma V^T$ 同时定义 $C = A^TA \in R^{n\times n}$

$$\begin{aligned} C &= V(\Sigma^T\Sigma)V^T = \Sigma\sigma_i^2v_iv_i^T \\ &\cdots \\ C^k &= V(\Sigma^T\Sigma)^2V^T = \Sigma\sigma_i^{2k}v_iv_i^T\end{aligned} \tag{2.12}$$

当 Σ_1 是本征值里最大的一项时。当 k 很大时候只有 σ_1 是有贡献的。

$$C^k = \sigma_1^{2k} v_1 v_1^T \tag{2.13}$$

当然这时候只要把 C^k 的第一项归一化就可以得到 v_1。
实际上一般不是这么做的, 而是右乘一个向量 x。$x = \Sigma c_i v_i$。

$$C^k x = \sigma_1^{2k} v_1 v_1^T c_1 v_1 = c_1 \sigma_1^{2k} v_1 \tag{2.14}$$

同样的，对这个量归一化可以得到 v_1.

那么这个分解方式在量子力学里面是什么样子的呢:
现在考虑有两个子空间, 叫做 1 空间和 2 空间。里面的基向量分别有 n 和 d 个。他的态这么描述:

$$|\psi\rangle = |\psi_1\rangle_i X_{i,j} |\psi_2\rangle_j \tag{2.15}$$

如果用 SVD 分解之后:

$$|\psi\rangle = |\psi_1\rangle_i X_{i,j} |\psi_2\rangle_j = |\psi_1\rangle_i U_{i\alpha} \Sigma_{\alpha\beta} (V^T)_{\beta j} |\psi_2\rangle_j \tag{2.16}$$

其中，U 是 $n \times n$ 的矩阵。V 是 $d \times d$ 的矩阵。Σ 是一个对角矩阵。这样对 α 和 β 的求和就变成了对一个量的求和。于是总的态就表示为:

$$|\psi\rangle = \Sigma_{\alpha=1}^{r} |\psi_1'\rangle_\alpha \Sigma_{\alpha\alpha} |\psi_2'\rangle_\alpha \tag{2.17}$$

r 是 X 的秩。

2.1.2. QR decomposition. 这里解释一下 QR 分解。首先考虑有一个矩阵 A, 是需要求的矩阵。

$$A = [a_1 | a_2 ... | a_n] \tag{2.18}$$

然后考虑一个找正交归一化基态矢量的方法。首先直接定义

$$u_1 = a_1 \tag{2.19}$$

于是第一个基底就说是

$$e_1 = \frac{u_1}{|u_1|} \tag{2.20}$$

然后找到一个和之前的基底正交的向量 u_2，然后就说第二个基底是 e_2

$$u_2 = a_2 - (a_2 \cdot e_1) e_1 \quad \rightarrow \quad e_2 = \frac{u_2}{|u_2|} \tag{2.21}$$

于是按照这个规律:

$$u_{k+1} = a_{k+1} - (a_{k+1} \cdot e_1) e_1 - ... - (a_{k+1} \cdot e_k) e_k \tag{2.22}$$

于是我们说 A 可以构造成这个样子:

$$A = [a_1 | a_2 ... | a_n] = [e_1 | ... | e_n] \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & ... & a_n \cdot e_1 \\ 0 & a_2 \cdot e_2 & ... & ... \\ \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & a_n \cdot e_n \end{bmatrix} \tag{2.23}$$

这个就叫做 QR 分解。一般写成:

$$A = QR \tag{2.24}$$

然后就有一些性质

- 如果说 A 是维数是 $m \times n$ 的一个矩阵。那么 Q 的维数是 $m \times n$。同时 R 的维数是 $n \times n$
- R 是上三角矩阵 (upper triangular Matrix)
- $Q^TQ = I$。并不能是使 $QQ^T = I$。因为实际上如果吧 Q^TQ 显式地写出来:

$$Q^TQ = \begin{bmatrix} e_1^T \\ e_2^T \\ e_3^T \\ \vdots \\ e_n^T \end{bmatrix} \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = I_{n\times n} \tag{2.25}$$

但是如果把 QQ^T 写出来，他实际上是:

$$QQ^T = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \begin{bmatrix} e_1^T \\ e_2^T \\ e_3^T \\ \vdots \\ e_n^T \end{bmatrix} = \Sigma_i^n e_i e_i^T \tag{2.26}$$

这个东西没有等于单位矩阵的性质。
- R 的模等于 A 的模。
这个是因为单纯对 R 取模的平方会有:

$$|R|^2 = \Sigma_{ij} R_{ij}^2 = |a_1|^2 + |a_2|^2 ... + |a_n|^2 = tr(A^TA) \tag{2.27}$$

这个是因为 a_i 确实只在 $e_1 \cdots e_i$ 上面有投影。(e_{i+1} 避免了和 $e_1 \cdots e_i$ 也就是 a_i 的正交性)

2.2. Density matrix

Difination of Density matrix:

$$\rho = \Sigma_i C_i |\psi_i\rangle\langle\psi_i| \tag{2.28}$$

in which, c_i is the probability for a state to be at $|\psi_i\rangle$. For a pure state, the state can be represent as:$|\psi\rangle = c_1|u_1\rangle ... + c_n|u_n\rangle$ in which, $|u_i\rangle$ is a set of basis vector.

For a pure state, the expectation value of the operator A can be represent as:

$$\begin{aligned} \langle A\rangle = \langle\psi|A|\psi\rangle \\ &= (c_1^*\langle u_1| ... + c_n^*\langle u_n|)A(c_1|u_1\rangle ... + c_n|u_n\rangle) \\ &= \Sigma_{ij} c_i^* c_j \langle u_i|A|u_j\rangle \end{aligned} \tag{2.29}$$

As the coefficient is determined by:

$$C_i = \langle u_i|\psi\rangle \quad C_i^* = \langle\psi|u_i\rangle \tag{2.30}$$

In this case:

$$\begin{aligned} \langle A\rangle &= \Sigma_{ij}\langle\psi|u_i\rangle\langle u_j|\psi\rangle A_{ij} \\ &= \Sigma_{ij}\langle u_j|\psi\rangle\langle\psi|u_i\rangle\langle u_i|A|u_j\rangle \\ &= Tr(\rho A) \end{aligned} \tag{2.31}$$

For a pure state, the density matrix satisfies:

$$Tr(\rho^2) = Tr(\rho) = 1 \tag{2.32}$$

As for a mixed state

$$Tr(\rho^2) \neq Tr(\rho) = 1 \tag{2.33}$$

The probability of measurement a_n:

$$p(a_n) = \langle u_n|\rho|u_n\rangle = Tr(|u_n\rangle\langle u_n|\rho) = Tr(P_n\rho) \tag{2.34}$$

In which $P_n = |u_n\rangle\langle u_n|$ is a projection operator.
After the measurement result in a_n. The density matrix would be:

$$\rho \to \frac{P_n\rho P_n}{Tr(P_n\rho)} = |u_n\rangle\langle u_n| \tag{2.35}$$